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A CONJECTURE CONCERNING THE EXISTENCE OF CERTAIN COMBINATORIAL DESIGNS

by

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ABSTRACT. This note presents a conjecture about the existence of a particular class of combinatorial designs; the consideration of this particular class of designs was suggested during the course of previous work on modular Hadamard matrices. It is hoped that work on the conjecture will lead to effective general techniques to study incidence matrices whose main features are: they have more columns than rows, and each pair of distinct rows has the same inner product.

A *combinatorial design* consists of a finite collection of subsets of a finite set satisfying certain prescribed conditions. A few examples of combinatorial designs are: (v,k,λ)-*designs* (*symmetric balanced incomplete block designs*) and

(b,v,r,k,λ)-*designs* (*balanced incomplete block designs*) [1,4]; *pseudo* (v,k,λ)-*designs* and $(m,v,k_1,\lambda_1,k_2,\lambda_2,f,\lambda_3)$-*designs* [2,3]. This article is concerned with a particular class of $(m,v,k_1,\lambda_1,k_2,\lambda_2,f,\lambda_3)$-designs; a conjecture about the existence of this particular class of designs is stated at the end of the paper.

Throughout the sequel, J will denote the matrix having all its entries equal to 1, and I will denote the identity matrix. Subscripts will be used whenever it is necessary or convenient to emphasize the order of a matrix; thus, $A_{m,n}$ will be an m by n matrix, and A_m will be a square matrix of order m. The transpose of the matrix A will be A^T. Finally, $|X|$ will denote the cardinality of the set X.

Each combinatorial design is completely determined by its *incidence matrix*. Let the subsets $X_1,X_2,\ldots,X_m$ of a set $X = \{x_1,x_2,\ldots,x_v\}$ form some combinatorial design; then the incidence matrix of this design is the m by v (0,1)-matrix $[a_{ij}]$ defined by taking $a_{ij} = 1$ if $x_j \in X_i$ and $a_{ij} = 0$ if $x_j \notin X_i$, for $1 \leqslant j \leqslant v$ and $1 \leqslant i \leqslant m$.

Let X be a set of v elements, and let $X_1,X_2,\ldots,X_m$ be subsets of X. For a fixed integer f ($1 \leqslant f \leqslant m$), the subsets $X_1,X_2,\ldots,X_m$ are said to form an $(m,v,k_1,\lambda_1,k_2,\lambda_2,f,\lambda_3)$-*design* if

$$|X_s| = \begin{cases} k_1 & \text{when } 1 \leqslant s \leqslant f; \\ k_2 & \text{when } f+1 \leqslant s \leqslant m; \end{cases}$$

$$|X_r \cap X_s| = \begin{cases} \delta_1 & \text{when } 1 \leqslant r,\ s \leqslant f \text{ and } r \neq s; \\ \delta_2 & \text{when } f+1 \leqslant r,\ s \leqslant n \text{ and } r \neq s; \\ \lambda_3 & \text{when } 1 \leqslant r \leqslant f \text{ and } f+1 \leqslant s \leqslant m; \end{cases}$$

and $0 < \lambda_1 < k_1 < v-1$; $0 < \lambda_2 < k_2 < v-1$; $0 \leqslant \lambda_3 < k_1, k_2$. If A is the incidence matrix of such a design, then

$$AA^T = \begin{pmatrix} MM^T & \lambda_3 J \\ \lambda_3 J & NN^T \end{pmatrix},$$

where $M_{f,v}$ is a (0,1)-matrix satisfying

$$MM^T = (k_1 - \lambda_1)I + \lambda_1 J,$$

and $N_{m-f,v}$ is a (0,1)-matrix satisfying

$$NN^T = (k_2 - \lambda_2)I + \lambda_2 J.$$

The class of $(m, v, k_1, \lambda_1, k_2, f, \lambda_3)$-designs is one of two classes (the other one is the class of pseudo (v,k,λ)-designs) which were defined and considered during the course of study of modular Hadamard matrices [2,3]. For the explicit relationship between each type of these designs and a corresponding class of modular Hadamard matrices, the reader is referred to [2,3]; in particular, Theorem 3.4 in [3] establishes the connection between the class of $(m, v, k_1, \lambda_1, k_2, \lambda_2, f, \lambda_3)$-designs considered in this article and a corresponding class of modular Hadamard matrices.

The particular class considered in this paper is the one having parameters $m = 12q+9$, $v = 12q+10$, $k_1 = 2q+2$, $\lambda_1 = q+1$, $k_2 = 6q+5$, $\lambda_2 = 5q+4$, $\lambda_3 = q+1$, and f some fixed integer satisfying $0 \leqslant f \leqslant 12q+9$, where q is a nonnegative integer. The existence of this particular class of designs has been determined only for $q = 0$, $q = 1$, and $q = 2$ [3, Theorem 4.1]. It is simple to show that these designs exist when $q = 0$ or $q = 1$ [3, Section 4]; however, the proof that this author wrote to show that these designs do not exist when $q = 2$ appears to be unduly complicated. It is known that the parameters f and q of a combinatorial design as described above must satisfy the following: if $x = (4q+3)((f+1) - (6q+5))$, then x must be a solution of the Diophantine equations

$$x^2 + y^2 = ((4q+3)(6q+5)+2(q+1))^2 \qquad (1)$$

[3, Lemma 4.1]. When $q = 2$, to prove the nonexistence of the particular class of designs under consideration, one determines first from Equation (1) that if such a design exists, then f must equal 16. In terms of the appropriate incidence matrix A, it follows that such a design exists if and only if there exist (0,1)-matrices $M_{16,34}$ and $N_{17,34}$ which satisfy

$$A = \begin{bmatrix} M \\ N \end{bmatrix} \qquad \begin{aligned} MM^T &= 3I_{16} + 3J_{16} , \\ NN^T &= 3I_{17} + 14J_{17} , \end{aligned}$$

and the inner product of each row vector of M with each row vector of N must be 3. The proof of nonexistence of such a matrix A is accomplished by showing that such a matrix $M_{16,34}$ cannot exist. Apparently, what makes the proof of the nonexistence of such a matrix so complicated is that it has more columns than rows; indeed, if M were of order 34 by 16 instead, then there is a very well known determinant method (cf. the proof of Theorem 1.1 on p.99 of [4]) which easily shows that it can not exist. But there seem to be no general techniques avaible to enable one to determine the existence of (0,1)-matrices $S_{m,n}$ having $m < n$ and satisfying

$$SS^T = (k-\lambda)I + \lambda J ;$$

thus, this author hopes that research on the problem which follows will lead to the discovery of new techniques that may be used to deal with such incidence matrices S.

PROBLEM. *Let* f *be a fixed integer satisfying* $0 \leq f \leq 12q+9$. *Determine precisely for which nonnegative integers* q *there exists a* (0,1)*-matrix*

$$A = \begin{bmatrix} M_{f,12q+10} \\ N_{12q+9-f,12q+10} \end{bmatrix} \qquad (2)$$

so that each of the following holds:

$$MM^T = (q+1)I + (q+1)J ; \qquad (3)$$

$$NN^T = (q+1)I + (5q+4)J ; \qquad (4)$$

and

$$AA^T = \begin{bmatrix} MM^T & (q+1)J \\ (q+1)J & NN^T \end{bmatrix} \qquad (5)$$

Of course, it follows from Theorem 3.4 in [3] that this Problem is equivalent to: *Determine precisely for which nonnegative integers* q *there exist* H(4q+3,12q+10) *matrices*. Some unpublished inconclusive research of this author has led him to formulate the following:

CONJECTURE. *A (0,1)-matrix* A *satisfying* (2), (3), (4) *and* (5) *exists if and only of* q = 0 *or* q = 1.

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