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# RANDOM SUM REPRESENTATION OF THE NONCENTRAL CHI-SQUARE RANDOM VARIABLE

by

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ABSTRACT. A new simple approach to the noncentral chi-square distribution is discussed in this paper. Different representations of this variable are given.

§1. Introduction. The noncentral chi-square distribution is of special importance in many areas of theoretical and applied statistics. It is useful in the chi-square goodness of fit, in evaluating the power of chi-square test of the homogeneity of means in a normal sample in multidimensional cases, and in the computation of approximate critical values of a test for the uniformity of a given sample.

The density function of the noncentral chi-square distribution has been derived by a number of authors. Fisher (1928) obtained it as a limiting case of another distribution, Tang (1938) used matrix theory to give an analytic derivation for this density. Patnaik (1949) gave a geometric derivation. Graybill (1961) and McNolty (1962) used moment generating function and characteristic function methods to derive this density. Kerridge (1965) gave a probabilistic derivation. Ifram (1970) used a mixture method to derive the density of the noncentral chi-square distribution. All these papers assume that the number of degrees of freedom is at least 1. Siegel (1978) defined a noncentral chi-square distribution with zero degrees of freedom.

The methods which have been used in the literature can not be applied to derive the density of the noncentral chi-square with zero degrees of freedom, because they define the noncentral chi-square random variable with n degrees of freedom as a sum of n independent normal random variables with means $\mu_j$ ($j = 1,\ldots,n$) and variance one. This paper suggests a new definition of the noncentral chi-square distribution with n degrees of freedom when $n \geqslant 0$. This new approach is very simple to teach in a first course in statistics dealing with the above applications of this distribution. It gives a simple unified way to derive the known properties of this distribution, and it provides some new ones. Moreover, it shows the connections between some of the approaches which have been used in the literature.

This new approach is motivated by a class of problems in which we are interested, related to the distribution of $\Sigma_{j=1}^{M} X_j$ where the number of the observations M follows a discrete distribution. For example, consider a system of a number of components which fail in a given period of time. This number is a random variable which may follow a Poisson distribution. These M components fail at different times. If $X_j$ is the survival time of the $j^{th}$ component then one may be interested in evaluating a probabily statement about the total survival time, $\Sigma_{j=1}^{M} X_j$, of these M components. Theorem 1 deals with such a problem since it gives the density function of $\Sigma_{j=1}^{M} X_j$.

§2. Random Sum Representation of $X_n^2(\theta)$. This section provides a general theorem for deriving the density of a lattice random sum of independent random variables. Then this theorem is applied to the noncentral chi-square distribution, $X_n^2(\theta)$. The following notation and definitions will be used in the paper.

A discrete random variable M is said to have a *lattice distribution* if there exist numbers a and b with b > 0 such that all possible values of M may be represented in the form a+bk where k=0,1,2,...

In particular, let M be a lattice random variable such that $P(M=a+bk) = e^{-\theta}\theta^k/k!$ provided that $\theta > 0$. Then M is said to have a *lattice Poisson* distribution with parameters $(a,b;\theta)$.

Let $Z_1, Z_2, \ldots, Z_M$ be a sequence of random variables such that M is a discrete random variable then $X_M = \Sigma_{j=1}^{M} Z_j$ is called a *random sum* of the $Z_j$'s. If M has a lattice distribution, $\Sigma_{j=1}^{M} Z_j$ is called a *lattice random sum*.

THEOREM 1. *Let* M *be an integrable lattice random variable taking values* a+bk, *let* $X_M$ *be a lattice random sum and for every given* k, *set* $F_k(x) = P(X_M \leq x | M = a+bk)$ *and suppose that:* (i) $F_k$ *is differentiable with* $F_k'(x) = f_k(x)$, *and* (ii) $\Sigma_{k=o}^{\infty} P(M=a+bk) f_k(x)$ *converges uniformly in* x. *Then the density of* $X_M$ *at* x *is* $g(x) = \Sigma_{k=o}^{\infty} P(M=a+bk) f_k(x)$.

*Proof*. Note that

$$P(X_M \leq x) = E(P(X_M \leq x \mid M))$$

$$= \Sigma_{k=o}^{\infty} P(M=a+bk) P(X_M \leq x \mid M=a+bk)$$

$$= \Sigma_{k=o}^{\infty} P(M=a+bk) F_k(x).$$

This series converges because it is bounded by $\Sigma_{k=o}^{\infty} P(M=a+bk) = 1$. Now, by a well known convergence criteria (Apostol, 1957, Theorem 13-14, p.403), this together with condition (ii) implies that the derivate of $P(X_M \leq x)$ with respect to x exists and equals $\Sigma_{k=o}^{\infty} P(M=a+bk) F_k'(x)$. ▲

THEOREM 2. *Let* M *be a lattice Poisson random variable with parameters* $(n,2,\theta/2)$, *and let* $\{Z_j\}$ *be i.i.d. normal with mean zero, variance one and*

N(0,1). *Then for any given* n, *the density of the random sum* $Y_M = \Sigma_{j=1}^{M} Z_j^2$ *at* x *is*

$$g(x) = \Sigma_{k=o}^{\infty} \frac{e^{-\theta/2}(\theta/2)^k}{k!} \frac{x^{(n/2)+k-1}e^{-x/2}}{2^{(n/2)+k}\Gamma((n/2)+k)}$$

*Proof*. Note that

$$\begin{aligned} F_k(x) &= P(Y_M \leqslant x \mid M=n+2k) \\ &= P(\chi^2_{n+2k} \leqslant x) \\ &= \int_o^x \frac{t^{(n/2)+k-1}e^{-t/2}}{2^{(n/2)+k}\Gamma((n/2)+k)} \\ &= \int_o^x f_k(t)dt, \text{ say, where } F_k'(x) = f_k(x). \end{aligned}$$

Now it will be shown that the assumptions of Theorem 1 hold . Assumption (i) is obvious. It can be shown that $\sup_x f_k(x) = f_k(n+2k-2)$. Take

$$R_k = P(M=a+bk)f_k(n+2k-2)$$

then it can be shown that

$$\lim_{k\to\infty}\left|\frac{R_{k+1}}{R_k}\right| = \lim_{k\to\infty}\frac{e^{-1}\theta}{2k}\left(1 + \frac{1}{(n/2)+k-2}\right)^{(n/2)+k-2} = 0,$$

i.e. $\Sigma_{k=o}^{\infty} R_k$ converges and hence condition (ii) follows by the Weirstrass test. ▲

It may be remarked that this theorem suggests that for any given n $\geqslant$ 0,

$$\chi_n^2(\theta) = \Sigma_{j=1}^{M} Z_j^2 = \Sigma_{j=1}^{n+2M'} Z_j^2$$

where $P(M' = k) = e^{-\theta/2}(\theta/2)^k/k!$ .

§3 Applications. The main properties of the $\chi_n^2(\theta)$ with $n > 0$ can be derived from the random sum representation given in Section 2. For example, the $r$-*th* moment is

$$E(\chi_n^2(\theta))^2 = E(E(\gamma_M^r|M)$$
$$= E\left[\Pi_{j=1}^{r}(2M+2j+n-2)\right].$$

The characteristic function of $\chi_n^2(\theta)$ at t is

$$h(t;\chi_n^2(\theta)) = E(\exp(itY_M)) = E(E(\exp(itY_M)|M))$$
$$= (1-2it)^{-n/2}E(1-2it)^{-M}$$
$$= (1-2it)^{-n/2}\exp(it\theta/(1-2it)). \qquad (3.1)$$

In particular, if $n = 0$ then

$$h(t;\chi_o^2(\theta)) = \exp(it\theta/(1-2it)). \qquad (3.2)$$

Formula (3.1) can be used to show that the $r$*th* cumulant of $\chi_n^2(\theta)$ is given by

$$c_r = 2^{r-1}(r-1)!(n+\theta r),$$

the skewness is $((n/2)+3\theta)/(n+\theta)^{3/2}$, the kurtosis (excess) is $6(2\theta+(n/2))/(\theta+(n/2))^2$, and if $\chi_n^2(\theta)$ and $\chi_m^2(\lambda)$ are independent then

$$\chi_n^2(\theta)+\chi_m^2(\lambda) = \chi_{n+m}^2(\theta+\lambda).$$

§4 Factorization of $\chi_n^2(\theta)$. Different factorizations of $\chi_n^2(\theta)$ have been given in the literature. This section gives relations between them and provides a new factorization of $\chi_n^2(\theta)$.

It has been shown that

$$\chi_n^2(\theta) = \Sigma_{j=1}^{n+2M'} Z_j^2 \qquad \text{(take } n \geqslant 2\text{)}$$

$$= \Sigma_{j=1}^{n-m} Z_j^2 + \Sigma_{j=n-m+1}^{n+2M'} Z_j^2 \qquad \text{for } 0 \leqslant m \leqslant n$$

$$= \Sigma_{j=1}^{n-m} Z_j^2 + \Sigma_{j=1}^{m+2M'} Z_j^2 \qquad \text{(independent)}$$

$$= \chi_{n-m}^2 + \chi_m^2(\theta).$$

In particular, if m = 1 then $\chi_n^2(\theta) = \chi_{n-1}^2 + \chi_1^2(\theta)$ which is a well known result given by Patnaik (1949). If m = 0 then $\chi_n^2(\theta) = \chi_n^2 + \chi_o^2(\theta)$ which is given by Siegel (1978). Another interesting factorization is

$$\chi_n^2(\theta) = \chi_n^2 + N(\theta, 4\theta),$$

where $\chi_n^2$ and $N(\theta,4\theta)$ are dependent with zero covariance. To prove this statement, note that if $\{X_j\}$ are independent $N(\mu_j,1)$ random variables for $j = 1,\ldots,n$, then $Y = \Sigma_{j=1}^n X_j^2$ has a $\chi_n^2(\theta)$ distribution with $\theta = \Sigma_{j=1}^n \mu_j^2$, also $U = \Sigma_{j=1}^n (X_j-\mu_j)^2$

has a $\chi_n^2$ distribution and $H = \Sigma_{j=1}^n (2\mu_j X_j - \mu_j^2)$ has a $N(\theta, 4\theta)$ distribution. Hence $\chi_n^2(\theta) = \chi_n^2 + N(\theta, 4\theta)$. Now use the fact that

$$Var(U+H) = Var(U) + Var(H) + 2Cov(H,U)$$

to get $Cov(H,U) = 0$. Finally, assume U and H are independent to get

$$h(t;U+H) = h(y;U)h(t,H) \neq h(t:\chi_n^2(\theta))$$

which is a contradiction and completes the proof. ▲

It may be remarked that Siegel (1978) has shown that $\chi_o^2(\theta) \xrightarrow{d} N(\theta, 4\theta)$ as $\theta \to \infty$. Compare this with $\chi_n^2(\theta) = \chi_n^2 + \chi_o^2(\theta)$ and $\chi_n^2(\theta) = \chi_n^2 + N(\theta, 4\theta)$.

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