A NOTE ON A SEQUENTIAL PROBABILITY RATIO TEST

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Summary. This is a type of problem that lies outside the scope of the exponential family. If the \( Z_i \) are real valued, with density \( \frac{1}{\sigma} \exp \left[ - \frac{Z_i - \mu}{\sigma} \right] h(z - \mu) \) (here \( h(z) = 1 \) or 0, according as \( z > 0 \) or \( z \leq 0 \)), and where one value of \( \sigma \) is tested against another, it is shown that

\[
\ln R_n = b + \sum_{i=1}^{n} (Z_i - U_n - a),
\]

where \( U = \min_{1 \leq i \leq n} Z_i \), \( a \) is a positive constant. Using this expression it is proved that for every non-degenerate distribution of the \( Z_i \), \( P(N > n) \) is exponentially bounded, which, of course, implies termination with probability 1.
§1. Introducción. In Abu-Salih [1], the following model was discussed. $Z, Z_1, Z_2, \ldots$ is a sequence of independent identically distributed (iid) m-vectors, with $k$-parameter exponential distribution $P$. $G^*$ is a group of transformations of the form $Z_n \rightarrow C(Z_n + b)$, where $C \in G$, $G$ is a Lie group of $m \times m$ nonsingular matrices, $\dim G \geq 1$, and $G$ is closed in the general linear group $GL(m, \mathbb{R})$; $b$ is an $m$-vector of reals, and the totality of vectors $b$ form an invariant subspace under $G$.

Let $P$ have the density

$$P^Z_{\Theta}(x) = B(\Theta) h(z) \exp \left( \sum_{j=1}^{k} \Theta_j S_j(z) \right)$$

with respect to Lebesgue measure on the $m$-dimensional Euclidean space $E^m$, and where $\Theta = (\Theta_1, \ldots, \Theta_k)$' belongs to the natural parameter space $\Omega$, and $S = (S_1, \ldots, S_k)$' is a continuously differentiable mapping of $E^m$ into $E^k$.

Let $U = (U_1, U_2, \ldots)$ be a maximal invariant under $G^*$ in the sample space, and $\gamma = \gamma(\Theta)$ a maximal invariant in $\Omega$. For given $\Theta_1, \Theta_2 \in \Omega$ such that $\gamma(\Theta_1) \neq \gamma(\Theta_2)$, write $U^n = (U_1, U_2, \ldots, U_n)$, and let $P_1^n$ be its density under $\gamma(\Theta_i)$, $i = 1, 2$, with respect to some $\sigma$-finite
measure. Let

\[(1.2) \quad r_n = P'_2n / P'_1n\]

and

\[(1.3) \quad R_n = r_n(U^n) ;\]

then \(R_n\) is the probability ratio at the \(n^{th}\) stage of sampling based on the maximal invariant \(U\). A sequential probability ratio test (SPRT) based on \(\{R_n\}\) continues sampling as long as \(B < R_n < A\) (\(B\) and \(A\) are two fixed stopping bounds), stops and accepts \(\Theta^1\) (resp. \(\Theta^2\)) the first time that \(R_n < B\) (resp. \(R_n > A\)). A SPRT based on \(\{R_n\}\) will be called in invariant SPRT.

The limiting behavior of \(R_n\) is studied in [1] under the assumption that the actual distribution belongs to certain family \(\mathcal{F}\), and it is proved that there are three cases:

(i) \(\lim_{n \to \infty} R_n = \infty, \text{ a.e.} P,\)

(ii) \(\lim_{n \to \infty} R_n = 0, \text{ a.e.} P,\)

(iii) \(\limsup_{n \to \infty} R_n = \infty, \text{ a.e.} P, \text{ or } \liminf_{n \to \infty} R_n = 0, \text{ a.e.} P,\)

each one corresponding to a subfamily of \(\mathcal{F}\). This
establishes termination with probability 1 of the (SPRT) based on \( \{R_n\} \).

The results obtained above form an extension of those of Wijsman in [2] and [3] in which the underlying model was assumed to be multivariate normal. Our methods of proof are closely modeled on those in [2] and [3].

§2. **Sequential probability ratio test based on negative exponential distribution with location parameter.** It is of interest to consider a model similar to the exponential one, except for a location parameter in the function \( h(z) \) of (1.1). We were unable to reduce this model to the one we have summarized in the introduction. Yet, we have worked a simple example for which we obtained an exponential bound on \( P(N > n) \) for any non-degenerate distribution \( P \).

Let \( Z,Z_1,Z_2,\ldots \) be iid random variables with density \( p_{\theta}^Z \) with respect to Lebesgue measure. Assume

\[
(2.1) \quad p_{\theta}^Z(z) = \frac{1}{\sigma} h(z-\mu) \exp\left(-\frac{1}{\sigma}(z-\mu)\right)
\]

where \( \Theta = (\mu,\sigma) \) and \( h(x) = 1 \) if \( x > 0 \),

\( h(x) = 0 \) if \( x \leq 0 \). \( \Omega = \{\Theta = (\mu,\sigma); -\infty < \mu < \infty, \sigma > 0\} \)

is the parameter space. The joint density of
(Z_1, \ldots, Z_n)\) is given by

\[
(2.2) \quad \frac{1}{\sigma} \left[ \prod_{i=1}^{n} h(z_i - \mu) \right] \exp\left[ -\frac{1}{\sigma} \sum_{i=1}^{n} (z_i - \mu) \right].
\]

Test about \(\sigma\), e.g. \(H_0: \sigma = \sigma_1\) vs \(H_1: \sigma = \sigma_2\), where \(\sigma_1 > \sigma_2\). Consider the group of translations \(G\) acting on the sample space as follows:

\[
g: Z_i \rightarrow Z_i + a \quad \text{for} \quad i = 1, 2, \ldots
\]

where \(-\infty < a < \infty\) and \(g \in G\). It is clear that \(G\) leaves the model invariant.

Using (2.11) in [1] ((3.3) in [2]) we get

\[
(2.3) \quad \frac{1}{\sigma_2^n} \int_0^1 \frac{1}{\sigma_1^n} \exp\left[ -\frac{1}{\sigma_1} \sum_{i=1}^{n} (z_i + a - \mu) \right] \prod_{i=1}^{n} h(z_i + a - \mu) \, da
\]

where \(\sigma_1 < \sigma_2\).
(Where \( u_n = \min_{1 \leq i \leq n} z_i \))

\[
= (\frac{\sigma_1}{\sigma_2})^{n-1} \exp \left[ \left( \frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right) \sum_{i=1}^{n} (z_i - u_n) \right].
\]

But \( R_n = r_n (Z_1, \ldots, Z_n) \), hence from (2.3):

\[
(2.4) \ln R_n = \ln \left( \frac{\sigma_1}{\sigma_2} \right)^{-1} + n \ln \frac{\sigma_1}{\sigma_2} \]
\[
+ \left( \frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right) \sum_{i=1}^{n} (Z_i - U_n).
\]

The SPRT mentioned above will continue sampling as long as \( \ln B < \ln R_n < \ln A \) and, from (2.4), it continues sampling if

\[
(2.5) \ln B + \ln \frac{\sigma_1}{\sigma_2} \]
\[
< n \ln \frac{\sigma_1}{\sigma_2} + \left( \frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right) \sum_{i=1}^{n} (Z_i - U_n) \]
\[
< \ln A + \ln \frac{\sigma_1}{\sigma_2},
\]

where \( U_n = \min_{1 \leq i \leq n} Z_i \). Let

\[
A_1 = \frac{\ln B + \ln \frac{\sigma_1}{\sigma_2}}{\left( \frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right)}
\]

(2.6)
\[
A_2 = \frac{\ln A + \ln \frac{\sigma_1}{\sigma_2}}{\frac{1}{\sigma_1} - \frac{1}{\sigma_2}}
\]
\[
a = -\ln \frac{\sigma_1}{\sigma_2} / \left( \frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right)
\]
\[ (a \text{ is positive since numerator and denominator have the same sign). Using (2.5) and (2.6) we continue sampling as long as:} \]

\[ (2.7) \quad A_1 < \sum_{i=1}^{n} (Z_i - U_n - a) < A_2 \]

From now on, we drop the assumption that \( Z, Z_1, Z_2, \ldots \) have the density (2.1) and instead consider \( Z_1, Z_2, \ldots \) iid random variables with distribution \( P \) and denote the distribution of \( (Z_1, \ldots, Z_n) \) also by \( P \). The only restriction we impose on \( P \) is that it be non-degenerate. Under this condition, we like to establish termination, with probability 1, of the SPRT based on (2.7) and find exponential bounds on \( P(N > n) \). Let

\[ (2.8) \quad E_n = \{A_1 < \sum_{i=1}^{n} (Z_i - U_n - a) < A_2\} \]

and let us compute \( P(E_{n+1} | E_1, \ldots, E_n) \). Given \( E_n \) with \( \sum_{i=1}^{n} (A_i - U_n - a) = d_n \), suppose \( Z_{n+1} > U_n \); then

\[ \sum_{i=1}^{n+1} (Z_i - U_{n+1} - a) = d_n + Z_{n+1} - U_n - a. \]

Hence, given \( E_n \) and \( Z_{n+1} > U_n \), \( E_{n+1} \) implies \( Z_{n+1} - U_n - a < D = A_2 - A_1 \), and, in particular

\[ (2.9) \quad U_n \leq Z_{n+1} < U_n + D + a. \]
On the other hand, given \( E_n \) and \( Z_{n+1} < U_n \), then \( E_{n+1} \) implies that

\[
\sum_{i=1}^{n+1} (Z_i - U_{n+1} - a) = d_n + n(U_n - Z_{n+1}) - a
\]

lies between \( A_1 \) and \( A_2 \) hence

\[
A_1 - d_n < n(U_n - Z_{n+1}) - a < A_2 - d_n,
\]

which implies

\[
U_n - \frac{D + a}{n} < Z_{n+1} < U_n + \frac{D - a}{n},
\]

Comparing (2.9) and (2.10) we have: given \( E_n \), then \( E_{n+1} \) implies

\[
U_n - \frac{L}{n} < Z_{n+1} < U_n + L,
\]

where \( L = D + a \). Furthermore, given \( E_1, \ldots, E_n \), then \( E_{n+1} \) implies (2.11), and so

\[
P(E_{n+1} \mid E_1, \ldots, E_n) = E[I_{F_{n+1}} \mid E_1, \ldots, E_n]
\]

\[
= E[E[I_{F_{n+1}} \mid U_n, E_1, \ldots, E_n] \mid E_1, \ldots, E_n]
\]

with probability 1, where \( I_{F_{n+1}} \) is the indicator function of \( F_{n+1} \) and \( F_{n+1} = \{ U_n - \frac{L}{n} < Z_{n+1} < U_n + L \} \).
Suppose that the support of $Z$ is not contained in an interval of length $2L$, and define 
\[ \rho = \sup_{-\infty < x < \infty} P(x - L < Z < x + L); \] then $\rho < 1$. Furthermore, since $Z_{n+1}$ is independent of $Z_1, \ldots, Z_n$ and hence independent of $E_1, \ldots, E_n$ and $U_n$, we have 
\[ E[I_{F_{n+1}} | E_1, \ldots, E_n, U_n = u_n] \]
\[ = P(u_n - \frac{L}{n} < Z_{n+1} < u_n + L | E_1, \ldots, E_n, U_n = u_n) \]
\[ = P(u_n - \frac{L}{n} < Z_{n+1} < u_n + L) \leq \rho < 1; \]
and therefore,
\[ E[E[I_{F_{n+1}} | E_1, \ldots, E_n, U_n] | E_1, \ldots, E_n] \leq \rho < 1; \]
hence (2.12) becomes:
\[ (2.13) \quad P(E_{n+1} | E_1, \ldots, E_n) \leq \rho < 1 \quad \text{for} \quad n = 1, 2, \ldots \]

But 
\[ P(N > n) = P(B < R_m < A, \quad 1 \leq m \leq n) \]
\[ = P(E_1E_2 \cdots E_n) \]
\[ = P(E_1)P(E_2 | E_1)P(E_3 | E_1E_2) \cdots P(E_n | E_1 \cdots E_{n-1}) \]
\[ \leq P(E_1) \rho^{n-1} < c \rho^n, \]
with $c > 0$, $\rho < 1$.

Therefore, we have established an exponential bound on $P(N > n)$, for $P$ with support not contain-
ed in an interval of length 2L.

The case of bounded support is considered in the rest of the paper. Without loss of generality, we assume that the support of \( P \) is \((0, b)\). We may do this because we are studying \[ \sum_{i=1}^n (Z_i - U_n - a) \] which is invariant under translations.

CASE 1: \( a \in (0, b) \). Let \( \delta \) be a positive constant such that \( a + 2\delta < b \). Hence

\[ \text{(2.14)} \quad P(Z > a + 2\delta) = p > 0. \]

Let

\[ m = \left\lceil \frac{D}{\delta} \right\rceil + 1 \quad (D = |A_2 - A_1|) \]

and

\[ v_k = \min_{1 \leq i \leq mk} Z_i \]

\[ \text{(2.15)} \quad E_k = \{ A_1 < \sum_{i=1}^{mk} (Z_i - v_k - a) < A_2 \} \]

(so \( E_k \) is the same as \( E_{mk} \) from (2.8))

\[ B_k = \{ Z_i > a + 2\delta, \quad i = m(k-1) + 1, \ldots, mk \} \]

\[ A_k = \{ v_k \leq \delta \}. \]

Since \( \delta > 0 \) then

\[ \text{(2.16)} \quad P(Z > \delta) = q_1 < 1. \]
Let
\[ \Delta_k = \sum_{i=1}^{m k} (Z_i - V_k - a) - \sum_{i=1}^{m (k-1)} (Z_i - V_{k-1} - a) \]
then
\[ \Delta_k = \sum_{i=m(k-1)+1}^{m k} (Z_i - a) - m V_k + m(k-1)(V_{k-1} - V_k) \]
\[ \geq \sum_{i=m(k-1)+1}^{m k} (Z_i - a) - m V_k \]
since \( V_k \leq V_{k-1} \). Also we have

(2.17) i) Given \( E_{k-1} \), then \( E_k \) implies \( |\Delta_k| < D \).

ii) Given \( A_k \), then \( B_{k+j} \) implies \( \Delta_{k+j} \geq 2m\delta \)
\[ - m\delta = m\delta \geq D, \text{ which implies} \]
\[ |\Delta_{k+j}| \geq D \text{ for } j = 1, 2, \ldots \]

Therefore, given \( E_{k+j} \) and \( A_k \) we have for any
\( j = 1, 2, \ldots \)

(2.18) \( B_{k+j+1} \) implies \( \bar{E}_{k+j+1} \) (\( \bar{E} \) denotes complementation)

(2.19) \( E_{k+j+1} \) implies \( \bar{B}_{k+j+1} \)

Now,

(2.20) \[
\Pr(E_1 E_2 \ldots E_{2k}) \leq \Pr(E_k E_{k+1} \ldots E_{2k}) = \Pr(E_k \ldots E_{2k} A_k) + \Pr(E_k \ldots E_{2k} \bar{A}_k)
\]
\[ \leq P(E_1 E_2 \ldots E_{2k} A_k) + P(\bar{A}_k) \]
\[ = P(E_1 E_2 \ldots E_{2k} A_k) + q_1^m, \]
by (2.16). And

\begin{equation}
(2.21) \quad P(E_1 E_2 \ldots E_{2k} A_k) = P(E_1 E_2 \ldots E_{2k} | A_k) P(A_k) \leq P(E_1 E_2 \ldots E_{2k} | A_k),
\end{equation}

because \( 0 < P(A_k) < 1 \) for \( k = 1, 2 \ldots \). Also

\begin{equation}
(2.22) \quad P(E_1 E_2 \ldots E_{2k} | A_k) = P(E_1 | A_k) P(E_2 + 1 | E_1, A_k) \ldots P(E_{2k} | E_{k-1}, E_{k+1} \ldots E_{2k-1}, A_k) \leq P(E_1 | A_k) P(B_2 + 1 | E_1, A_k) \ldots P(B_{2k} | E_{k-1}, A_k)
\end{equation}

by (2.19). But, since \( B_{k+j} \) is independent of \( A_k \) and \( E_{k+j-i} \) for \( i = 1, \ldots, j \) then

\begin{equation}
(2.23) \quad P(B_{k+j} | A_k, E_{k+j-i}, \ i = 1, \ldots, j) = P(B_{k+j}) = 1 - p^m = p_1
\end{equation}

where \( p \) is given by (2.14) and so \( p_1 < 1 \).

Using (2.23), (2.22), and (2.21), then (2.20) becomes:

\[ P(E_1 E_2 \ldots E_{2k}) \leq c_1 p_1^k + q_1^m \leq c_2 p_2^{2k} \]
where $\rho_2^2 = \max(p_1, q_1^m) < 1$, and $c_1, c_2$ are easily determined positive constants. From (2.7)

$$P(N > n) = P(A_1 < \sum_{i=1}^{\nu} (Z_i - U_{\nu} - a) < A_2, \text{ for } \nu = 1, \ldots, n)$$

$$\leq P(A_1 < \sum_{i=1}^{\nu} (Z_i - U_{\nu} - a) < A_2, \text{ for } \nu = m, 2m, \ldots, 2mk)$$

(where $k$ is the largest integer such that $2mk < n$)

$$\leq P(E_1 \ldots E_{2k})$$

$$\leq c_2 \rho_2^{2k} < c \rho^n$$

where $\rho = \rho_2^{1/m} < 1$, and $c > 0$. Hence,

$$P(N > n) < c \rho^n, \ c > 0, \ \rho < 1.$$  \hspace{1cm} (2.24)

**CASE 2:** $a = b$.

$$\sum_{i=1}^{n} (Z_i - U_n - a) = \sum_{i=1}^{n} (Z_i - U_n - b) \leq \sum_{i=1}^{n} (Z_i - b)$$

with probability 1.

Let $N^*$ be the smallest integer $\ell$ for which

$$\sum_{i=1}^{\ell} (Z_i - b) \leq A_1,$$

then $N \leq N^*$. But $Z < b$ with probability 1, therefore we need to consider $A_1 < 0$ only.
Let $\delta$ be a constant, $0 < \delta < b$, and 
$m = \left\lceil \frac{d}{\delta} \right\rceil + 1$, where $d = |A_1|$. Let

$\text{(2.25)} \quad E_k = \{A_1 < \sum_{i=1}^{m k} (Z_i - b) \leq 0\}$

$B_k = \{Z_i - b < -\delta, \ i = m(k-1)+1, \ldots, mk\}.$

Given $E_{k-1}$, then $E_k$ implies

$\text{(2.26)} \quad |\Delta_k| = \left| \sum_{i=1}^{m k} (Z_i - b) - \sum_{i=1}^{m(k-1)} (Z_i - b) \right|$

$= \left| \sum_{i=m(k-1)+1}^{m k} (Z_i - b) \right| < d.$

But, $B_k$ implies

$\text{(2.27)} \quad |\Delta_k| = \left| \sum_{m(k-1)+1}^{m k} (Z_i - b) \right| > m\delta \geq d,$

which implies $\overline{E_{k-1}} \rightarrow E_k$, and therefore $E_{k-1}E_k$ implies $B_k$. Hence, for $k = 2, 3, \ldots$

$\text{(2.28)} \quad P(E_k | E_{k-1}, \ldots, E_1) \leq P(\overline{B_k} | E_{k-1}, \ldots, E_1)$

$= P(\overline{B_k}) = 1 - (P(Z < b - \delta))^m = q < 1,$

and therefore,

$\text{(2.29)} \quad P(E_1, \ldots, E_k) = P(E_1)P(E_2 | E_1) \ldots P(E_k | E_{k-1} \ldots E_1) \leq c q^k,$
with $c > 0$, $q < 1$.

(2.30) $P(N^* > n) = P(A_1 < \sum_{i=1}^{\nu} (Z_i - b) \leq 0, \nu = 1, \ldots, n)$

$\leq P(A_1 < \sum_{i=1}^{m^l} (Z_i - b) \leq 0, \lambda = 1, 2, \ldots, [n/m])$

$= P(E_1 \cdots E_k) \leq c_1 q^k < c \rho^n$,

where $c > 0$ (properly defined), and $\rho^n = q^{\lfloor n/m \rfloor} < 1$.

But $N \leq N^*$, hence

(2.31) $P(N > n) \leq P(N^* > n) < c \rho^n$,

with $c > 0$, $\rho < 1$.

CASE 3: $a > b$. We observe that $Z_i - U_n - a \leq b - a < 0$ with probability 1, and hence we have to terminate sampling with Probability 1 after at most $[-A_1/a-b] + 1$ steps, when $A_1$ is negative.

When $A_1$ is positive, we have termination after the first observation. This completes the proof.

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BIBLIOGRAPHY


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