ON A PROBLEM OF SAMUEL

by

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RESUMEN. Se demuestra que si $A$ es un dominio euclideano con respecto a una función que vale 1 en los primos de $A$, y además $A$ es una $k$-álgebra finitamente generada sobre un cuerpo $k^A$ que contiene todos los invertibles de $A$, entonces $k$ es algebraicamente cerrado y $A = k[x]$, el anillo de polinomios en $x$ con coeficientes en $k$. Por otra parte, el cuerpo de fracciones de $A$ debe tener genus 0.

Interest in euclidean rings was revived with the appearance of an excellent and interesting paper of Samuel [3]. Since then more than 20 papers have appeared on this topic. In this paper we consider the following problem of Samuel. Let $A$ be a unique factorization domain. Then every non-zero element $a$ of $A$ is of the form $a = u \Pi_{i=1}^{e_i} \Pi_{i=1}^{r}$, where $u$ is a unit of $A$, the $\Pi_i$ are primes of $A$ and the $e_i \geq 0$ are integers for $1 \leq i \leq r$. Set $\phi(a) =$
e_1 + \ldots + e_r. Under what conditions is A euclidean with respect to \( \phi \)?

Before considering this question, we give some examples of domains which are euclidean with respect to a function of the type \( \phi \).

1. \( A = k[x] \), the polynomial ring with coefficients in an algebraically closed field \( k \).

2. A semilocal principal ideal domain (see Prop. 5 in [3]).

3. A principal ideal domain \( A \) such that \( A^* \to (\frac{A}{Aa})^* \) is surjective for all \( a \) in \( A \), where \( A^* \) is the set of all units of \( A \).

4. If \( A \) is euclidean for a function \( \theta \) then localizing \( A \) at all primes \( \Pi \) such that \( \theta(\Pi) \geq 2 \), we find that the localized ring is euclidean for a function of the type \( \phi \).

Now we consider the following two general cases.

**CASE 1. A contains a field \( k \).** In this case we suppose that \( A \) is a finitely-generated \( k \)-algebra. This also includes the case when characteristic of \( A \) is not 0. Since \( A \) is euclidean we find that the transcendental degree of \( K \) over \( k \) is 0 or 1, i.e. either \( A \) is a field or \( K \) is an algebraic function field in one variable over \( k \). Thus \( A = \bigcap_{P \in S} v_P \), where \( S \) is a finite set of primes of \( K \) and \( v_P \) is the valuation ring of \( K \) at the prime \( P \).
CASE 2. A does not contain a field. Thus the characteristic of A is 0 and \( \mathbb{Z} \subseteq A \), we now assume that A is a finitely-generated \( \mathbb{Z} \)-algebra. Since A is euclidean, we find that K, the quotient field of A, is a number field. Thus \( A = \bigcap_{P \not\in S} v_P \), where S is a finite set of primes of K containing all the archimedean primes A and \( v_P \) is the valuation ring of K at the prime P.

In view of these examples we may assume that A is contained in all but a finite number of valuation rings of K, where K is the field of fractions of A.

Next we state, without proof, a theorem of Queen and Weinberger (pag.68 in [2]). Let A = \( \bigcap_{P \not\in S} v_P \) be a principal ideal domain, \( \#(S) \geq 2 \), such that its quotient field K is a global field. We also assume a certain generalized Riemann hypothesis if K is a number field. Then A is euclidean and the smallest algorithm \( \theta \) on A is given by

\[
\theta(x) = \sum_{P \not\in S} \text{ord}_P(x)n_P, \quad x \neq 0
\]

where \( n_P = 1 \) if \( A^* \to (A_P)^* \) is surjective, and \( n_P = 2 \) otherwise.

In view of this we find that if a subring A of a global field K is euclidean for a function of the type \( \phi \) such that \( \phi(\mathfrak{m}) = 1 \) for all primes \( \mathfrak{m} \) of A, then A is a localization at a large number of primes of K, i.e. S is infinite. We also need the following.
THEOREM [Cunnea, 1]. Let $K$ be an algebraic function field in one variable over an algebraically closed field $k$. Let $A$ be a subring of $K$ such that $k \subseteq A$, $K$ is a field of fractions of $A$ and $A$ is contained in all but a finite number of valuation rings of $K$. Then $A$ is a unique factorization domain if and only if genus of $K$ is 0.

Using this result we prove the following theorem.

MAIN THEOREM. Let $A$ be a domain which is not a field and such that $k = \{0\} \cup \{\text{units of } A\}$ is a field. Let $K$ be the quotient field of $A$. Suppose now that $A$ is a finitely-generated $k$-algebra which is euclidean for a function $\phi$ such that $\phi(\Pi) = 1$ for all primes $\Pi$ of $A$. Then $K$ is algebraically closed and the genus of $K$ is 0. Moreover, $A = k[x]$, the polynomial ring in $x$ with coefficients in $k$.

Proof. Since $A$ is euclidean and $K$ is the field of fractions of $A$, we find that the transcendental degree of $K$ over $k$ is less than or equal to 1. Now $\text{trans.deg.}(K/k) = 0$ implies that $A$ is integral over $k$ and thus a field, a contradiction to our hypothesis. Thus $\text{trans.deg.}(K/k) = 1$. Choose $x$ in $A$ such that $x$ is transcendental over $k$. Let $f(x)$ be an irreducible polynomial in $k[x]$ and let

$$f(x) = u\Pi_1^{e_1} \ldots \Pi_r^{e_r}$$

be its prime decomposition in $A$, where $u$ is a unit of $A$ and the $\Pi_i$ are primes of $A$, $1 \leq i \leq r$. Now $k = \{0\} \cup \{\text{units of } A\}$ and $\phi(\Pi_1) = 1$ imply that
Thus we see that $k$ is algebraically closed. It now follows that $K$ is an algebraic function field over an algebraically closed field $k$. Since $A$ is euclidean, using the result of Cunnea we find that the genus of $K$ is 0 and thus $K = k(x)$, a rational field. Since $k[x] \subseteq A \subseteq k(x)$ and all the units of $A$ are in $k$, we find that $A = k[x]$ and whence the result.

REFERENCES