

MOMENTS AND DIFFERENTIAL EQUATIONS

by

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RESUMEN. Se presenta un método elemental, basado en la teoría de momentos de Hausdorff, para construir soluciones de sistemas lineales de ecuaciones diferenciales con coeficientes polinomiales y funciones forzadoras de variación acotada.

This work presents an elementary method of constructing solutions to systems of linear differential equations with real polynomial coefficients and forcing functions of bounded variation, using results from Hausdorff moment theory [6].

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§1. Approximation by step functions associated to moments. Suppose f is a real function with domain the number interval $[0,1]$ which is continuous or of bounded variation. If n is a nonnegative integer, then the statement that c_n is the n^{th} moment of f means

$$c_n = \int_0^1 j^n df \quad (1)$$

where the integral is the subdivision refinement Stieljes integral [1] and j is the identity function. The number sequence so defined, is the *moment sequence generate by f .*

The functions that will be used to approximate the solutions to the differential equations are step functions that are described next. If $\{c_n\}_{n=0}^{\infty}$ is a number sequence and a is a number, then there is an associated step function sequence $\{f_n\}_{n=1}^{\infty}$ each term of which is defined on $[0,1]$ as follows.

If x is in $[0,1]$, then there is a unique integer k such that x is in $[k/n, (k+1)/n)$,

$$f_n(x) = a + \sum_{t=0}^k C(n,t) \sum_{i=0}^{n-t} C(n-t,i) (-1)^i c_{i+t}$$

where $C(u,v)$ is the binomial coefficient. The number a is called the *initial point* [2].

If the number sequence $\{c_n\}_{n=0}^{\infty}$ is a moment sequence generated by a function f and $a = f(0)$, then $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f on $[0,1]$

if f is bounded variation on $[0,1]$, see [3], [4], and [9]. The following lemma is found in [8], and with its help it is proved that the convergence is uniform in case f is continuous.

LEMMA. If $\varepsilon > 0$ and $0 < d \leq 1/2$, there exists a positive integer N such that, if $n > N$, and $z \in (d, 1-d)$, then

$$P_n^z(x) < \varepsilon \text{ for } 0 \leq x \leq z-d$$

$$1-P_n^z(x) < \varepsilon \text{ for } z+d \leq x \leq 1,$$

where $P_n^z(x) = \sum_{t=k+1}^n C(n,t)x^t(1-x)^{n-t}$ and $z \in [k/n, (k+1)/n)$.

THEOREM. If f is continuous on $[0,1]$ and $\{c_n\}_{n=0}^{\infty}$ is the moment sequence generated by f , then the associated step function sequence, $\{f_n\}_{n=1}^{\infty}$, converges uniformly to f on $[0,1]$.

Proof. Let $\varepsilon > 0$. There is a $\delta > 0$ such that $|f(x)-f(y)| < \varepsilon$ if $x, y \in [0,1]$ and $|x-y| < \delta$. Moreover if $d = \delta/3$, then there is a $N > 0$ such that if $z \in [d, 1-d]$ and $n > N$, then by the lemma

$$1-Q_n^z(x) < \varepsilon/(4\|f\|+1) \text{ if } 0 \leq x \leq z-d$$

$$Q_n^z(x) < \varepsilon/(4\|f\|+1) \text{ if } z+d \leq x \leq 1$$

where $P_n^z = 1-Q_n^z$ and $\| \cdot \|$ is the norm of the supremum. Let

$$S_w(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq w \\ 0 & \text{if } w < x \leq 1 \end{cases}$$

for each $w \in (0,1)$. Since $\int_0^1 f dS_z = -f(z)$ and a simple integration by parts gives $\int_0^1 f dQ_n^z = -f_n(z)$, then

$$\begin{aligned} |f(z) - f_n(z)| &= \left| \int_0^1 f dS_z - \int_0^1 f dQ_n^z \right| \leq \left| \int_0^{z-d} f d(S_z - Q_n^z) \right| \\ &+ \left| \int_{z-d}^{z+d} \{f(z) + f - f(z)\} d(S_z - Q_n^z) \right| + \left| \int_{z+d}^1 f d(S_z - Q_n^z) \right| \\ &\leq \|f\| V_0^{z-d}(S_z - Q_n^z) + \|f\| V_{z+d}^1(S_z - Q_n^z) \\ &+ \left| \int_{z-d}^{z+d} f(z) d(S_z - Q_n^z) \right| + \left| \int_{z-d}^{z+d} \{f - f(z)\} d(S_z - Q_n^z) \right| \\ &\leq 2\|f\| \varepsilon / (4\|f\| + 1) + f(z)(S_z - Q_n^z) \Big|_{z-d}^{z+d} + \varepsilon V_{z-d}^{z+d}(S_z - Q_n^z) \\ &< \varepsilon/4 + \varepsilon/4 + \varepsilon/2 + 2\varepsilon = 3\varepsilon, \end{aligned}$$

where V denotes the total variation function. Hence $\{f_n\}_{n=1}^\infty$ converges uniformly to f on $[d, 1-d]$. Note that $f(0) = f_n(0)$ and $f(1) = f_n(1)$ for every positive integer n . Let $z \in [0, d]$ and n a positive integer. Then

$$\begin{aligned} |f(z) - f_n(z)| &= \left| \int_0^1 f d(S_z - Q_n^z) \right| \\ &\leq \left| \int_0^1 f(0) d(S_z - Q_n^z) \right| + \left| \int_0^1 \{f(0) - f\} d(S_z - Q_n^z) \right| \\ &\leq |f(0)(S_z - Q_n^z)|_0^1 + \varepsilon V_0^1(S_z - Q_n^z) \\ &< 2\varepsilon. \end{aligned}$$

Similarly for $z \in [1-d, 1]$. Hence $\{f_n\}_{n=1}^\infty$ converges uniformly to f on $[0, 1]$. \blacktriangle

§2. Construction of solutions to linear systems.

Consider now the system $F' = PF + G$, where F and G are column vectors and P is a square matrix with real polynomial entries. We shall construct the solution of the two dimensional case only, and note that the higher dimensional cases are entirely similar.

Suppose g_1 and g_2 are functions of bounded variation on $[0,1]$ and P is a 2×2 matrix each entry of which is a polynomial, $P_{uv} = \sum_{t=0}^m a_{tuv} j^t$, $1 \leq u, v \leq 2$. Note that some a_{tuv} may be zero. Then the system is

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}' = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}.$$

For each non-negative integer n , let

$$C_{n,1} = \int_0^1 j^n dF_1, \quad C_{n,2} = \int_0^1 j^n dF_2,$$

$$d_{n,1} = \int_0^1 j^n dg_1, \quad d_{n,2} = \int_0^1 j^n dg_2.$$

Then for each non-negative integer n

$$\begin{aligned} c_{n,1} &= \int_0^1 j^n F_1' dj = \int_0^1 j^n (P_{11}F_1 + P_{12}F_2 + g_1) dj \\ &= \int_0^1 \sum_{t=0}^m a_{t11} j^{n+1} F_1 dj + \int_0^1 \sum_{t=0}^m a_{t12} j^{n+d} F_2 dj \\ &\quad + \int_0^1 j^n g_1 dj \end{aligned}$$

$$\begin{aligned}
c_{n,1} = & F_1(1) \sum_{t=0}^m a_{t11}/(n+t+1) \\
& + F_2(1) \sum_{t=0}^m a_{t12}/(n+t+1) + g_1(1)/(n+1) \\
& - \sum_{t=0}^m (a_{t11}/(n+t+1)) c_{n+t+1,1} \\
& - \sum_{t=0}^m (a_{t12}/(n+t+1)) c_{n+t+1,2} - d_{n+1,1}/(n+1)
\end{aligned}$$

Similarly

$$\begin{aligned}
c_{n,2} = & F_1(1) \sum_{t=0}^m a_{t22}/(n+t+1) \\
& + F_2(1) \sum_{t=0}^m a_{t22}/(n+t+1) + g_2(1)/(n+1) \\
& - \sum_{t=0}^m (a_{t21}/(n+t+1)) c_{n+t+1,1} \\
& - \sum_{t=0}^m (a_{t22}/(n+t+1)) c_{n+t+1,2} - d_{n+1,2}/(n+1)
\end{aligned}$$

Rewriting the expression for $c_{n,1}$ and $c_{n,2}$ we have for each non-negative integer n ,

$$\begin{aligned}
c_{n,1} + & \sum_{t=0}^m (a_{t11}/(n+t+1)) c_{n+t+1,1} \\
& + \sum_{t=0}^m (a_{t12}/(n+t+1)) c_{n+t+1,2} \\
= & \left(\sum_{t=0}^m a_{t11}/(n+t+1) \right) F_1(1) \quad (A) \\
& + \left(\sum_{t=0}^m a_{t12}/(n+t+1) \right) F_2(1) \\
& + g_1(1)/(n+1) - d_{n+1,1}/(n+1)
\end{aligned}$$

and

$$\begin{aligned}
c_{n,2} &+ \sum_{t=0}^m (a_{t21}/(n+t+1))c_{n+t+1,2} \\
&+ \sum_{t=0}^m (a_{t22}/(n+t+1))c_{n+t+1,1} \\
&= \left(\sum_{t=0}^m a_{t21}/(n+t+1) \right) F_1(1) \\
&+ \left(\sum_{t=0}^m a_{t22}/(n+t+1) \right) F_2(1) \\
&+ g_2(1)/(n+1) - d_{n+1,2}/(n+1).
\end{aligned} \tag{B}$$

For each integer t in $[0, m]$ let

$$A_t = \begin{bmatrix} a_{t11} & a_{t12} \\ a_{t21} & a_{t22} \end{bmatrix}$$

and for each positive integer i and each integer k , where $i < k \leq m+i$, let $A_{i,k} = (1/(k-1))A_{k-i-1}$. Let $A_{u,v}$ be the 2×2 zero matrix, if $u \geq v \geq 1$ or if $v \geq m+i$; and let $\bar{A} = (A_{u,v})$.

Let I denote the 2×2 identity matrix and let \bar{I} denote the infinite matrix having I at each entry on the main diagonal and the 2×2 zero matrix elsewhere. Let

$$\bar{F} = \begin{bmatrix} F_1(1) \\ F_1(2) \end{bmatrix}$$

and for each positive integer i ,

$$H_i = \sum_{t=0}^m (1/(t+i))A_t \bar{F},$$

and let \bar{H} denote the column vector $[H_i]$

For each positive integer i define

$$C_i = \begin{bmatrix} c_{i-1,2} \\ c_{i-1,2} \end{bmatrix} \quad D_i = \begin{bmatrix} d_{i,1}/i \\ d_{i,2}/i \end{bmatrix}$$

and

$$G_i = \begin{bmatrix} g_1(1)/i \\ g_2(1)/i \end{bmatrix}.$$

Define column vectors $\bar{C} = (C_i)$, $\bar{D} = (D_i)$ and $\bar{G} = (G_i)$. We then are able to write equations A and B as

$$(\bar{I} + \bar{A})\bar{C} = \bar{G} - \bar{D} + \bar{H}.$$

Note that each of \bar{G} , \bar{D} and \bar{H} is in $l_\infty \times l_\infty$ [7], [10], and $\bar{I} + \bar{A}$ is a bounded linear transformation from $l_\infty \times l_\infty$ into $l_\infty \times l_\infty$. Hence if $\bar{I} + \bar{A}$ has a bounded inverse, then

$$\bar{C} = (\bar{I} + \bar{A})^{-1}(\bar{G} - \bar{D} + \bar{H})$$

and the moments for F_1 and F_2 would be determined, from which we could construct the associated step function sequence, which will converge uniformly to F_1 and F_2 on $[0,1]$, with $F_1(1)$ and $F_2(1)$ specified. The initial points a_1 and a_2 are determined from $C_{0,1}$, $C_{0,2}$, $F_1(1)$ and $F_2(1)$.

Let us now demonstrate the $\bar{I} + \bar{A}$ is invertible. Observe the $[A_{uv}] \rightarrow 0$ as $u \rightarrow \infty$ and hence there is a U such that if u is an integer greater than U , then

$$\begin{bmatrix} 0 & A_{u,u+1} & \cdots & A_{u,m+u} & 0 \\ 0 & 0 & A_{u+1,u+1} \cdots & A_{u+1,m+u+1} & 0 \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

has norm less than 1. It then follows that

$$B = \begin{bmatrix} I & A_{u,u+1} & \cdots & A_{u,m+n} & 0 \\ 0 & I & A_{u+1,u+2} \cdots & A_{u+1,m+n+1} & 0 \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

has a bounded inverse [5], and then it is easily established that $(I+A)^{-1}$ exists and is a bounded linear transformation.

It should be noted that the technique, with little modification, holds if the entries in matrix P are real analytic functions with domain containing $[0,1]$.

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