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NON-STANDARD ANALYSIS AND GENERALIZED FUNCTIONS*

by

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ABSTRACT. A preliminary theory of generalized functions as an alternative to distributions is developed using non-standard analysis. Various of the elementary results in distribution theory are proven using these non-standard generalized functions. In particular, non-standard stochastic fields which are alternatives to generalized stochastic processes are introduced.

RESUMEN. Se desarrolla una teoría preliminar de funciones generalizadas como aternativa a distribuciones, utilizando el análisis noestandar. Se demuestran varios de los resultados elementales de la teoría de distribuciones utilizando estas funciones generalizadas noestandar.

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En particular, campos estocásticos noestandar que son alternativas a procesos estocásticos generalizados son estudiados.

§1. <u>Introducción</u>. Distribution theory [1] gives a concrete and rigorous realization of Dirac's calculus of generalized functions [2]. In his treatment of quantum mecanics, Dirac needed a "function" $\delta(y-x)$ which satisfies

$$\int \delta(y-x) f(y) dy = f(x) \tag{1}$$

for any test function f. As it turns out, Dirac's calculus really did not solve the problem of the continuous spectrum, but it did motivate mathematicians to develop distribution theory.

Let us take our test functions to be elements of the Schwartz space $S(\mathbb{R})$ i.e. functions f with the property that f and all its derivatives decrease for $|x| \to \infty$ faster than any negative power of $|x|$. This set is dense in $L^2(R)$ and complete orthonormal sets for the latter can be selected from the former. The distribution corresponding to $\delta(y-x)$ is the linear functional δ_x defined on $S(\mathbb{R})$ by

$$\delta_x f = f(x). \tag{2}$$

If we put the appropriate locally convex topology on $S(\mathbb{R})$, then δ_x is continuous i.e. an element of the dual space $S(\mathbb{R})'$.

It is worthwhile to make the observation that many physicists still use Dirac's ideas without recurring to the theory of distributions, although they may recognize the latter as justifying the former. Many, who prefer some degree of rigor in their work without sophisticated mathematics, refer to Lighthill's work [3]. Here generalized functions are obtained as equivalence classes of sequences of functions. For $\delta(y-x)$, for example, considered as a generalized function, one may consider the equivalence class corresponding to the sequence

$$\sum_{i\leqslant N} \overline{e_i(y)}e_i(x) \qquad N = 1,2,3,\ldots \tag{3}$$

where $\{e_i\}$ is a complete orthonormal set in $L^2(\mathbb{R})$ formed by elements of $S(\mathbb{R})$. This converges to $\delta(y-x)$ in the sense that the sequence of functions $f^{(N)}$ in $L^2(\mathbb{R})$ defined by

$$f^{(N)}(x) = \int\Big(\sum_{i\leqslant N} \overline{e_i(y)}e_i(x)\Big)f(y)dy \tag{4}$$

converges to f in $L^2(\mathbb{R})$.

The main problem with Lighthill's definition is that one has to work in installments, i.e. work with the N-th term of the sequence and at the end take limits. A statement like

$$\lim_{N\to\infty} \sum_{i\leqslant N} \overline{e_i(y)}e_i(x) = \delta(y-x) \tag{5}$$

has no meaning, and, even if it did, it still involves a limit. (2) is a more concise definition in the sense that no limiting process is involved.

The purpose of this paper is to develop in a preliminary form a theory of generalized functions in the spirit of Lighthill [3], which needs no limiting process as in the theory of distributions [1]. Our main tool will be non-standard analysis [4]. These results represent the logical foundations for the work by Yasue [5] and the author [6,7] in stochastic field theory where many of the same ideas were used without a complete mathematical discussion.

The organization of this paper is as follows: In the next section we consider special classes of functions from $\mathbb{R}$ to ${}^*\mathbb{R}$ and we study the relation of these to Yasue's space of functions [5]. We will show that neither Yasue nor the author when he wrote [6,7] considered the structure of the spaces involved. In particular, derivatives and products of generalized elements in a class of functions considered by Yasue may lie outside that class. Even the question of equality of functions was not made clear before. Moreover, the relation between ordinay L^2-functions and generalized functions defined by the author [6] is studied in more detail. In Sections 3 and 4 we consider specifically operations on the generalized functions. We restrict our attention to operations on functions in the restricted class defined by Yasue [5], although we now understand that these operations may take the functions out of his class to a larger class. There is much new material here, since Yasue did not realize that one can embed generalized elements in his class of functions (e.g. the delta function). Finally, in Sections 5 and 6 we consider the application of this preliminary theory of generalized functions to stochastic fields, which was the motivation for their study in the first place [5-7].

§2 Basic Principles. Let I be a regular ultrafilter in the natural numbers. Then the set of non-standard real numbers

$$^{*}\mathbb{R} = \prod_{N \in \mathbb{N}} \mathbb{R}/I \tag{6}$$

is well-defined by Robinson's non-standard analysis [4]. The equivalence class of the infinite sequence $(a^{(N)})$ of real numbers will be denoted by $[a^{(N)}]$. We recall that the $[a^{(N)}]$ have a *arithmetic that models exactly ordinary arithmetic.

Regularity of I guarantees that I is free since $\mathbb{N}$ is infinite. Moreover, this implies that each $A \in I$ is infinite. This gives a direct way to embed $\mathbb{R}$ in $^{*}\mathbb{R}$. For $a \in \mathbb{R}$, we take $[a^{(N)}]$ so that $a^{(N)} = a$ except possibly for a finite number of N (Since I is an ultrafilter, $\{N: a^{(N)} \neq a\} \notin I$ implies that $\{N: a^{(N)} = a\} \in I$). It also gives a quick way to check when two equivalence classes are equal: for example, if $a^{(N)} \neq b^{(N)}$ for only a finite number of N, then $[a^{(N)}] = [b^{(N)}]$.

Another interesting point is the relation between convergent sequences and their limits. In fact, the following lemma will be extremely useful in the sequel.

LEMMA. *Let* $(a^{(N)})$ *be a sequence in* $\mathbb{R}$ *such that* $a^{(N)} \to a$ *as* $N \to \infty$. *Then* $[a^{(N)}]$ *and* a *differ by at most an infinitesimal.*

Proof. Let $\varepsilon > 0$, $\varepsilon \in \mathbb{R}$. Then there exists an M such that $N \geqslant M$ implies that $|a^{(N)}-a| < \varepsilon$. Hence $\{N:|a^{(N)}-a| \geqslant \varepsilon\} \notin I$. Thus $\{N:|a^{(N)}-a| < \varepsilon\} \in I$, so that $|[a^{(N)}]-a| < \varepsilon$. This is true for every ε, so $|[a^{(N)}]-a|$ is infinitesimal. Q.E.D.

It is amusing that this cannot be made stronger: $(\frac{1}{N})$ is a sequence which converges to zero, but $[\frac{1}{N}] \neq 0$ since $\{N:\frac{1}{N} = 0\} \notin I$. Thus $[\frac{1}{N}]$ is a true infinitesimal.

Functions from $\mathbb{R}$ to ${}^*\mathbb{R}$ are defined easly enough: given a sequence of funtions $(f^{(N)})$, $x \mapsto [f^{(N)}(x)]$ defines a functions from $\mathbb{R}$ to ${}^*\mathbb{R}$ which we will denote by F. However, ${}^*\mathbb{R}^{\mathbb{R}}$ is so huge that to use it in its totality would cause problems. In order to obtain generalized functions, we can dispose of many of these functions by simply noting that there are two definitions of equality possible between functions. The first is a weak form stating that $F = G$ iff $F(x) = G(x)$. This would mean that for all x, $\{N:f^{(N)}(x) = g^{(N)}(x)\} \in I$. However, this set of N's is dependent on x and $\bigcap_x \{N:f^{(N)}(x) = g^{(N)}(x)\}$ could possible be a set not in I. The second form of equality is a strong form and will be the one used in this paper: $F = G$ iff $\{N:f^{(N)} = g^{(N)}\} \in I$. In other words, $F = G$ in the strong sense iff the intersection previously considered is indeed an element of I.

We further restrict our attention to those functions F for which $\{N: f^{(N)} \in L^2(\mathbb{R})\} \in I$. Note that when we write $\{N: f^{(N)} \in L^2(\mathbb{R})\} \in I$, we are stating something about the whole function $f^{(N)}$ and not its pointwise values. This is another motivation for selecting the strong equality above. In fact, since f and g in $L^2(\mathbb{R})$ are identified if $\|f-g\| = 0$, this suggests that we identify F and G if $\{N: \|f^{(N)} - g^{(N)}\| = 0\} \in I$.

We denote the set of functions F for which $\{N: f^{(N)} \in L^2(\mathbb{R})\} \in I$ and F = G if $\{N: \|f^{(N)} - g^{(N)}\| = 0\} \in I$ by ${}^*K$, the *space of generalized functions*. On ${}^*K$ we may define a *norm by

$$\|F\|^2 = [\|f^{(N)}\|^2] \tag{7}$$

where $\|f^{(N)}\|$ is the L^2-norm of $f^{(N)}$ for those N for which it exists and zero for the others. Note that $\|F\|$ is welldefined for strong equality but is not necessarily well-defined for weak equality. Also note that now F = G simply means that $\|F-G\| = 0$.

We now make the connection between the ${}^*K$ defined here and the space of functions used by Yasue [5]. The ultraproduct

$${}^*E = \prod_{N \in \mathbb{N}} \mathbb{R}^N / I \tag{8}$$

is also well-defined by non-standard analysis [4]. Note that, in the sense of model theory, this is not the * of any standard set; in particular, it is not the * of any $\mathbb{R}^N$. However, ${}^*E$ is a vector space over ${}^*\mathbb{R}$ and also a *Euclidean space with respect to the inner product

$$[x^{(N)}]\cdot[y^{(N)}] = [x^{(N)}\cdot y^{(N)}] = [\sum_{i\leq N} x_i^{(N)} y_i^{(N)}] \qquad (9)$$

The inner product on $^*E$ can be used to define a *norm, of course, i.e. a function from $^*E$ to the non-negative non-standar real numbers:

$$\|[x^{(N)}]\|^2 \equiv [\sum_{i<N} |x_i^{(N)}|^2] \qquad (10)$$

Note that this is indeed well-defined: $[x^{(N)}] = [y^{(N)}]$ iff $\{N: x^{(N)} = y^{(N)}\} \in I$ and this is equivalent to $[\sum_{i\leq N} |x_i^{(N)} - y_i^{(N)}|^2] = 0$. In a similar manner, other (equivalent) norms on the $\mathbb{R}^N$ can be used to define (in general, inequivalent) *norms on $^*E$.

PROPOSITION 1. *The function which sends a sequence* (a_i) *in the real Banach space* ℓ^p, $p < \infty$, *to* $[x^{(N)}]$ in $^*E$ where $x_i^{(N)} = a_i$, $i \leq N$, *is an embedding of* $\mathbb{R}$*-rector spaces which almost preserves* p*-norms in the sense that* $[\sum_{i<N} |x_i^{(N)}|^p]$ *and* $\|(a_i)\|_p^p$ *differ by at most an infinitesimal.*

<u>*Proof*</u>. If $[x^{(N)}] = 0$ in $^*E$ then $\{N: x^{(N)} = 0\} \in I$ and so it is infinite. Given any n, take $N \in I$ such that $N \geq n$; then $x^{(N)} = (a_1, \ldots, a_n, \ldots, a_N) = 0$, and so $a_n = 0$. This shows that $(a_i) = 0$ and the assignment is one to one. $^*E$ is a $^*\mathbb{R}$-vector space, hence it is a $\mathbb{R}$-vector space and it is clear that the $\mathbb{R}$-vector space operations are preserved. Finally $\sum_{i\leq N} |a_i|^p = \sum_{i\leq N} |x_i^{(N)}|^p$ converges to

$\|(a_i)\|_p^p$; hence by the Lemma the last condition follows. Q.E.D.

Thus for the effects of applications (i.e. up to infinitesimals), one can work with $^*E$ just as well as with ℓ^p. Now ℓ^2 is one of the important Hilbert spaces in quantum mechanics. $L^2(\mathbb{R})$ (and its tensor products) is another. The equivalence of the Heisenberg and Schrödinger formulations of quantum mechanics from the mathematical point of view is simply due to the fact that there exists an isometry of $L^2(\mathbb{R})$ onto ℓ^2 where each function in $L^2(\mathbb{R})$ is mapped onto the sequence of its Fourier coefficients with respect to the orthonormal set $\{e_i\}$. It is interesting to study whether there exists an extension of this isometry to the case of $^*E$.

The following proposition and the discussion following it show that the space of functions considered by Yasue [5] is contained in $^*K$.

PROPOSITION 2. *Suppose* $\{N: f^{(N)} \in L^2(\mathbb{R})\} \in I$ *and, for these* N, $\sum_{i \leq N} |a_i^{(N)}|^2 \to 0$, *where the* $a_i^{(N)}$ *are the Fourier coefficients of* $f^{(N)}$. *Then there exists an* $[x^{(N)}] \in {}^*E$ *such that* F *and* G, *the latter defined by*

$$G(x) = \left[\sum_{i \leq N} x_i^{(N)} e_i(x) \right] \tag{11}$$

where $x_i^{(N)} = a_i^{(N)}$, *are such that* $\|F-G\|$ *is infinitesimal.*

Proof. We have (with zeros understood for the entries where $N \notin \{N : f^{(N)} \in L^2(\mathbb{R})\}$):

$$\|F-G\|^2 = [\int dx |f^{(N)}(x) - \sum_{i \leq N} x_i^{(N)} e_i(x)|^2]$$

$$= [\int dx | \sum_{i=1}^{\infty} a_i^{(N)} e_i(x) - \sum_{i \leq N} x_i^{(N)} e_i(x)|^2]$$

$$= [\int dx | \sum_{i>N} a_i^{(N)} e_i(x)|^2]$$

$$= [\sum_{i>N} |a_i^{(N)}|^2].$$

Using the lemma, the hypotheses for the $f^{(N)}$ indicate that $\|F-G\|$ is infinitesimal. Q.E.D.

There is then a one-to-one *isometric mapping of $^*E$ into $^*K$ defined by

$$[x^{(N)}] \mapsto [\sum_{i \leq N} x_i^{(N)} e_i] .$$

By *isometry we mean that

$$\|[x^{(N)}]\|^2 = [\sum_{i \leq N} |x_i^{(N)}|^2]$$

$$= [\int dx | \sum_{i \leq N} x_i^{(N)} e_i(x)|^2]$$

$$= \|[\sum_{i \leq N} x_i^{(N)} e_i]\|^2$$

There is a definite practical advantage in our definition of $^*K$: this space of funtions does not depend on the orthonormal set $\{e_i\}$. This is not obvious in Yasue's paper [5] since he only considers functions of the type (11).

We say that $F \in {}^*K$ is integrable if $\{N : f^{(N)} \in L^1(\mathbb{R})\} \in I$. In this case the integral is defined by

$$\int F(x)dx = [\int f^{(N)}(x)dx]. \qquad (12)$$

This is well-defined since $F = G$ only if $\{N : f^{(N)} = g^{(N)}\} \in I$, only if $\{N : \int f^{(N)} = \int g^{(N)}\} \in I$, only if $\int F(x)dx = \int G(x)dx$.

Any G of the form (11) is integrable. It is also infinitely differentiable if we define.

$$G^{(k)}(x) = [\sum_{i \leqslant N} x_i^{(N)} e_i^{(k)}(x)]. \qquad (13)$$

The derivative (13) may not be of the form (11), but it is an element of ${}^*K$, i.e. it is a well-defined function from $\mathbb{R}$ to ${}^*\mathbb{R}$.

Finally we note that one may define a product for functions of the form (11) by

$$[\sum_{i<N} x_i^{(N)} e_i(x)][\sum_{j<N} y_i^{(N)} e_j(x)]$$
$$= [\sum_{i,j \leqslant N} x_i^{(N)} y_j^{(N)} e_i(x) e_j(x)]$$

Again this may not be equal to a function of the form (11). Nevertheless, the product of two functions of the form (11) is in ${}^*K$ because $\{e_i\}$ is in $S(\mathbb{R})$. This is precisely the product defined by Yasue [5].

§3. Operations with Generalized Functions. Throughout the rest of this paper we will concentrate only on those F which have the form (11). Except for occasional operations which take us out of this class (but not out of ${}^*K$), this is all that we will need. The question of extending specific results to all of ${}^*K$ will not be considered.

Besides the operations we have already defined, we also wish to define the convolution and Fourier transform. To do so, it is convenient to extend the results of the previous section to complex functions; this is trivial since we may simply replace ${}^*\mathbb{R}$ by ${}^*\mathbb{C}$. Hence there is no longer any need to assume that the e_i are real.

We fix a complete orthonormal set $\{e_i\}$ in (the now complex) $S(\mathbb{R})$. All of results may be dependent on this set. The question of whether these are actually independent of the selection of $\{e_i\}$ will not be considered in this paper.

The convolution of two elements F, G in (the now complex) ${}^*K$ is defined by

$$(F*G)(x) \equiv \int F(\xi-x)\overline{G(\xi)}d\xi \quad . \tag{14}$$

PROPOSITION 3. $F*G$ *exists in* ${}^*K$ *for all elements* $F,G \in {}^*K$ *of the form* (11).

Proof. One verifies that

$$(F*G)(x) = \Big[\sum_{i,j\leq N} x_i^{(N)}\overline{y_j^{(N)}}\int e_i(\xi-x)\overline{e_j(\xi)}d\xi\Big].$$

Since the e_i are L^1-functions, the result follows. Q.E.D.

The Fourier transform is a little more difficult. Let $\hat{e}_i(k)$ be the Fourier transform of e_i (these also form a complete orthonormal set), and define

$$\hat{F}(k) = \Big[\sum_{i\leq N} x_i^{(N)}\hat{e}_i(k)\Big] . \qquad (15)$$

These functions are elements of an entirely different space, a copy of $^*K$ which we may denote by $^*\hat{K}$. Our first step is to show that (15) is consistent.

PROPOSITION 4. *Suppose the ordinary transform* $\hat{f}$ *of the function* f *exists. Then* $\hat{f}$ *and the right-hand side of* (15) *differ by at most an infinitesimal.*

Proof. Our hypothesis says that

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int f(x)e^{-ikx}dx$$

exists. By Propositions 2, this differs from

$$\Big[\sum_{i\leq N} y_i^{(N)}\hat{e}_i(k)\Big]$$

by at most an infinitesimal, where

$$y_i^{(N)} = \int\hat{f}(k)\overline{\hat{e}_i(k)}dk = \frac{1}{\sqrt{2\pi}}\int dk\overline{\hat{e}_i(k)}\int dxf(x)e^{-ikx}$$

$$= \int dxf(x)\frac{1}{\sqrt{2\pi}}\overline{\int dk\ \hat{e}_i(k)e^{ikx}} = \int dxf(x)\overline{e_i(x)}$$

$$= x_i^{(N)}$$ Q.E.D.

PROPOSITION 5. *The Fourier-Plancherel theorem is valid:*

$$\int \hat{F}(k)\overline{\hat{G}(k)}dk = \int F(x)\overline{G(x)}dx. \qquad (16)$$

Proof. This is a direct calculation:

$$\int \hat{F}(k)\overline{\hat{G}(k)}dx = \Big[\sum_{i,j\leqslant N} x_i^{(N)}\overline{y_j^{(N)}}\int \hat{e}_i(k)\overline{\hat{e}_j(k)}dk\Big]$$

$$= \Big[\sum_{i,j\leqslant N} x_i^{(N)}\overline{y_j^{(N)}}\; e_i(x)\overline{e_j(x)}dx\Big]$$

$$= \int F(x)\overline{G(x)}dx.$$

Q.E.D.

§4. Dirac's Delta function. In this section we finally return to the ideas of Section 1 to show that Section 2 and 3 are really consistent with the informal use of Dirac's delta function.

PROPOSITION 6. *The generalized function.*

$$\delta(y-x) = \Big[\sum_{i\leqslant N} \overline{e_i(y)}e_i(x)\Big] \qquad (17)$$

has the property that

$$\int F(y)\delta(y-x)dy = F(x) \qquad (18)$$

for any element F *in* $^*K$ *of the form* (11).

The proof is trivial. Note that δ has the shifting property (18) for any element of $^*K$ of the form (11), which reflects the fact that we can multiply these non-standard generalized functions.

Any physicist uses formally the relation

$$\frac{1}{2\pi}\int dx e^{i(k-k')x} = \delta(k-k'). \qquad (19)$$

By Proposition 6, we know the right-hand side is

$$\delta(k-k') = \Big[\sum_{i\leqslant N} \overline{\hat{e}_i(k')}\hat{e}_i(k)\Big]. \qquad (20)$$

Can this be made consistent with the left-hand side?

The plane wave $\frac{1}{\sqrt{2\pi}}e^{ikx}$ is not an element of $L^2(\mathbb{R})$. To make sense out of (19), one needs an element of $^*K$ which has the properties of the plane wave. We will see that the element with these properties is

$$\Big[\sum_{i\leqslant N} \overline{\hat{e}_i(k)}e_i(x)\Big]. \qquad (21)$$

We calculate that the left-hand side of (19) reduces to

$$\Big[\sum_{i,j\leqslant N} \overline{\hat{e}_i(k)}\hat{e}_j(k')\int e_i(x)\overline{e_j(x)}dx\Big]$$

$$= \Big[\sum_{i\leqslant N} \overline{\hat{e}_i(k)}e_i(k')\Big].$$

Hence $\Big[\sum_{i\leqslant N} \overline{\hat{e}_i(k)}e_i(x)\Big]$ is the correct substitute for the plane wave. Moreover, (21) is consistent with the Fourier transform:

$$\frac{1}{\sqrt{2\pi}} F(x)e^{-ikx}dx =$$

$$= [\sum_{i,j\leq N} x_i^{(N)}\hat{e}_j(k)\int e_i(x)\overline{e_j(x)}dx]$$

$$= [\sum_{i\leq N} x_i^{(N)}\hat{e}_i(k)] .$$

§5. Stochastic fields. Stochastic fields are easy to define: φ is a stochastic field on some manifold M if for each $x \in M$, $\varphi(x):\Omega \to \mathbb{R}$ (or $\mathbb{C}$) where $(\Omega, P, \mathcal{B})$ is a probability space, thus $\varphi(x)$ is also a function of $\omega \in \Omega$, but we will leave ω as understood in the following. Stochastic fields are the natural extension of stochastic processes.

Nevertheless, stochastic fields have not been studied very much in mathematics, except indirectly, The reason seems to be that mathematicians consider more manageable the derived generalized stochastic process

$$\varphi(f) = \int dx f(x)\varphi(x) \qquad (22)$$

where $f \in \mathcal{S}(M)$. This preference stems from the influence of the theory of distributions, of course. It is safe to say that most work in this area uses Gel'fand's treatise (Volume 4 of the five volume set indicated in [1]) as a starting point, and not the ideas associated with generalized functions, primarily because the limiting process is less manageable. But now, equipped with non-standard generalized functions, no limiting process is necessary.

We take $M = \mathbb{R}^3$ and add a time component. From the previous sections, we know that

$$\varphi(\vec{x}) = [\sum_{i \leq N} x_i^{(N)} e_i(\vec{x})] \tag{23}$$

defines a function from $\mathbb{R}^3$ to ${}^*\mathbb{R}$ where e_i now forms a complete set in $S(\mathbb{R}^3)$. Let $x^{(N)}$ be a stochastic process on $\mathbb{R}^N$ for each N. Then

$$\varphi(\vec{x},t) = [\sum_{i \leq N} x_i^{(N)}(t) e_i(\vec{x})] \tag{24}$$

is a stochastic field. The following proposition is a trivial consequence of our previous work.

PROPOSITION 7. *Given a standard stochastic field* φ *with sample functions in* $L^2(\mathbb{R}^3)$, *its sample functions differ with those of a stochastic field of the form* (24) *by at most an infinitesimal.*

The distribution functions of a field of the form (24) are defined by

$$F(\varphi_1,t_1,\ldots,\varphi_p t_p) = [F^{(N)}(x_1^{(N)},t_1,\ldots,x_p^{(N)},t_p)] \tag{25}$$

where the $F^{(N)}$ are the distribution functions of the component processes $x^{(N)}(t)$ and the φ_j are time-independent "classical" fields in ${}^*K$ of the form

$$\varphi_j(\vec{x}) = [\sum_{i \leq N} x_{ji}^{(N)} e_i(\vec{x})]. \tag{26}$$

Along with the F we can also define probability densities. Suppose each $x^{(N)}(t)$ is such that there exist $\rho^{(N)}(x_1^{(N)},t_1,\ldots,x_p^{(N)},t_p)$ with the property that

$$\int d^N x_1^{(N)} \ldots d^{(N)} x_p^{(N)} \rho^{(N)} = F^{(N)} \tag{27}$$

Then $\rho(\varphi_1, t_1, \ldots, \varphi_p, t_p)$ defined by

$$\rho(\varphi_1, t_1, \ldots, \varphi_p, t_p) = [\rho^{(N)}(x_1^{(N)}, t_1, \ldots, x_p^{(N)}, t_p)] \tag{28}$$

is called the p-th order probabily density of the non-standard stochastic field. Equation (27) has its analog in

$$\int \delta\varphi_1 \ldots \delta\varphi_p \; \rho(\varphi_1, t_1, \ldots, \varphi_p, t_p) = F(\varphi_1, t_1, \ldots, \varphi_p, t_p) \tag{29}$$

where the functional integral is defined by iteration from

$$\int f\{\varphi\} \delta\varphi = [\int d^{(N)} x^{(N)} f^{(N)}\{x^{(N)}\}] \tag{30}$$

for any non-standard field of the form (23).

§6. Markov Field. We say that the stochastic field φ is Markov if the component $x^{(N)}(t)$ are Markov processes. In this case, let us suppose that all component processes have an associated transition probability $p^{(N)}(x^{(N)}, t | x_o^{(N)}, t_o)$. Then the transition probability for φ is given by

$$p(\varphi, t | \varphi_o, t_o) = [p^{(N)}(x^{(N)}, t | x_o^{(N)}, t_o)]. \tag{31}$$

PROPOSICION 8. $p(\varphi,t|\varphi_o,t_o)$ *satisfies a Chapman-Kolmogorov equation.*

Proof. This is trivial using (30). The required equation is

$$\int p(\varphi,t|\varphi_1,t_1)p(\varphi_1,t_1|\varphi_o,t_o)\delta\varphi_1 = p(\varphi,t|\varphi_o,t_o) \qquad (32)$$

Q.E.D.

Equation (32) is characteristic of Markov fields. Using it, one can show:

PROPOSITION 9. *Given that the componente transition probabilities exist, we have*

$$\rho(\varphi_1,t_1,\ldots,\varphi_p,t_p) = \prod_{j=2}^{p} p(\varphi_j,t_j)\varphi_{j-1},t_{j-1})\rho(\varphi_1,t_1). \qquad (33)$$

Equation (33) shows that the basic objects are the transition probabilities. Just as in the case of ordinary Markov processes, once one knows them one knows the probability densities and hence the moments

$$<\varphi(\vec{x}_1,t_1)\ldots\varphi(\vec{x}_p,t_p)>=\int\delta\varphi_1\ldots\delta\varphi_p\rho(\varphi_1,t_1,\ldots,\varphi_p,t_p). \qquad (34)$$

These may be also obtained by differential and integral methods for a special type of Markov field where each component $x^{(N)}(t)$ is a diffusion process. This is the subject of the papers by Yasue [5] and the author [6,7].

§7. Concluding remarks. We have seen that $^*K$ contains enough elements to make it interesting as a candidate for a space of generalized functions. Some applications of these ideas are hinted at in Sections 6 and 7 and their consequences have already appeared [5-7]. Doubtlessly many another applications are possible.

The ideas reported on here were stimulated by the work of Yasue [5]. Although he did not study generalized functions, it is clear that the moments of stochastic fields will in general be generalized functions. This is what motivated the author to consider $^*K$ as a space of generalized functions in the first place. In all fairness it must be stated that the results in Sections 2 though 5 may have appeared earlier, independently of the author's work, in a paper in Japanese to which Yasue makes references [8]. Since the author does not know Japanese, there is no way for him verify this.

There are certainly many open questions that one may consider here besides applications of the theory. Firstly, from the viewpoint of aesthetics al least, we would like to find a minimal subset of $^*K$ which contains functions of the type (11), their products, their derivatives and their convolutions as well as the delta function and its derivatives; moreover, we would like to make this subset independent of the choice of $\{e_i\}$. Although the existence of such a minimal subset is not necessary for applications (for applications we only need to have the operations well-defined with respect to one $\{e_i\}$), the discovery of such a subset with the desired closure properties would allow us to better understand the generalized functions. Secondly, $^*K$ would appear to contain duplicated elements if, for example, the definition of $\delta(y-x)$ (eq. (17)) does indeed depend on the choice of $\{e_i\}$. The author conjectures that this is not the case, but the conjecture remains to be proven.

We also have avoided questions of convergence (i.e. topological questions) in the space $^*K$. There are two justifications for this. The first is that these questions have not affected the applications (so far). The second is that until one finds a minimal subset like that described above, the study of topology would seem to be only of remote interest. However, there is no doubt that convergence in $^*K$ or the minimal subset will be a topic of future research.

Hence there is still much work to be done and many new and interesting ideas to explore within the context of the theory of non-standard generalized functions.

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