Revista Colombiana de Matemáticas
Vol. XIII (1979), págs. 193 - 206

CHARACTERIZATION OF CERTAIN $\mathfrak{A}$-SELF-ADJOINT OPERATORS BY MEANS OF THEIR EXPONENTIAL FUNCTION

by

Klaus Gero KALB

§1. Introduction. Let X be a complex Banach space. Let $T \in \mathcal{L}(X)$ be a continuous linear operator with spectrum $\sigma(T) \subset (a,b) \subset \mathbb{R}$. For natural n, $1 \leqslant n < \infty$, denote $V_p^n[a,b]$ the algebra (of *Sobolev type*) of functions $f \in C^{n-1}[a,b]$ for which $f^{(n-1)}$ is absolutely continuous, and $f^{(n)} \in L^p(a,b)$ for $1 \leqslant p < \infty$ or $f^{(n)} \in C^n[a,b]$ for $p = \infty$. $V_p^n[a,b]$ is provided with the norm

$$\|f\|_{n,p;[a,b]} = \max\{|f^{(k)}(a)| : k=0,\ldots,n-1\} + \|f^{(n)}\|_{0,p;[a,b]} ,$$

where $\|\cdot\|_{0,p;[a,b]}$ denotes the norm of $L^p(a,b)$ for $1 \leqslant p < \infty$ and the *maximum-norm* of $C[a,b]$ for

$p = \infty$. For $n = 0$, only $V^{o}[a,b] = C[a,b]$, with the norm $\|\cdot\|_{o,\infty;[a,b]}$, is considered.

T is called V_p^n-*self-adjoint* (denoted: $T \in [V_p^n]$) if there exists a continuos homomorphism ϕ: $V_p^n[a,b] \to \mathcal{L}(X)$ with $\phi(e_o) = I$, $\phi(e_1) = T$, where $e_o(t) = 1$, $e_1(t) = t$ $(t \in \mathbb{R})$. It turns out that this notion does not depend on the selection of the interval $[a,b]$. The V_p^n-self-adjoint operators are interesting examples of $\mathfrak{A}$-self-adjoint (particularly C^{∞}-self-adjoint) operators in the sense of Colojoara-Foiaş [2] and extrapolate the C^n-selt-adjoint operators of Kantorovitz [7], [8]. As is well known, an operator $T \in \mathcal{L}(X)$ is C^{∞}-self-adjoint if and only if the exponential function $R \ni \xi \mapsto e^{i\xi T} \in \mathcal{L}(X)$ of T satisfies a growth condition

$$(1.1) \qquad \|e^{i\xi T}\| = O(|\xi|^k) \quad (|\xi| \to \infty)$$

with $k \in \mathbb{N} = \{0,1,\ldots\}$ (cf. [7], Lemma 2.11; [2], Thm. 4.5, or section 4 of this note). In Kalb [6] the V_p^n-self-adjoint operators are characterized by conditions on their resolvent function $z \mapsto R(z) = (T-zI)^{-1}$. In this work the V_p^n-self-adjointness shall be described by means of the exponential function of T (and another related function introduced by Kantorovitz [8]). Thereby, among other things, results of Kantorovitz [7], [8] are generalized and newly proven by a very elementary and natural method. This method consists in developing suitable representation formulas for the analytic functional calculus of T on certain subalgebras of the algebra $\mathcal{H}(\mathbb{C})$ of entire functions (cf. section 3 and Lemma 4.1).

Sections 2 and 3 of this work generalize a portion of chapter 2 of my Habilitationsschrift [5] (where only the case $n = \infty$ is treated); section 4 was essentially written during a visiting professorship at the Departamento de Matemáticas of the Universidad de los Andes in Bogotá/Colombia (February - April 1979), which was partially supported by COLCIENCIAS.

§2. Preliminaries. For the sake of selfcontainedness we present here some facts from [3], [7], [6], which we shall need later. For the time being let $T \in \mathcal{L}(X)$ be arbitrary.

2.1 LEMMA. *Let* $\alpha_+ = \sup\{\mathrm{Im}(z) : z \in \sigma(T)\}$, $\alpha_- = \inf\{\mathrm{Im}(z) : z \in \sigma(T)\}$. *Then*

$$(2.1) \qquad R(z) = \begin{cases} i\int_{-\infty}^{0} e^{-i\xi z}\, e^{i\xi T}\, d\xi & \text{for } \mathrm{Im}(z)>\alpha_+ \\ -i\int^{\infty} e^{-i\xi z}\, e^{i\xi T}\, d\xi & \text{for } \mathrm{Im}(z)<\alpha_- \end{cases}$$

where the integrals converge conditionally.

Proof. Application of the analytic functional calculus of T to the formula

$$\frac{1}{w-z} = \begin{cases} i\cdot\int_{-\infty}^{0} e^{-i\xi z}\, e^{i\xi w}\, d\xi & \text{for } \mathrm{Im}(w)<\mathrm{Im}(z) \\ -i\cdot\int_{0}^{\infty} e^{-i\xi z}\, e^{i\xi w}\, d\xi & \text{for } \mathrm{Im}(w)>\mathrm{Im}(z). \end{cases} \blacksquare$$

2.2 COROLLARY. *If* T *satisfies the condition* (1.1), *then* $\sigma(T)\subset\mathbb{R}$.

Proof. Under (1.1), the integrals in (2.1) converge absolutely for $z\in\mathbb{C}\setminus\mathbb{R}$ and the affirmation follows from Lemma 2.1 by the identity theorem for analytic functions. ■

Now let be $\sigma(T)\subset(a,b)\subset\mathbb{R}$. For $\varepsilon>0$ and $n\in\mathbb{N}$ let

$$A_o(b,\varepsilon;t) = \frac{1}{2\pi i}\left[R(t+i\varepsilon)-R(t-i\varepsilon)\right] \quad (t\in\mathbb{R})$$

(2.2)

$$A_n(b,\varepsilon;t) = \int_t^b \frac{(s-t)^{n-1}}{(n-1)!} A_o(b,\varepsilon;s)ds \quad (t\in\mathbb{R}\ ;\ n\geqslant 1)$$

$$(2.3)\qquad I_\varepsilon^{[-n]}(h) = \int_a^b h(t)A_n(b,\varepsilon;t)dt \quad (h\in C[a,b]).$$

2.3 LEMMA (cf. [6], [9]). *For every bounded set* $\mathcal{M}\subset\mathcal{H}(D)$ *and function* $g\in\mathcal{M}$,

$$g(T) = \sum_{k=0}^{n-1} \frac{g^{(k)}(a)}{k!}(T-aI)^k + \lim_{\varepsilon\to 0+} I_\varepsilon^{[-n]}(g^{(n)})$$

uniformly on $\mathcal{M}$ *with respecto to* g.

Finally we shall need the following formula, which yields from (2.1):

$$I_\varepsilon^{[-0]}(h) = \frac{1}{2\pi i}\int_a^b h(s)\ R(s+i\varepsilon)-R(s-i\varepsilon)\ ds$$

(2.4)

$$= \int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_a^b h(s).e^{-i\xi}ds\right]e^{-\xi|\xi|}e^{i\xi T}d\xi$$

where the integral converges conditionally.

§3. Description of the analytic functional calculus by means of the exponential function. Let $T \in \mathcal{L}(X)$ with $\sigma(T) \subset (a,b) \subset \mathbb{R}$ and let $n \in \mathbb{N}$.

3.1. Definition (cf. Kantorovitz [8]): let $E_n(\xi,T;a) = e_n(\xi,z;a)\big|_{z=T}$ where

$$e_n(\xi,z;a) = e^{i\xi a} \sum_{\rho=0}^{\infty} (i\xi)^{\rho} \frac{(z-a)^{\rho+n}}{(\rho+n)!} \quad (\xi \in \mathbb{R};\ z \in \mathbb{C}),$$

$e_n(\xi,\cdot;a) \in \mathcal{H}(\mathbb{C})$.

3.2 Remark.

(a) $e_o(\xi,z;a) = e^{i\xi z}$.

(b) $\dfrac{d^k}{dz^k} e_n(\xi,z;a) = e_{n-k}(\xi,z;a) \qquad (0 \leq k \leq n)$.

(c) $e_n(\xi,a;a) = \begin{cases} e^{i\xi a} & \text{for } n = 0 \\ 0 & \text{for } n > 0 . \end{cases}$

Particularly $E_o(\xi,T;a) = e^{i\xi T}$.

For $f \in L^1(\mathbb{R})$ let $\hat{f}(s) = \int_{-\infty}^{\infty} f(\xi) e^{i\xi s} d\xi$, $s \in \mathbb{R}$, denote the Fourier transform of f. For $f \in C_c(\mathbb{R})$ we have $\hat{f} \in \mathcal{H}(\mathbb{C})$.

3.3 LEMMA. *For* $f \in C_c(\mathbb{R})$:

$$\hat{f}(T) = \sum_{k=o}^{n-1} \frac{\hat{f}^{(k)}(a)}{k!} (T-aI)^k + \int_{-\infty}^{\infty} (i\xi)^n f(\xi) E_n(\xi,T;a) d\xi$$

Proof. First we obtain by application of Lemma 2.3 and Remark 3.2:

$$(3.1)\quad E_n(\xi,T;a) = \lim_{\varepsilon\to o+} \int_a^b e^{i\xi s} A_n(b,\xi;s)ds \qquad (\xi \in \mathbb{R}),$$

uniformly with respect to ξ on compact subsets of $\mathbb{R}$.

Let $f \in C_c(\mathbb{R})$, again from Lemma 2.3 ensues

$$(3.2)\quad \hat{f}(T) = \sum_{k=o}^{n-1} \frac{\hat{f}^{(k)}(a)}{k!}(T-aI)^k + \lim_{\varepsilon\to o+}\int_b^b \hat{f}^{(n)}(s)A_n(b,\varepsilon;s)ds$$

Let $\operatorname{supp}(f) \subset [\alpha,\beta]$. Then

$$\lim_{\varepsilon\to o+}\int_a^b \hat{f}^{(n)}(s)A_n(b,\varepsilon;s)ds$$

$$= \lim_{\varepsilon\to o+}\int_a^b \int_\alpha^\beta (i\xi)^n f(\xi)e^{i\xi s}d\xi \; A_n(b,\varepsilon;s)ds$$

$$= \int_\alpha^\beta (i\xi)^n \; f(\xi) \lim_{\varepsilon\to o+}\int_a^b e^{i\xi s}A_n(b,\varepsilon;s)ds \; d\xi$$

$$= \int_{-\infty}^{\infty} (i\xi)^n f(\xi)E_n(\xi,T;a)\; d\xi \;,$$

from where the affirmation of the lemma follows, using (3.2). ■

From Lemma 3.3 ensues the following characterization of V_p^n-self-adjointness of T, which in the particular case $p = \infty$ was given by Kantorovitz [7], [8] in a similar way.

3.4 THEOREM. *For* $T \in \mathcal{L}(X)$ *with* $(T) \subset (a,b)$ *the following statements are equivalent:*

(1) $T \in [V_p^n]$.

(2) $\|\int_{-\infty}^{\infty} f(\xi)\, e^{i\xi T} d\xi\| \leq M \|\hat{f}\|_{n,p;[a,b]}$, *for all* $f \in C_c(\mathbb{R})$.

(3) $\|\int_{-\infty}^{\infty} f(\xi) E_n(\xi, T; a) d\xi\| \leq N \|\hat{f}\|_{o,p;[a,b]}$, *for all* $f \in C_c(\mathbb{R})$.

Proof. Note that $Z = \{\hat{f}: f \in C_c^{\infty}(\mathbb{R})\}$ is dense in $V_p^n[a,b]$.

(1) $\Leftrightarrow$ (2). Because of Lemma 3.3 applied to $n = 0$ the condition (2) is equivalent to the continuity of the restriction $\phi_o: Z \to \mathcal{L}(X)$ of the $\mathcal{H}(\mathbb{C})$-functional calculus of T onto Z, with respect to the topology induced on Z by $V_p^n[a,b]$.

(1) $\Rightarrow$ (3). If $T \in [V_p^n]$, then there exists a constant N such that

$$\|g(T)\| \leq N . \|g\|_{n,p;[a,b]}, \text{ for all } g \in \mathcal{H}(\mathbb{C}).$$

If $f \in C_c^{\infty}(\mathbb{R})$ is given, choose $g \in \mathcal{H}(C)$ such that $g^{(k)}(a) = 0$ for $k = o, \ldots, n-1$, $g^{(n)} = \hat{f}$. Then

$$\|\lim_{\varepsilon \to o+} I_{\varepsilon}^{[-n]}(\hat{f})\| = \|g(T)\| \leq N . \|g\|_{n,p;[a,b]}$$

$$= N . \|\hat{f}\|_{o,p;[a,b]}, \text{ (Lemma 2.3).}$$

But it holds, by a calculation similar to that in the proof of lemma 3.3, that:

$$\lim_{\varepsilon \to o+} I_{\varepsilon}^{[-n]}(f) = \int_{-\infty}^{\infty} f(\xi)\, E_n(\xi, T; a) d\xi .$$

(3) ⇒ (2). From Lemma 3.3 it follows

$$\left\|\int_{-\infty}^{\infty} f(\xi)e^{i\xi T}d\xi\right\| = \left\| \sum_{k=0}^{n-1} \frac{\hat{f}^{(k)}(a)}{k!}(T-aI)^k + \int_{-\infty}^{\infty}(i\xi)^n f(\xi)E_n(\xi,T;a)d\xi \right\|$$

$$\leqslant M_1 . \max\{|\hat{f}^{(k)}(a)| : k=0,\dots,n-1\} + N . \|(i\cdot)^n f(\cdot)\|_{o,p;[a,b]}$$

$$= M_1 . \max\{|\hat{f}^{(k)}(a)| : k=0,\dots,n-1\} + N . \|\hat{f}^{(k)}\|_{o,p;[a,b]}$$

$$\leqslant M \cdot \|\hat{f}\|_{n,p;[a,b]} . \quad \blacksquare$$

3.5 Remark. Instead of the interval $[a,b]$ with $\sigma(T) \subset (a,b)$, an interval $[-\alpha,\alpha]$ with $\sigma(T) \subset (-\alpha,\alpha)$ can be considered and the expansion point a can be substituted by 0. Then Lemma 3.3 and Theorem 3.4 hold with $E_n(\cdot,T;0)$ instead of $E_n(\cdot,T;a)$.

To prove this, only Lemma 2.3 has to be adapted to this situation, usign another kernel function $\tilde{A}_n(\varepsilon,t)$ (cf. the proof of Lemma 2.3 in [6]).

§4. Estimation of the order of C^∞-self-adjoint operators. Let $T \in \mathcal{L}(X)$ be an operator whose exponential function satisfies the growth condition (1.1), then $\sigma(T) \subset \mathbb{R}$. Let $\mathcal{S}$ be the space of rapidly decreasing functions of Schwartz; the Fourier transform is a topological isomorphism of $\mathcal{S}$ onto itself (c.f. e.g. [4]). Define a continuous linear transformation $\Psi : \mathcal{S} \to \mathcal{L}(X)$ by

$$\Psi(\hat{f}) = \int_{-\infty}^{\infty} f(\xi)\, e^{i\xi T}\, d\xi \qquad (f \in \mathcal{S})$$

(This and the following integrals converge absolutely).

4.1 LEMMA. *Let* $\sigma(T) \subset (a,b)$ *and let* $\chi \in C^{\infty}(\mathbb{R})$ *be a function with* $\operatorname{supp}(\chi) \subset (a,b)$ *and* $\chi \equiv 1$ *on an interval* $[a_1, b_1]$ *such that* $\sigma(T) \subset (a_1,b_1) \subset [a_1,b_1] \subset (a,b)$. *Then* $\hat{f}(T) = \Psi(\chi \cdot \hat{f})$ *for all* $\hat{f} \in C_c^{\infty}(\mathbb{R})$.

Proof. Choose $\varphi \in \mathcal{S}$ such that $\hat{\varphi} = \chi \cdot \hat{f} \in \mathcal{S}$. Then

$$\Psi(\chi \cdot \hat{f}) = \Psi(\hat{\varphi}) = \int_{-\infty}^{\infty} \varphi(\xi) e^{i\xi T} d\xi$$

$$= \lim_{\varepsilon \to 0+} \int_{-\infty}^{\infty} \varphi(\xi)\, e^{-\varepsilon|\xi|}\, e^{i\xi T}\, d\xi$$

$$\overset{(1)}{=} \lim_{\varepsilon \to 0+} I_{\varepsilon}^{[-0]}(\hat{\varphi})$$

$$= \lim_{\varepsilon \to 0+} I_{\varepsilon}^{[-0]}(\chi \cdot \hat{f})$$

$$\overset{(2)}{=} \lim_{\varepsilon \to 0+} \frac{1}{2\pi i} \int_{a_1}^{b_1} \hat{f}(t)\left[R(t+i\varepsilon) - R(t-i\varepsilon)\right] dt$$

$$\overset{(3)}{=} \hat{f}(T).$$

Here, (1) ensues from formula (2.4) because

$\frac{1}{2\pi}\int_a^b \hat{\varphi}(s)\, e^{-is\xi}ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\varphi}(s)\, e^{-is\xi}ds = \varphi(\xi)$ (Fourier inversion formula), (2) follows from the inclusion $[([a,a_1]\cup[b_1,b])\times\mathbb{R}]\subset\mathbb{C}\setminus\sigma(T)$, and (3) from Lemma 2.3 with $n = 0$. ∎

This lemma permit us to characterize the V_p^n-self-adjointness of T by conditions of the type of those in Theorem 3.4, in which the interval (a,b) no longer appears. For $1 \leq n < \infty$, $1 \leq p < \infty$, denote $\|\cdot\|_{n,p}$ the norm

$$\|f\|_{n,p} = \max\{|f^{(k)}(t)| : t\in\mathbb{R},\ 0\leq k\leq n-1\} + \|f^{(n)}\|_{L^p(\mathbb{R})}$$

and for $0 \leq n < \infty$, $p = \infty$, the norm

$$\|f\|_{n,\infty} = \max\{|f^{(k)}(t)| : t\in\mathbb{R},\ 0\leq k\leq n\} \quad (f\in\mathcal{S}).$$

4.2 THEOREM. *The following affirmations are equivalent:*

(1) $T\in[V_n^p]$.

(2) $\|\int_{-\infty}^{\infty} f(\xi)e^{i\xi T}d\xi\| \leq M.\|\hat{f}\|_{n,p}$, *for all* $f\in C_c^\infty(\mathbb{R})$.

(3) $\|\int_{-\infty}^{\infty} f(\xi)E_n(\xi,T;0)d\xi\| \leq M.\|\hat{f}\|_{L^p(\mathbb{R})}$, *for all* $f\in C_c^\infty(\mathbb{R})$.

Proof. (1) ⇒ (2) and (1) ⇒ (3) ensue from

Theorem 3.4 and remark 3.5 respectively, because $\|\hat{f}\|_{n,p;[a,b]} \leq \|\hat{f}\|_{n,p}$.

(2) ⇒ (1). Let Ψ be as in Lemma 4.2. First we have because of hypothesis (2) that

$$\|\Psi(\hat{f})\| = \|\int_{-\infty}^{\infty} f(\xi)e^{i\xi T}d\xi\| \leq M\cdot\|\hat{f}\|_{n,p} \text{ , for all}$$

$f \in C_c^\infty(\mathbb{R})$; as $C_c^\infty(\mathbb{R})$ is dense in $\mathcal{S}$ it follows by reasons of continuity that

$$\|\Psi(g)\| \leq M\cdot\|g\|_{n,p} \text{ , for all } g \in \mathcal{S}.$$

Therefore, for every $f \in C_c^\infty(\mathbb{R})$ it holds

$$\|\hat{f}(T)\| = \|\Psi(\chi\cdot\hat{f})\| \leq M\cdot\|\chi\hat{f}\|_{n,p} \leq \tilde{M}\cdot\|\hat{f}\|_{n,p;[a,b]}.$$

(3) ⇒ (2). The affirmation ensues from

$$\int_{-\infty}^{\infty} f(\xi)e^{i\xi T}d\xi = \sum_{k=0}^{n-1}\frac{\hat{f}^{(k)}(0)}{k!}T^k + \int_{-\infty}^{\infty}(i\xi)^n f(\xi)E_n(\xi,T;0)d\xi$$

(cf. note 3.5) the same as in the demostration of Theorem 3.4. ∎

The growth condition (1.1) for the exponential function of T is used here to dominate the infinite integrals. In the case $p = \infty$ a somewhat finer argument (using conditionally convergent integrals) shows that the affirmation of Theorem 4.2 remains valid for *arbitrary* $T \in \mathcal{L}(X)$, if $C_c^\infty(\mathbb{R})$ is substituted by $\overset{o}{C}_c(\mathbb{R})$. (cf. also Kantorovitz [8], Thm. 1).

Note that the implication (2) $\Rightarrow$ (1) remains valid, if the norm $\|\cdot\|_{n,p}$ is substituted by

$$|||f|||_{n,p} = \max\{\|f^{(k)}\|_{L^p(\mathbb{R})} : \; k=0,1,\ldots, n\} .$$

4.3 COROLLARY. (cf. Albrecht [1], Satz 3.3). *Let* $T \in \mathcal{L}(X)$ *be an operator whose exponential function satisfies the growth condition* (1.1). *Then* T *is* V_2^{k+1}*-self-adjoint (all the more* C^{k+1}*-self-adjoint)*

Proof. Choose $M > 0$ such that $\|e^{i\xi T}\| \leq M \cdot |\xi|^k$ for $|\xi| \geq 1$. Then it holds for every $f \in C_c^\infty(\mathbb{R})$, applying the $L^2(\mathbb{R})$-isometry of the Fourier transformation, that :

$$\|\int_{-\infty}^{\infty} f(\xi) e^{i\xi T} d\xi\|$$

$$\leq \int_{|\xi|\leq 1} |f(\xi)| \cdot \|e^{i\xi T}\| d\xi + \int_{|\xi|\geq 1} \frac{\|e^{i\xi T}\|}{|\xi|^k} \frac{1}{|\xi|} |\xi^{k+1} f(\xi)| d\xi$$

$$\leq C_1 \cdot \|\hat{f}\|_{L^2(\mathbb{R})} + M\left(\int_{\mathbb{R}} \frac{d\xi}{|\xi|^2} \int_{\mathbb{R}} |\xi^{k+1} f(\xi)|^2 \, d\xi\right)^{\frac{1}{2}}$$

$$\leq C_1 \cdot \|\hat{f}\|_{L^2(\mathbb{R})} + C_2 \|\cdot \hat{f}^{(k+1)}\|_{L^2(\mathbb{R})}$$

$$\leq C |||\hat{f}|||_{k+1,2}$$

***

BIBLIOGRAPHY

[1] E. Albrecht, "Funktionalkalküle in mehreren Veränderlichen für stetige lineare Operatoren auf Banachräumen". Manuscripta Math. 14 (1974), 1-40.

[2] I. Colojoara and C. Foiaş, *Theory of generalized spectral operators*. New York: Gordon and Breach 1968.

[3] E. Hille and R.S. Philips, *Functional analysis and semigroups*. Providence; Amer. Math. Soc. Coll. Publ. 31, 1957.

[4] J. Horvâth, *Topological vector spaces and distributions*. Reading: Addison-Wesley 1966.

[5] K.G. Kalb, "Charakterisierung von Operatoren mit $C^n(\mathbb{R})$-Funktionalkalkül". Habilitationsschrift, Mainz 1977.

[6] K.G. Kalb, "Charakterisierung gewisser C^∞-selbstadjungierter Operatoren durch ihre Resolventenfunktion". To appear (Preprint).

[7] S. Kantorovitz, "Classification of operators by means of their functional calculus". Trans. Amer. Math. Soc. 115 (1965), 194-224.

[8] S. Kantorovitz, "Characterization of C^n-operators". Indiana Univ. Math. Jour 25 (1976) 119-133.

[9] E. Marschall, "Vektorwertige Ultradistributionen und Funktionalkalküle". Diplomarbeit, Mainz 1976.

***

Fachbereich Mathematik
der Universität
D 6500 Mainz
República Federal de Alemania.

(Recibido en mayo de 1979).