Revista Colombiana de Matemáticas
Vol. XIII (1979), págs. 139 - 154

A NOTE ON A SEQUENTIAL PROBABILITY RATIO TEST

por

M. S. ABU-SALIH

Summary. This is a type of problem that lies outside the scope of the exponential family. If the Z_i are real valued, with density $\frac{1}{\sigma}\exp\left[-\frac{z-\mu}{\sigma}\right]h(z-\mu)$ (here $h(z) = 1$ or 0, according as $z > 0$ or $z \leqslant 0$), and where one value of σ is tested against another, it is shown that

$$\ell n\ R_n = b + \sum_{i=1}^{n} (Z_i - U_n - a),$$

where $U_n = \min_{1\leqslant i\leqslant n} Z_i$, a is a positive constant. Using this expression it is proved that for every non-degenerate distribution of the Z_i, $P(N > n)$ is exponentially bounded, which, of course, implies termination with probability 1.

§1. Introducción. In Abu-Salih [1], the following model was discussed. $Z, Z_1, Z_2, \ldots$ is a sequence of independent identically distributed (iid) m-vectors, with k-parameter exponential distribution P. G^* is a group of transformations of the form $Z_n \mapsto C(Z_n + b)$, where $C \in G$, G is a Lie group of $m \times m$ nonsingular matrices, $\dim G \geq 1$, and G is closed in the general linear group $GL(m,R)$; b is an m-vector of reals, and the totality of vectors b form an invariant subspace under G.

Let P have the density

$$(1.1) \qquad P_\Theta^Z(x) = B(\Theta) h(z) \exp\left(\sum_{j=1}^{k} \Theta_j S_j(z)\right)$$

with respect to Lebesgue measure on the m-dimensional Euclidian space E^m, and where $\Theta = (\Theta_1, \ldots, \Theta_k)'$ belongs to the natural parameter space Ω, and $S = (S_1, \ldots, S_k)'$ is a continuously differentiable mapping of E^m into E^k.

Let $U = (U_1, U_2, \ldots)$ be a maximal invariant under G^* in the sample space, and $\gamma = \gamma(\Theta)$ a maximal invariant in Ω. For given $\Theta^1, \Theta^2 \in \Omega$ such that $\gamma(\Theta^1) \neq \gamma(\Theta^2)$, write $U^n = (U_1, U_2, \ldots, U_n)$, and let P'_{in} be its density under $\gamma(\Theta^i)$, $i = 1,2$, with respect to some σ-finite measure. Let

(1.2) $$r_n = P'_{2n} / P'_{1n}$$

and

(1.3) $$R_n = r_n(U^n) ;$$

then R_n is the probability ratio at the n^{th} stage of sampling based on the maximal invariant U. A sequential probability ratio test (SPRT) based on $\{R_n\}$ continues sampling as long as $B < R_n < A$ (B and A are two fixed stopping bounds), stops and accepts Θ^1 (resp. Θ^2) the first time that $R_n < B$ (resp. $R_n > A$). A SPRT based on $\{R_n\}$ will be called in *invariant* SPRT.

The limiting behavior of R_n is studies in [1] under the assumption that the actual distribution belongs to certain family $\mathcal{F}$, and it is proved that there are three cases:

(i) $\lim_{n\to\infty} R_n = \infty$, a.e.P,

(ii) $\lim_{n\to\infty} R_n = 0$, a.e.P,

(iii) $\limsup_{n\to\infty} R_n = \infty$, a.e.P, or $\liminf_{n\to\infty} R_n = 0$, a,e,P,

each one corresponding to a subfamily of $\mathcal{F}$. This establishes termination with probability 1 of the (SPRT) based on $\{R_n\}$.

The results obtained above form an extension of those of Wijsman in [2] and [3] in which the underlying model was assumed to be multivariate normal. Our methods of proof are closely modeled on those in [2] and [3].

§2. <u>Sequential probability ratio test based on negative exponential distribution with location parameter</u>. It is of interest to consider a model similar to the exponential one, except for a location parameter in the function $h(z)$ of (1.1). We were unable to reduce this model to the one we have summarized in the introduction. Yet, we have worked a simple example for which we obtained an exponential bound on $P(N > n)$ for any non-degenarate distribution P.

Let $Z, Z_1, Z_2, \ldots$ be <u>iid</u> random variables with density p_Θ^Z with respect to Lebesgue measure. Assume

$$(2.1) \qquad p_\Theta^Z(z) = \frac{1}{\sigma} h(z-\mu)\exp(-\frac{1}{\sigma}(z-\mu))$$

where $\Theta = (\mu,\sigma)$ and $h(x) = 1$ if $x > 0$, $h(x) = 0$ if $x \leqslant 0$. $\Omega = \{\Theta = (\mu,\sigma): -\infty < \mu<\infty, \sigma > 0\}$ is the parameter space. The joint density of

$(Z_1, \ldots, Z_n)$ is given by

$$(2.2) \qquad P_\Theta^{Z_1,\ldots,Z_n}(z_1,\ldots,z_n)$$

$$= \frac{1}{\sigma^n} \left[\prod_{i=1}^{n} h(z_i - \mu) \right] \exp\left[- \frac{1}{\sigma} \sum_{i=1}^{n} (z_i - \mu) \right].$$

Test about σ, e.g. H_o: $\sigma = \sigma_1$ vs H_1: $\sigma = \sigma_2$, where $\sigma_1 > \sigma_2$. Consider the group of translations G acting on the sample space as follows:

$$g: Z_i \to Z_i + a \quad \text{for} \quad i = 1,2,\ldots$$

where $-\infty < a < \infty$ and $g \in G$. It is clear that G leaves the model invariant.

Using (2.11) in [1] ((3.3) in [2]) we get

$$(2.3) \qquad r_n(z_1,\ldots,z_n)$$

$$= \frac{\int \frac{1}{\sigma_2^b} \exp\left[- \frac{1}{\sigma_2} \sum_{i=1}^{n} (z_i + a - \mu) \right] \prod_{i=1}^{n} h(z_i + a - \mu)\, da}{\int \frac{1}{\sigma_1^n} \exp\left[- \frac{1}{\sigma_1} \sum_{i=1}^{n} (z_i + a - \mu) \right] \prod_{i=1}^{n} h(z_i + a - \mu)\, da}$$

$$= \frac{\frac{1}{\sigma_2^n} \frac{\sigma_2}{n} \exp\left[- \frac{1}{\sigma_2} \sum_{i=1}^{n} (z_i - \mu) \right] \exp\left[- \frac{n}{\sigma_2} (\mu - u_n) \right]}{\frac{1}{\sigma_1^n} \frac{\sigma_1}{n} \exp\left[- \frac{1}{\sigma_1} \sum_{i=1}^{n} (z_i - \mu) \right] \exp\left[- \frac{n}{\sigma 1} (\mu - u_n) \right]}$$

(Where $u_n = \min_{1\leq i\leq n} z_i$)

$$= \left(\frac{\sigma_1}{\sigma_2}\right)^{n-1} \exp\left[\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) \sum_{n=1}^{n} (z_i - u_n)\right].$$

But $R_n = r_n(Z_1, \ldots, Z_n)$, hence from (2.3):

$$(2.4) \quad \ln R_n = \ln \left(\frac{\sigma_1}{\sigma_2}\right)^{-1} + n \ln \frac{\sigma_1}{\sigma_2} + \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) \sum_{i=1}^{n} (Z_i - U_n)$$

The SPRT mentioned above will continue sampling as long as $\ln B < \ln R_n < \ln A$ and, from (2.4), it continues sampling if

$$(2.5) \quad \ln B + \ln \frac{\sigma_1}{\sigma_2} < n \ln \frac{\sigma_1}{\sigma_2} + \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right) \sum_{i=1}^{n} (Z_i - U_n) < \ln A + \ln \frac{\sigma_1}{\sigma_2},$$

where $U_n = \min_{1\leq i\leq n} Z_i$. Let

$$A_1 = \left(\ln B + \ln \frac{\sigma_1}{\sigma_2}\right) / \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right)$$

$$(2.6) \quad A_2 = \left(\ln A + \ln \frac{\sigma_1}{\sigma_2}\right) / \frac{1}{\sigma_1} - \frac{1}{\sigma_2}$$

$$a = - \ln \frac{\sigma_1}{\sigma_2} / \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right)$$

$$= \left(\ln \frac{1}{\sigma_1} - \ln \frac{1}{\sigma_2}\right) / \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right)$$

(a is positive since numerator and denominator have the same sing). Using (2.5) and (2.6) we continue sampling as long as:

$$(2.7) \qquad A_1 < \sum_{i=1}^{n} (Z_i - U_n - a) < A_2$$

From now on, we drop the assumption that Z, Z_1, $Z_2, \ldots$ have the density (2.1) and instead consider $Z_1, Z_2, \ldots$ iid random variables with distribution P and denote the distribution of $(Z_1, \ldots, Z_n)$ also by P. The only restriction we impose on P is that it be non-degenerate. *Under this conditions, we like to establish termination, with probability 1, of the* SPRT *based on* (2.7) *and find exponential bounds on* $P(N > n)$. Let

$$(2.8) \qquad E_n = \{A_1 < \sum_{i=1}^{n} (Z_i - U_n - a) < A_2\} .$$

and let us compute $P(E_{n+1} | E_1, \ldots, E_n)$. *Given* E_n with $\sum_{i=1}^{n} (A_i - U_n - a) = d_n$, suppose $Z_{n+1} \geqslant U_n$; then

$$\sum_{i=1}^{n+1} (Z_i - U_{n+1} - a) = d_n + Z_{n+1} - U_n - a .$$

Hence, *given* E_n *and* $Z_{n+1} \geqslant U_n$, E_{n+1} implies $Z_{n+1} - U_n - a < D = A_2 - A_1$, and, in particular

$$(2.9) \qquad U_n \leqslant Z_{n+1} < U_n + D + a .$$

On the other hand, *given* E_n *and* $Z_{n+1} < U_n$, then E_{n+1} implies that

$$\sum_{i=1}^{n+1} (Z_i - U_{n+1} - a) = d_n + n(U_n - Z_{n+1}) - a$$

lies between A_1 and A_2 hence

$$A_1 - d_n < n(U_n - Z_{n+1}) - a < A_2 - d_n ,$$

which implies

$$(2.10) \qquad U_n - \frac{D}{n} - \frac{a}{n} < Z_{n+1} < U_n + \frac{D}{n} - \frac{a}{n} .$$

Comparing (2.9) and (2.10) we have: *given* E_n, then E_{n+1} implies

$$(2.11) \qquad U_n - \frac{L}{n} < Z_{n+1} < U_n + L ,$$

where $L = D + a$. Furthermore, given $E_1, \ldots E_n$, then E_{n+1} implies (2.11), and so

$$(2.12) \quad P(E_{n+1} | E_1, \ldots, E_n)$$

$$\leqslant P(U_n - \frac{L}{n} < Z_{n+1} < U_n + L | E_1, \ldots, E_n)$$

$$= E[I_{F_{n+1}} | E_1, \ldots, E_n]$$

$$= E[E[I_{F_{n+1}} | U_n, E_1, \ldots, E_n] | E_1, \ldots, E_n]$$

with probability 1, where $I_{F_{n+1}}$ is the indicator function of F_{n+1} and $F_{n+1} = \{U_n - \frac{L}{n} < Z_{n+1} < U_n + L\}$.

Suppose that the support of Z is *not* contained in an interval of length 2L, and define $\rho = \sup_{-\infty<x<\infty} P(x-L < Z < x+L)$; then $\rho < 1$. Furthermore, since Z_{n+1} is independent of $Z_1, \ldots, Z_n$ and hence independent of $E_1, \ldots, E_n$ and U_n, we have

$$E[I_{F_{n+1}} | E_1, \ldots, E_n, U_n = u_n]$$

$$= P(u_n - \frac{L}{n} < Z_{n+1} < u_n + L | E_1, \ldots, E_n, U_n = u_n)$$

$$= P(u_n - \frac{L}{n} < Z_{n+1} < u_n + L) \leq \rho < 1 ;$$

and therefore,

$$E[E[I_{F_{n+1}} | E_1, \ldots, E_n, U_n] | E_1, \ldots, E_n] \leq \rho < 1 ;$$

hence (2.12) becomes:

$$(2.13) \quad P(E_{n+1} | E_1, \ldots E_n) \leq \rho < 1 \quad \text{for } n = 1,2,\ldots$$

But

$$P(N>n) = P(B < R_m < A, \ 1 \leq m \leq n)$$

$$= P(E_1 E_2 \ldots E_n)$$

$$= P(E_1)P(E_2|E_1)P(E_3|E_1E_2) \ldots P(E_n|E_1 \ldots E_{n-1})$$

$$\leq P(E_1)\, \rho^{n-1} < c\, \rho^n ,$$

with $c > 0$, $\rho < 1$.

Therefore, *we have established an exponential bound on* $P(N>n)$, *for* P *with support not contain-*

ed in an interval of length 2L.

The case of bounded support is considered in the rest of the paper. Without loss of generality, we assume that the support of P is (0,b). We may do this because we are studying $\sum_{i=1}^{n}(Z_i - U_n - a)$ which is invariant under translations.

CASE 1: $a \in (0,b)$. Let δ be a positive constant such that $a + 2\delta < b$. Hence

(2.14) $$P(Z > a + 2\delta) = p > 0.$$

Let

$$m = \left[\frac{D}{\delta}\right] + 1 \qquad (D = |A_2 - A_1|)$$

$$V_k = \min_{1 \leq i \leq mk} Z_i$$

(2.15) $$E_k = \{A_1 < \sum_{i=1}^{mk} (Z_i - V_k - a) < A_2\}$$

(so E_k is the same as E_{mk} from (2.8))

$$B_k = \{Z_i > a+2\delta, \quad i = m(k-1)+1,\dots,mk\}$$

$$A_k = \{V_k \leq \delta\}.$$

Since $\delta > 0$ then

(2.16) $$P(Z > \delta) = q_1 < 1 .$$

Let

$$\Delta_k = \sum_{i=1}^{mk} (Z_i - V_k - a) - \sum_{i=1}^{m(k-1)} (Z_i - V_{k-1} - a) ;$$

then

$$\Delta_k = \sum_{i=m(k-1)+1}^{mk} (Z_i - a) - mV_k + m(k-1)(V_{k-1} - V_k)$$

$$\geqslant \sum_{i=m(k-1)+1}^{mk} (Z_i - a) - mV_k ,$$

since $V_k \leqslant V_{k-1}$. Also we have

(2.17) i) *Given* E_{k-1}, *then* E_k *implies* $|\Delta_k| < D$.

ii) *Given* A_k, *then* B_{k+j} *implies* $\Delta_{k+j} \geqslant 2m\delta - m\delta = m\delta \geqslant D$, *which implies* $|\Delta_{k+j}| \geqslant D$ for $j = 1,2,\ldots$.

Therefore, *given* E_{k+j} *and* A_k we have for any $j = 1,2,\ldots$

(2.18) B_{k+j+1} *implies* $\tilde{E}_{k+j+1}$ (~ *denotes complementation*)

(2.19) E_{k+j+1} *implies* $\tilde{B}_{k+j+1}$.

Now,

(2.20) $P(E_1 E_2 \ldots E_{2k})$

$$\leqslant P(E_k E_{k+1} \ldots E_{2k})$$

$$= P(E_k \ldots E_{2k} A_k) + P(E_k \ldots E_{2k} \tilde{A}_k)$$

$$\leqslant P(E_k \ldots E_{2k} A_k) + P(\tilde{A}_k)$$

$$= P(E_k \ldots E_{2k} A_k) + q_1^{mk} \quad ,$$

by (2.16). And

$$(2.21) \quad P(E_k \ldots E_{2k} A_k) = P(E_k \ldots E_{2k} | A_k) P(A_k)$$

$$\leqslant P(E_k \ldots E_{2k} | A_k) \ ,$$

because $0 < P(A_k) < 1$ for $k = 1, 2 \ldots$ Also

$$(2.22) \ P(E_k \ldots E_{2k} | A_k)$$

$$= P(E_k | A_k) P(E_{k+1} | E_k, A_k) \ldots P(E_{2k} | E_k, E_{k+1}, \ldots$$

$$\ldots, E_{2k-1}, A_k)$$

$$\leqslant P(E_k | A_k) P(\tilde{B}_{k+1} | E_k, A_k) \ldots P(\tilde{B}_{2k} | E_k, \ldots$$

$$\ldots, E_{2k-1}, A_k)$$

by (2.19). But, since B_{k+j} is independent of A_k and E_{k+j-i} for $i = 1, \ldots, j$ then

$$(2.23) \quad P(\tilde{B}_{k+j} | A_k, \ E_{k+j-i}, \quad i = 1, \ldots, j)$$

$$= P(\tilde{B}_{k+j}) = 1 - p^m = p_1$$

where p is given by (2.14) and so $p_1 < 1$.

Using (2.23), (2.22), and (2.21), then (2.20) becomes: $P(E_1 E_2 \ldots E_{2k}) \leqslant c_1 p_1^k + q_1^{mk} \leqslant c_2 \ p_2^{2k}$, where $\rho_2^2 = \max(p_1, q_1^m) < 1$, and c_1, c_2 are easily determined positive constants. From (2.7)

$$P(N>n) = P(A_1 < \sum_{i=1}^{\nu}(Z_i - U_\nu - a) < A_2, \text{ for } \nu = 1,\ldots,n)$$

$$\leqslant P(A_1 < \sum_{i=1}^{\nu}(Z_i - U_\nu - a) < A_2, \text{ for } \nu = m, 2m, \ldots$$

$$\ldots, 2mk).$$

(where k is the largest integer such that $2mk \leq n$)

$$\leqslant P(E_1 \ldots E_{2k})$$

$$\leqslant c_2\, \rho_2^{2k} < c\rho^n$$

where $\rho = \rho_2^{1/m} < 1$, and $c > 0$. Hence,

$$(2.24) \qquad P(N > n) < c\rho^n, \quad c > 0, \quad \rho < 1.$$

<u>CASE 2</u>: $a = b$.

$$\sum_{i=1}^{n}(Z_i - U_n - a) = \sum_{i=1}^{n}(Z_i - U_n - b) \leqslant \sum_{i=1}^{n}(Z_i - b)$$

with probability 1.

Let N^* be the smallest integer ℓ for which $\sum_{i=1}^{\ell}(Z_i - b) \leqslant A_1$, then $N \leqslant N^*$. But $Z < b$ with probability 1, therefore we need to consider $A_1 < 0$ only.

Let δ be a constant, $0 < \delta < b$, and $m = [\frac{d}{\delta}] + 1$, where $d = |A_1|$. Let

$$(2.25) \quad E_k = \{A_1 < \sum_{i=1}^{mk} (Z_i - b) \leq 0\}$$

$$B_k = \{Z_i - b < -\delta, \quad i = m(k-1)+1, \ldots, mk\}.$$

Given E_{k-1}, then E_k implies

$$(2.26) \quad |\Delta_k| = |\sum_{i=1}^{mk} (Z_i - b) - \sum_{i=1}^{m(k-1)} (Z_i - b)|$$

$$= |\sum_{i=m(k-1)+1}^{mk} (Z_i - b)| < d .$$

But, B_k implies

$$(2.27) \quad |\Delta_k| = |\sum_{m(k-1)+1}^{mk} (Z_i - b)| > m\delta \geq d ,$$

which implies $\widetilde{E_{k-1}E_k}$, and therefore $E_{k-1}E_k$ implies $\tilde{B}_k$. Hence, for $k = 2,3,\ldots$

$$(2.28) \quad P(E_k|E_{k-1},\ldots,E_1) \leq P(\tilde{B}_k|E_{k-1},\ldots E_1)$$

$$= P(\tilde{B}_k) = 1 - (P(Z < b - \delta))^m = q < 1,$$

and therefore,

$$(2.29) \quad P(E_1,\ldots,E_k) = P(E_1)P(E_2|E_1)\ldots P(E_k|E_{k-1},\ldots,E_1) \leq c\, q^k ,$$

with $c > 0$, $q < 1$.

$$(2.30)\quad P(N^*>n) = P(A_1 < \sum_{i=1}^{\nu} (Z_i - b) \leq 0,\ \nu = 1,\ldots,n)$$

$$\leq P(A_1 < \sum_{i=1}^{m\ell} (Z_i - b) \leq 0,\ \ell = 1,2,\ldots,[n/m])$$

$$= P(E_1 \ldots E_k) \leq c_1 q^k < c\rho^n ,$$

where $c > 0$ (properly defined), and $\rho^n = q^{[n/m]} < 1$.

But $N \leq N^*$, hence

$$(2.31)\quad P(N>n) \leq P(N^*>n) < c\rho^n,$$

with $c > 0$, $\rho < 1$.

CASE 3: $a > b$. We observe that $Z_i - U_n - a \leq b - a < 0$ with probability 1, and hence we have to terminate sampling with Probability 1 after at most $[-A_1/a-b] + 1$ steps, when A_1 is negative. When A_1 is positive, we have termination after the first observation. This completes the proof.

§3. Acknowledgment. The author is grateful to his thesis advisor, Professor R.A. Wijsman, for suggesting this problem which is mainly chapter 8 of his thesis.

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Department of Mathematics
Yarmouk University
Irbid, JORDAN.

(Recibido en agosto de 1978).