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RELATIONS AMONG SOME CLASSES OF QUASIGROUPS

by

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ABSTRACT. We show relations among some classes of quasigroups. A quasigroup (G,∘) is called unipotent if it contains an element x such that a∘a = x, for every a in G; subtractive if b∘(b∘a) = a and a∘(b∘c) = c∘(b∘a) for all a,b,c in G; medial if (a∘b)∘(c∘d) = (a∘c)∘(b∘d) for all a,b,c,d in G. We define a Ward quasigroup as any quasigroup (G,∘) containing an element $i \in G$ such that a∘a = i and (a∘b)∘c = a∘(c∘(i∘b)) for all a,b,c in G. A quasigroup (G,∘) which contains an element i that satisfies the axioms a∘x = b ↔ x = (i∘b)∘(i∘a) and y∘a = b ↔ y = b∘(i∘a) for all a,b in G, is called a Cardoso quasigroup by A. Sade [4]. If the class of the quasigroup is denoted by the initial letter of the respective name, then: (1) $S \subset W \subset C \subset U$; (2) $M \cap C = S$. There is no relation of inclusion between the class of loops and any of the other classes; we exhibit examples to evidence this fact. Furthermore, we establish nec-

essary and sufficient conditions for a Cardoso quasigroup to be a loop.

This paper is concerned with relations among the classes of the following quasigroups: subtractive, medial, Cardoso, Ward, unipotent and loop. In a sense, it is a continuation of [5], where some types of unipotent quasigroups were studied. Ward quasigroups are important because there is a conection between these quasigroups and groups [2]. Namely, if (G,∘) is a Ward quasigroup, then (G,*) is a group under the operation * defined by a*b = a.(i.b); conversely, if (G,*) is agroup, then (G,.) is a Ward quasigroup with respect to the operation · defined by $a \circ b = a*b^{-1}$. In particular, if the group (G,*) is abelian, then the quasigroup is subtractive.

DEFINITION 1. A *subtractive* quasigroup is a quasigroup (G,.) such that:

(S1) b.(b.a) = a, for every a,b in G.

(S2) a.(b.c) = c.(b.a), for every a,b,c in G.

Example 1. The set of all integers with usual subtraction.

The axioms (S1) and (S2) are independent. For example, the set of all integers, with usual addition, satisfies (S2) but does not satisfy (S1); and the quasigroup given by the table

∘	i	a	b
i	i	b	a
a	b	a	i
b	a	i	b

satisfies (S1) but does not satisfy (S2); for example, $i\circ(i\circ a) \neq a\circ(i\circ i)$.

DEFINITION 2. A *medial* [1] quasigroup is a quasigroup $(G,\circ)$ which satisfies the axiom:

(M) $(a\circ b)\circ(c\circ d) = (a\circ c)\circ(b\circ d)$, for every a,b, c,d in G.

Example 2. The set of all integers with the operation $\circ$ defined by $a\circ b = b-a$.

DEFINITION 3. A *Ward* quasigroup is a quasigroup $(G,\circ)$ which satisfies the axioms:

(W1) there is an element $i\in G$ such that $a\circ a = i$, for every $a\in G$.

(W2) $(a\circ b)\circ c = a\circ(c\circ(i\circ b))$, for every a,b,c in G.

Example 3. The set $G = \{i,a,b,c,d,e\}$ with the operation defined by the table

$\circ$	i	a	b	c	d	e
i	i	c	b	a	d	e
a	a	i	e	c	b	d
b	b	e	i	d	a	c
c	c	a	d	i	e	b
d	d	b	c	e	i	a
e	e	d	a	b	c	i

The axioms (W1) and (W1) are independent. For example, in the quasigroup of all integers with usual addition, zero satisfies (W2) bu does not satisfy (W1). And in the quasigroup of example 2, zero satisfies (W1) but does not satisfy (W2).

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(1) Stein [6] uses this terminology. Murdoch [3] calls it an abelian quasigroup.

DEFINITION 4. A *Cardoso*[2] quasigroup is a quasigroup $(G,\circ)$ which contains an element i such that:

(C1) $a \circ x = b \leftrightarrow x = (i \circ b).(i \circ a)$,

(C2) $y \circ a = b \leftrightarrow y = b \circ (i \circ a)$,

for every a,b in G.

Example 4. The set $G = \{0,1,2,3,4,5,6,7,8\}$ with the operation defined by the table:

∘	0	1	2	3	4	5	6	7	8
0	0	2	1	4	3	6	5	8	7
1	1	0	3	2	8	7	4	6	5
2	2	4	0	5	1	8	7	3	6
3	3	1	6	0	5	4	8	7	2
4	4	7	2	6	0	1	3	5	8
5	5	8	7	3	2	0	6	1	4
6	6	3	8	7	4	5	0	2	1
7	7	5	4	8	6	2	1	0	3
8	8	6	5	1	7	3	2	4	0

The axioms (C1) and (C2) are independent. In the example 2, zero satisfies (C2) but does not satisfy (C1). In the quasigroup defined by the following table, the element i satisfies (C1) but does not satisfy (C2).

.	i	a	b	c
i	a	i	c	b
a	b	a	i	c
b	c	b	a	i
c	i	c	b	a

(2) The terminology is of A. Sade. In [4] this quasigroup is defined a a groupoid which satisfies the axioms (C1), (C2) above and (C3): $a \circ i = a, \forall a \in G$. But in [5], it was shown that (C3) is a consequence of (C1) and (C2).

For example, $c \circ a = c$ and $c \neq c.(i.a)$.

DEFINITION 5. A *unipotent*(3) quasigroup is a quasigroup $(G,.)$ such that there is an element $x \in G$ with $a \circ a = x$, for every $a \in G$.

Example. The quasigroup of example 1.

THEOREM 1. *Every subtractive quasigroup* $(G,.)$ *is unipotent.*

Proof. If a,b are elements of a subtractive quasigroup, then $a \circ a = a.(b.(b.a)) = a.(a.(b.b))$ $= b \circ b$. ∎

The converse of theorem 1 does not hold. In example 3, we have: $a \circ (a \circ b) = d$. For subtractive quasigroup, we will denote by i the element equal to $a.a$ for every $a \in G$.

THEOREM 2. *If* $(G,.)$ *is a subtractive quasigroup, the element* i *is a right identity.*

Proof: For every $c \in G$, $c \circ i = c \circ (a \circ a) = a.(a \circ c)$ $= c$. ∎

THEOREM 3. *Let* $(G,.)$ *be a subtractive quasigroup; then*

1) $i \circ (a \circ b) = b.a$, *for every* a,b *in* G.

2) $(a \circ b) \circ c = (a \circ c) \circ b$, *for every* a,b,c *in* G.

Proof: If a,b are elements of G, we have:

1) $i \circ (a \circ b) = b \circ (a \circ i) = b \circ a$.

2) $(a \circ b).c = i \circ (c \circ (a \circ b)) = i \circ (b \circ (a \circ c)) = (a.c) \circ (b \circ i) = (a \circ c) \circ b$. ∎

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(3) The terminology is of R.H.Bruck, for loops. See [1].

The converse of theorem 4 does not hold: for example, the set G = i,a,b with the operation defined by the table

∘	i	a	b
i	i	b	a
a	b	a	i
b	a	i	b

is a medial quasigroup, but does not satisfy the condition of the theorem; for example, $a\circ(b\circ b) \neq b\circ(b\circ a)$. It follows from theorem 4 that every subtractive quasigroup is medial.

THEOREM 5. *Every subtractive quasigroup is a Ward quasigroup.*

<u>Proof</u>: Let $(G,\circ)$ be a subtractive quasigroup. The validity of (W1) was proved in theorem 1; we show now that (W2) holds: $(a\circ b)\circ c = (a\circ c)\circ b = (a\circ c)\circ(i\circ(i\circ b)) = (a\circ i)\circ(c\circ(i\circ b)) = a\circ(c\circ(i\circ b))$. ∎

The converse of theorem 5 does not hold. The quasigroup of example 3 is not subtractive: $a\circ(a\circ b) \neq b$.

THEOREM 6. *Let* $(G,\circ)$ *be a Ward quasigroup. The element* i *whose existence was postulated in* (W1) *is such that for all* a,b *in* G:

1) $a\circ i = a$
2) $i\circ(i\circ a) = a$
3) $i\circ(a\circ b) = b\circ a$.

<u>Proof:</u> Let a,b,c be arbitrary elements in G.

1) $i\circ a = (i\circ i)\circ a = i\circ(a\circ(i\circ i)) = i\circ(a\circ i)$; hence $a = a\circ i$

2) $i.a = (i.a).i = i.(i.(i.a))$; hence $a = i.(i.a)$.
3) $c.(a.b) = c.(a.(i.(i.b))) = (c.(i.b)).a$; for $c = i$, we have $i.(a.b) = (i.(i.b)).a = b.a$. ∎

THEOREM 7. *Every ward quasigroup is a Cardoso quasigroup.*

Proof: We will show that if G is a Ward quasigroup, and a,b,c are alements of G, then the following statements are equivalent: (1) $a.b = c$, (2) $a = c.(i.b)$, (3) $b = (i.c).(i.a)$.

(1) → (2): $c.(i.b) = (a.b).(i.b) = a.((i.b).(i.b)) = a.i = a$.

(2) → (3): If $a = c.(i.b)$ then $i.a = (i.b).c$. Hence $(i.c).(i.a) = (i.c).((i.b).c) = ((i.c).(i.c)).(i.b) = i.(i.b) = b$.

(3) → (1): $a.b = a.((i.c).(i.a)) = (a.a).(i.c) = i.(i.c) = c$. ∎

The converse of theorem 7 does not hold. In example 4, the quasigroup is Cardoso, but is not Ward; for example, $(1.2).4 \neq 1.(4.(0.2))$.

THEOREM 8. *Every Cardoso quasigroup is unipotent.*

Proof: Consider $a \in G$. There is an element $t \in G$ such that $i.t = a$; (C2) implies $i = a.(i.t)$. Hence $i = a.a$. ∎

The converse of theorem 8 does not hold. The following quasigroup is not a Cardoso quasigroup, though it is unipotent:

.	i	a	b	c	d	e
i	i	a	b	c	d	e
a	a	i	c	b	e	d
b	b	e	i	d	c	a
c	c	d	e	i	a	b
d	d	b	a	e	i	c
e	e	c	d	a	b	i

because (C1) does not hold; for example, a.b = c but b ≠ (i.c).(i.a).

THEOREM 9. *Let* (G,.) *a Cardoso quasigroup. The element* i *of definition 4 is a right identity.*

Proof: Let $a \in G$. The solution of equation a.x = a is x = (i.a).(i.a). Theorem 8 implies that x = i. ∎

THEOREM 10. *If* (G,.) *is a medial quasigroup and has right identity* i, *then* (a.b).c = (a.c).b, *for every* a,b,c *in* G.

Proof: Let a,b,c be arbitrary elements of G. Then (a.b).c = (a.b).(c.i) = (a.c).(b.i) = (a.c).b. ∎

THEOREM 11. *Let* (G,.) *be a medial quasigroup which is unipotent, where* i = a.a, *for every* $a \in G$. *If* i *is right identity, then, for all* a,b,c *in* G:

1) i.(b.a) = a.b
2) a.(a.b) = b
3) a.(b.c) = c.(b.a).

Proof: Let a,b,c arbitrary elements of G. Then

1) a.b = (a.b).(a.a) = (a.a).(b.a) = i.(b.a).
2) a.(a.b) = (a.i).(a.b) = (a.a).(i.b) = i.(i.b) = b.i = b.
3) a.(b.c) = (b.(b.a)).(b.c) = (b.b).((b.a).c) =

$i.((b.a).c) = c.(b.a)$. ∎

It follows from theorem 11 that if a quasigroup is medial and Cardoso, then it is also subtractive.

THEOREM 12. *If* (G,.) *is a Cardoso quasigroup, then the following conditions are equivalent:*

(1) (G,.) *is commutative*

(2) (G,.) *is a loop*

(3) $(a.b).b = a$, *for all* a,b *in* G.

Proof:

(1) → (2): As G is a Cardoso quasigroup, there is an element $i \in G$ which is right identity; but G is a commutative quasigroup, and consequently this element is also left identity.

(2) → (3): Consider a,b in G. As G is a Cardoso quasigroup, the equation $y.b = a$ has the unique solution $y = a.(i.b)$. By hypothesis, G is a loop; so $y = a.b$. Substituting in the equation, we have $(a.b).b = a$.

(3) → (1): $(a.b).b = a$, $\forall a,b$ in G, implies that $a.b = a.(i.b)$. Hence $b = i.b$, $\forall b \in G$. Then $a.b = i.(a.b) = b.a$. ∎

In the following, we exhibit examples to show that the class of loops is not contained in any of the other classes of quasigroups studies here, and that any of these classes is not contained in the class of loops.

Example 5. A loop which is not a Cardoso quasigroup: the loop in the remark that follows Theorem 8.

Example 6. A loop which is not a medial quasigroup:

·	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	4	5	6	1
3	3	6	5	2	1	4
4	4	1	2	6	3	5
5	5	4	6	1	2	3
6	6	5	1	3	4	2

In order to verify that it is not a medial quasigroup, note that $(1.2).(3.4) \neq (1.3).(2.4)$.

Example 7. A subtractive quasigroup which is not a loop: the quasigroup of example 1; just note that zero is rigth identity but is not left identity.

Example 8. A medial quasigroup which is not a loop: the quasigroup of example 2. Note that zero is left identity but is not right identity.

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