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NECESSARY AND SUFFICIENT CONDITIONS FOR EXISTENCE OF SOLUTIONS TO EQUATIONS WITH NONINVERTIBLE LINEAR PART

by

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RESUMEN. Demostramos (véase Teorema 2.1) que la existencia de puntos críticos de cierto funcional $J:H \to \mathbb{R}$, donde H es un espacio de Hilbert, se reduce a la existencia de puntos críticos de un nuevo funcional $\tilde{J}:N \to \mathbb{R}$, donde N es un subespacio finitodimensional de H. Las propiedades variacionales de $\tilde{J}$ son usadas en las secciones 3 y 4 para dar aplicaciones a ecuaciones diferenciales ordinarias y parciales.

§1. Introducción. This paper is concerned with variational methods which give necessary and sufficient conditions for the existence of solutions of the nonlinear operator equation in Hilbert space

$$Au + Bu = p \ , \tag{1.1}$$

where A is linear, selfadjoint and noninvertible, and B is nonlinear but satisfies certain compatibility conditions which we make precise later. Suffice it to say that (1.1) may represent a nonlinar system of elliptic partial differential equations at resonance. For clarity, the theory developed in section 2 is for resonance at the first eigenvalue and in section 5 the extension for resonance at other eigenvalues is given. Our methods give important variational estimates useful in concrete problems. We illustrate this applicability in section 4 by giving an extension of a classical result due to Ambrosetti and Prodi [19]. The primary motivation for this work was to get a deeper understanding of the variational result for the pendulum equation given in [5]. Since this paper was completed a paper by Amann [18] has appeared containing results related to ours. Howeverm Amann's results, being based on a lemma due to the second author (see [6]), do not apply to the problems considered here. In particular our hypotheses do not imply the μ-monotonicity which he requires.

The results obtained here are related to the classical Landesman-Lazer results [16], but the assumptions on B allow more general necessity and sufficiency conditions for the existence of solutions. In [16] as well as in the work of De Figueiredo and Gossez [9], Dancer [7,8] Berger and Podolak [4], Podolak [17], and Kazdan and Warner [15] the authors studied the Dirichlet problem for scalar equations and obtained solvability conditions in terms of the asymptotic behaviour of the nonlinearity. The abstract theorem given here includes results for the Dirichlet, Neumann, mixed and periodic problems for higher order systems of elliptic partial differential equations. The solvability condition given here contains that given in [16] and, in general, is not given in terms of the asymptotic behaviour of the nonlinearity. The reader is encouraged to study [2], [4], [7], [8], [9], [10], [12], [14], [15], and the recent paper of Hess [13] for more on the resonance problem.

§2. Abstract Results. Let H be a real separable infinite dimensional Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $A:\text{dom } A \subset H \to H$ be a linear selfadjoint operator such that $N = \ker A$ is finite dimensional. Suppose that the restriction A_1 of A to $N^{\perp}$ is a positive operator with compact inverse, i.e., if $\{y_n\} \subset N^{\perp}$ is such that $\{Ay_n\}$ is bounded then $\{y_n\}$ has a convergent subsequence. Thus, the eigenvalues of A_1 form an unbounded sequence $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots$ We note that the positive square root of A_1 has a compact inverse and spectrum $\lambda_1^{\frac{1}{2}} \leqslant \lambda_2^{\frac{1}{2}} \leqslant \ldots$. Let $A^{\frac{1}{2}}$ denote the nonnegative selfadjoint square root of A, then $H_1 \equiv \text{dom}A^{\frac{1}{2}}$ is a Hilbert space when given the inner product $(u,v)_1 \equiv (A^{\frac{1}{2}}u, A^{\frac{1}{2}}v) + (u_0, v_0)$, where w_0 is the orthogonal projection of w on N for each $w \in H$. The norm in H_1 will be denoted by $\|\cdot\|_1$. Note that the inclusion map $H_1 \to H$ is compact and continuous.

Let $B:H \to H$ be a continuous gradient operator with potential $\Psi:H \to R$, i.e. $\lim_{t\to 0} (\Psi(u+tv)-\Psi(u))/t = (B(u),v)$ for all $u,v \in H$. Assume that B takes bounded sets into bounded sets and that for some $\gamma \in [0,1)$ and $D > 0$,

$$(Bu,u) \geqslant -\gamma\|u\|_1^2 - D\|u\|_1, \qquad \text{for all } u \in H_1. \tag{B1}$$

Using the fundamental theorem of calculus and (B1), we see that $\Psi(u) \geqslant -(\gamma/2)\|u\|_1^2 - D\|u\|_1 - \Psi(0)$. Hence, if $\gamma' > (\gamma/2)$ then there exists $C \in \mathbb{R}$ such that

$$\Psi(u) \geqslant -\gamma'\|u\|_1^2 - C, \qquad \text{for all } u \in H_1. \tag{2.1}$$

Suppose that B also satisfies

$$(B(x+u)-B(x+v),u-v) > -\|u-v\|_1^2 \tag{B2}$$

for all $u,v \in Y$, $u \neq v$ and $x \in N$.

We seek solutions of (1.1) by looking for critical points of the functional J defined on H_1 by

$$J(u) = \|A^{\frac{1}{2}}u\|^2/2 + \Psi(u) - (p,u) \tag{2.2}$$

Letting <,> denote duality pairing, one sees that for $u \in \operatorname{dom} A$ and $v \in H_1$,

$$\begin{aligned} \langle \nabla J(u),v\rangle &= (A^{\frac{1}{2}}u, A^{\frac{1}{2}}v) + (Bu,v) - (p,v) \\ &= (Au+Bu-p,v). \end{aligned} \tag{2.3}$$

For this reason we define u to be a weak solution of (1.1) if and only if

$$\nabla J(u) = 0. \tag{2.4}$$

Critical points will be sought by first minimizing J over the subspace $Y = N^{\perp} \cap H_1$ taken to be the orthogonal complement of N in H_1. For each $x \in N$ define $J_x(Y) \equiv J(x+y)$, where $y \in Y$.

We write $p = p_0 + p_1$ (see (1.1)) with $p_0 \in N$ and $p_1 \in N^{\perp}$. From hypotheses (B1) and (B2) it follows that J_x is strictly convex. Taking $\gamma' \in (\gamma/2, 1/2)$ and replacing (2.1) in (2.2) we see that $J_x(Y) \to \infty$ as $\|Y\|_1 \to \infty$, for each $x \in N$. Since J is of class C^1, for each $x \in N$ there exists a unique $\phi(x,p) \in Y$ such that $J_x(\phi(x,p)) = \min\{J_x(y): y \in Y\}$. Moreover, $\phi(x,p)$ is the only critical point of J_x. This implies that $\phi(x,p)$ is independet of p_0. Using the compactness of the embedding $H_1 \to H$, the fact that $\phi(x,p)$ is the minimum of J_x, and the weak lower semicontinuity of the norm, one can show that $\phi(x,p)$ is continuous in x. Arguing as in Lemma 2.1 of [6], we see that the functional $\tilde{J}: N \to \mathbb{R}$, sending $x \mapsto J(x+\phi(x,p))$ is of class C^1 and

$$\begin{aligned} \langle \nabla\tilde{J}(x), x_1 \rangle &\equiv \lim_{t\to 0}((J(x+tx_1) - \tilde{J}(x))/t)) \\ &= \langle \nabla J(x+\phi(x,p)), x_1 \rangle \end{aligned} \tag{2.5}$$

for all $x, x_1 \in N$. A simple computation shows that for $x \in N$ and $y \in Y$, $z = x+y$ is a critical point of J iff $y = \phi(x,p)$ and x is a critical point of $\tilde{J}$.

Thus, from (2.5), se see that $z = x+y$ is a critical point of J iff $y = (x,p)$ and

$$\begin{aligned} 0 &= \langle A(x+\phi(x,p)) + B(x+\phi(x,p)) - p, x_1 \rangle \\ &\langle B(x+\phi(x,p_1)) - p_0, x_1 \rangle \quad \text{for all } x_1 \in N^{\perp} \end{aligned} \tag{2.6}$$

In (2.6) we have used that A is selfadjoint and that $p_1 \in N^{\perp}$. We restate the above observations as

THEOREM 2.1. *Equation* (1.1) *has a weak solution if and only if* J *has a critical point. Hence* (1.1) *has a solution if and only if there exist* $x \in N$ *such that* (2.6) *holds.*

REMARK 2.2. Is is convenient to point out that even though solving (2.6) is equivalent to finding a critical point of $\tilde{J}$, in many cases checking that (2.6) has a solution is not easy whereas verifying that $\tilde{J}$ has a critical point may be simpler. We provide typical examples in the next section.

Also we note that since the function $\phi(x,p)$ depends only on the projection of p on Y, from (2.6) we see that (1.1) is solvable iff the projection of p on N lies in the range of $P(B(x+\phi(x,p)))$, where P denotes the orthogonal projection on N. Hence for $p_1 \in Y$ fixed, the solvability of (1.1) is reduced to computing the range of $P(B(x+\phi(x,p)))$. If N is one dimensional the range is just an interval. In the equation treated in [5] the interval is always closed and in an example of [3, theorem 4.2] the interval is always open.

§3. Applications to ordinary differential equations.

As a first application of Theorem 2.1 we consider the problem of finding weak solution to

$$u''(t)+\sin^2(t)\,\mathrm{sgn}(u)\ln(1+u^2) = f(t)$$

$$t \in (0,2\pi) \tag{3.1}$$

$$u(0) = u(2\pi),\ u'(0) = u'(2\pi).$$

Here we take $H = L_2(0,2\pi)$. We put $Au = u''$ for all functions u satisfying $u(0) = u(2\pi)$ and $u'(0) = u'(2\pi)$. We define

$$\Psi(u) = -\int_0^{2\pi} G(t,u(t))dt$$

with

$$G(t,u) = \sin^2(t)\int_0^u \mathrm{sgn}(s)\ln(1+s^2)ds.$$

Hence, $N = \mathrm{Ker}A$ is the one dimensional subspace of H generated by the constant functions and $\lambda_1 = 1$.

For each constant function $c \in N$ (of value c) we have

$$\tilde{J}(c) = J(c+\phi(c,f)) \leq J(c)$$

$$= \int_0^{2\pi} (-G(t,c)dt + c\int_0^{2\pi} f(t)dt.$$

Since $\lim_{|u|\to\infty} \ln(1+u^2) = \infty$, the latter inequality implies that $\tilde{J}(c) \to -\infty$ as $\|c\|_1 \to \infty$, for each $f \in L_2(0,2\pi)$. Hence, for each $f \in L_2(0,2\pi)$, $\tilde{J}$ has a point of maximum c_0. Thus $c_0+\phi(c_0,f)$ is a weak solution of (3.1). Consequently we have proved

LEMMA 3.1. *For each* $f \in L_2(0,2\pi)$ *the boundary value problem* (3.1) *has a weak solution.*

Next we consider the boundary value problem

$$u''(t)+(u(t)/(1+u^2(t))) = f(t)$$

$$t \in (0,2\pi) \tag{3.2}$$

$$u(0) = u(2\pi),\ u'(0) = u'(2\pi)$$

For this problem A, H, N and λ_1 coincide with those of the problem (3.1). For (3.2) we put

$$\Psi(u) = -(\int_0^{2\pi} \ln(1+u^2(t))dt)/2$$

By Theorem 2.1 we see that (3.2) has a weak solution iff there exists a constant function $c \in N$ such that

$$\begin{aligned} g(c) &\equiv \int_0^{2\pi} (c+\phi(c,f_1))/(1+(c+\phi(c,f_1))^2)(t)dt \\ &= \int_0^{2\pi} f_0(t)dt \end{aligned} \tag{3.3}$$

where $f_1 = f-(\int_0^{2\pi} f(t)dt)/2$ and $f_0 = f-f_1$. Now we are ready to prove.

LEMMA 3.2. *Let* g, f_0 *and* f_1 *be as above. The equation* (3.2) *has a weak solution iff* $\int_0^{2\pi} f_0(t)dt$ *lies in the closed interval* $[\alpha,\beta]$ *where* $\alpha \equiv \min\{g(c):c \in N\}$, $\beta \equiv \max\{g(c):c \in N\}$ *and* $\alpha < 0 < \beta$.

Proof: Since $\phi(c,f_1)$ is a critical point of $J_c : Y \to \mathbb{R}$, $y \mapsto J(c+y)$ we have

$$\begin{aligned} \int_0^{2\pi} ((\phi(c,f_1)'(t))^2 dt &\equiv \int_0^{2\pi} (h((c+\phi(c,f_1))(t)) \\ &+ f_1(t))\phi(c,f_1)(t)dt - \int f_1(t)\phi(c,f_1)(t)dt \end{aligned} \tag{3.4}$$

where $h(s) = s/(1+s^2)$. Since h is a bounded function, from (3.4) and the Sobolev embedding Theorem [1,p.97] we see that there exists a real number M, which depends on f_1, such that

$$\max\{|\phi(c,f_1)(t)| : t \in \mathbb{R}\} \leqslant M \tag{3.5}$$

for all $c \in N$. From (3.5) it follows that $g(c) \to 0$ as $\|c\|_1 \to \infty$. Hence g has a maximum and a minimum. Also (3.5) implies that $g(c)c > 0$ for $\|c\|_1$ suficiently large. Hence, by (3.3), the assertions of the Lemma have been proved.

REMARK. Since $g(c) \to 0$ as $\|c\|_1 \to \infty$, we see that if

$$\alpha < \int_0^{2\pi} f_0(t)dt < \beta \quad \text{and} \quad \int_0^{2\pi} f_0(t)dt \neq 0$$

then (3.2) actually has at least two weak solutions. If f(t) is a nonzero constant function with $\alpha < f < \beta$, then (3.2) has two weak solutions. On the other hand, if $f(t) \equiv 0$ then the only solution of (3.2) is $u(t) \equiv 0$. This illustrates the sharpness of the result.

§4. __An Application to a nonlinear Dirichlet problem__. Let $\Omega \subset \mathbb{R}^n$ be a bounded region and let $H = L^2(\Omega)$. Let (λ_i,ϕ_i) be the i-th eigenvalue-normalized eigenfunction pair for the problem

$$\Delta u + \lambda u = 0 \quad \text{in} \quad \Omega$$

$$u = 0 \quad \text{in} \quad \partial\Omega$$

where Δ denotes the Laplacian operator $\partial^2/\partial x_1^2 + \ldots$

$+\partial^2/\partial x_n^2$.

In [19], Ambrosetti and Prodi considered the problem

$$\Delta u + g(u) = \rho\phi_1 + h \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega \tag{4.1}$$

where ρ is a real parameter, $h \quad Y_0 = \langle\phi_1\rangle^{\perp}$ is continuous and g is strictly convex of class C^2 and satisfies other technical conditions. We will show how our variational information can be used to extend the results of [19] by weakening the conditions on g.

Suppose that g is continuous and satisfies

(I) $\lim_{x\to-\infty} g(x)/x = \mu < \lambda_1$

(II) $\lim_{x\to\infty} g(x)/x = \nu \in (\lambda_1,\lambda_2)$

(III) $(g(u)-g(v))/(u-v) \leq \gamma < \lambda_2$ if $u \neq v$.

THEOREM 4.1. *If* g *is as above, then for each* $h \in Y_0$ *there exists* $\rho(h)$ *such that problem* (4.1) *has* (A) *at least two solution for* $\rho > \rho(h)$, (B) *at least one solution for* $\rho = \rho(h)$, (C) *no solution for* $\rho < \rho(h)$. *Further, if* $h_n \to h$ *weakly in* L^2 *then* $\rho(h_n) \to \rho(h)$. *If, in addition,*

(IV) g *is strictly convex* ,

then (A) *and* (B) *are valid with "at least" replaced by "precisely".*

Proof. By choosing $\varepsilon < \lambda_1-\mu$ and C large and considering the equation

$$\Delta u+(\lambda_1-\varepsilon)u+(g(u)-(\lambda_1-\varepsilon)u+C) = \rho_1\phi_1+h_1,$$

it is clear that there is no loss of generality by assuming $\mu < 0$ and $g \geqslant 0$. We will also take $\phi_1 \geqslant 0$ in Ω. Define the function G and functional $J:H_0^1 \to R$ by

$$G(x) = \int_0^x g(s)ds$$

and

$$J(u) = \int(|\nabla u|^2/2-G(u)+\rho\phi_1 u+hu).$$

For fixed t let $\phi(t)$ be the unique element of $Y = Y_0 \cap H_0^1$ such that $J(t\phi_1+\phi(t)) = \min\{J(t\phi_1+Y): y \in Y\}$. We wish to show that

$$J'(t) \equiv \frac{d}{dt}J(t\phi_1+\phi(t)) \to -\infty \quad \text{as } |t| \to \infty.$$

Note that

$$\frac{d}{dt}J(t\phi_1+\phi(t)) = \langle\nabla J(t\phi_1+\phi(t)),\phi_1\rangle$$
$$= t\lambda_1-\int g(t\phi_1+\phi(t))\phi_1+\rho. \tag{4.2}$$

Since g is nonnegative, $J'(t) \to -\infty$ as $t \to -\infty$. Also

$$J(t\phi_1+\phi(t)) \leqslant J(t\phi_1) = t^2\lambda_1/2+t\rho-\int G(t\phi_1), \tag{4.3}$$

and for $t > 0$ and $\varepsilon > 0$ there is a constant C such that

$$J(t\phi_1) \leqslant t^2\lambda_1/2-(\nu-\varepsilon)t^2/2+C \to -\infty \quad \text{as } t \to +\infty.$$

Hence, if $J'(t)$ does not tend to $-\infty$ as $t \to \infty$, there must exist a sequence $t_n \to \infty$ such that $J'(t_n)$ is bounded. From (4.3) we have $\int\|\nabla\phi(t_n)\|^2$

$\leq \gamma\int(t_n\phi_1+\phi(t_n))^2+g(0)\int(t_n\phi_1+\phi(t_n)) \leq \gamma t_n^2+\gamma\int(\phi(t_n))^2$ $|g(0)|(t_n\int\phi_1+(\text{meas}(\Omega)\|\nabla\phi(t_n)\|/\sqrt{\lambda_2}))$. This implies that $\|\nabla\phi(t_n)\|/t_n$ is bounded and by taking a subsequence, we can suppose that $\phi(t_n)/t_n$ converges weakly in Y to Ψ say. Replacing t by t_n in (4.2), dividing by t_n, and taking the limit as $n \to \infty$ gives

$$0 = \int(|\nabla\phi_1|^2-g_1(\phi_1+\Psi)\phi_1), \qquad (4.4)$$

where

$$g_1(s) = \begin{cases} \mu s & \text{if } s \leq 0 \\ \nu s & \text{if } s \geq 0. \end{cases}$$

Notice that $\lim_{n\to\infty}\langle\nabla J(t_n\phi_1+\phi(t_n))/t_n, y\rangle = \lim_{n\to\infty} 0 = 0$ for all $y \in Y$, i.e.,

$$0 = \lim_{n\to\infty}\int(\nabla\phi(t_n)\cdot\nabla y-g(t_n\phi_1+\phi(t_n))y+hy)/t_n$$

$$= \int\nabla\Psi\cdot\nabla y-g_1(\phi_1+\Psi)y.$$

But putting $y = \Psi$ gives

$$\lambda_2\|\Psi\|^2 \leq \|\nabla\Psi\|^2 = \int g_1(\phi_1+\Psi)\Psi \leq \int\nu\Psi^2 = \nu\|\Psi\|^2$$

which implies that $\Psi = 0$. Now (4.4) becomes $0 = \lambda_1-\nu$, a contradiction. Thus, $J'(t) \to -\infty$ as $|t| \to \infty$, which implies the existence of $\rho(h)$ satisfying (A), (B) and (C). Observe that this also implies that the set of zeros of J' (i.e. the solutions of (4.1)) is bounded. Suppose that (IV) in the statement of Theorem 4.1 holds and that for $\rho \geq \rho(h)$ there are three distinct solutions u, v and w of (4.1). The function $U = u-v$ satisfies

$$\Delta U + pU = 0 \quad \text{in } \Omega$$
$$U = 0 \quad \text{on } \partial\Omega, \tag{4.5}$$

where $p = (g(u)-g(v))/(u-v)$ when $u(x) \neq v(x)$, and 0 otherwise. Since $\rho < \lambda_2$ in Ω, U does not vanish in Ω. We may suppose, therefore, that $u \geqslant v \geqslant w$ in $\bar{\Omega}$. If we define $q = (g(v)-g(w))/(v-w)$ then by the convexity of g, $q \leqslant p$ and $q \neq p$ on a set of positive measure. But this gives a contradiction since $V = v-w$ satisfies

$$\Delta V + qV = 0 \quad \text{in } \Omega, \quad V = 0 \quad \text{on } \partial\Omega.$$

Hence, there are at most two solutions for $\rho \geqslant \rho(h)$, that is, horizontal lines cut the graph of $\frac{dJ'}{dt}(t\phi_1+\phi(t))$ in at most two places. This implies that for $\rho = \rho(h)$ there is precisely one solution. Finally, the fact that $\rho(h)$ depends continuously on h will follow by showing that $\phi(t)$ depends continuosly on h. Let $h_1, h_2 \in Y_0$ and for t fixed let Ψ_1 and Ψ_2 be the $\phi(t)$ corresponding to replacing h in J by h_1 and h_2, respectively. Then

$$0 = \int\{(\nabla\Psi_1\cdot\nabla(\Psi_1-\Psi_2)-g(t\Psi_1+\Psi_1)(\Psi_1-\Psi_2)+h_1(\Psi_1-\Psi_2))\}$$

and

$$0 = \int\{(\nabla\Psi_2\cdot\nabla(\Psi_1-\Psi_2)-g(t\phi_1+\Psi_2)(\Psi_1-\Psi_2)+h_2(\Psi_1-\Psi_2))\}$$

Subtraction gives

$$0 = \|\nabla(\Psi_1-\Psi_2)\|^2-\int\{(g(t\phi_1+\Psi_1)-g(t\phi_1+\Psi_2))((t\phi_1+\Psi_1) -$$

$$- (t\phi_1+\Psi_2))\} + \int(h_1-h_2)(\Psi_1-\Psi_2), \tag{4.6}$$
$$0 \geqslant \|\nabla(\Psi_1-\Psi_2)\|^2-\gamma\|\Psi_1-\Psi_2\|^2-\|h_1-h_2\|\|\Psi_1-\Psi_2\|.$$

This may be written

$$\|\nabla(\Psi_1-\Psi_2)\| \leqslant \sqrt{\lambda_2}/(\lambda_2-\gamma)\|h_1-h_2\|. \tag{4.7}$$

Thus, for fixed t, the mapping h $\phi(t)$ is globally Lipschitzian from L^2 into H_0^1. Now (4.7) shows that if $h_1 \to h$ weakly in L^2, and if Ψ_n, Ψ denote the corresponding $\phi(t)$'s, then Ψ_n is bounded in H_0^1. Hence, by the Sobolev embedding Theorem [1,p.97] we can assume that $\Psi_n \to \Psi$ in L^2. Finally (4.6) shows that $\Psi_n \to \Psi$ in H_0^1 and the proof is complete.

§5. __Resonance at eigenvalues other than the first one__. Let A,B and $\{\lambda_i;\ i = 1,2,...\}$ be as in section 2. We consider the problem

$$Au-\lambda_k u+Bu = p \tag{5.1}$$

and, instead of (B1) and (B2), we assume that there exist real numbers γ and γ_1 such that $\lambda_k-\lambda_{k+1} < \gamma \leqslant \gamma_1 < \lambda_k-\lambda_{k-1}$ and

$$\gamma\|u-v\|_2 \leqslant (B(u)-B(v),\ u-v) \leqslant \gamma_1\|u-v\|^2 \tag{5.2}$$

Let X_1 denote the linear subspace generated by the eigenfunctions corresponding to the eigenvalues 0, $\lambda_1,...,\lambda_{k-1}$, let Y denote the closed subspace generated by the eigenfunctions corresponding to the eigenvalues λ_{k+1},... and let N denote the kernel of $A-\lambda_k I$. Define J as in section 2 replacing A by $A-\lambda_k I$ and B by $B+\lambda_k I$.

For each $x = x_1+x_2$, $x_1 \in X_1$, $x_2 \in N$ there exists a unique element $\phi(x) \in Y$ such that $J(x+\phi(x)) = \min\{J(x+y): y \in Y\}$. It is easily proved that the critical points of J coincide with the critical points of the functional $F: H_1 \to \mathbb{R}$ defined by

$$F(u) = 2J(x+\phi(x))+J(x+y) ;$$

where x denotes the orthogonal projection of u on $X_1 \oplus N$, and $y = u-x$.

Following a procedure similar to that of section 2, it is easily shown that for each $x \in N$ there exists a unique $\bar{\phi}(x) = \phi_1(x)+\phi_2(x) \in X_1 \oplus Y$ such that

$$F(x+\bar{\phi}(x)) = \min_{z \in X_1 \oplus N} F(x+z) \equiv \tilde{F}(x).$$

From theorem 2.1 we have:

THEOREM 5.1. *The equation* (5.1) *is solvable iff* F *has a critical point. Hence* (5.1) *is solvable iff there exists* $x \in N$ *such that* $\langle \nabla F(x+\bar{\phi}(x)), x_o \rangle = 0$ *for all* $x_o \in N$.

Theorem 5.1 together with remark 2.10 generalize theorem 4.1 of [3]. Considerations as those given in section 3 permit simplifications to give explicit solvability conditions for (5.1) in terms of B and the elements of N.

REFERENCES

[1] Adams, R., *Sobolev Spaces*, Academic Press, New York, San Francisco, London (1975).

[2] Amann, H., Ambrosetti,A. and Mancini,G., *Elliptic equations with noninvertible Fredholm linear part and bounded nonlinearities*, Math. Z., 158 (1978), 179-194.

[3] Ambrosetti,A. and Mancini,G., *Existence and multiplicity results for nonlinear elliptic problems with linear part at resonance. The case of simple eigenvalue*, J. Differential Equations, 28 (1978), 220-245.

[4] Berger, M.S. and Podolak,E., *On the solutions of a nonlinear Dirichlet problem*, Indiana Univ. Math. J., 24 (1975), 837-846.

[5] Castro, A., *Periodic solutions of the forced pendulum equation*, Differential Equations, Editors S. Ahmad, M. Keener and A.C. Lazer, Academic Press (1980), 149-160.

[6] Castro, A., *Hammerstein integral equations with indefinite Kernel*, Internat. J. Math. and Math. Sci., 1 (1978), 187-201.

[7] Dancer, E.N., *Boundary-value problems for weakly nonlinear ordinary differential equations*, Bull. Austral. Math. Soc., 15 (1976), 321-328.

[8] Dancer, E.N. *On the Dirichlet problem for weakly nonlinear elliptic partial differential equations*, Proc. Roy. Soc. Edin. 76 A (1977), 283-300.

[9] de Figueiredo,D.G. and Gossez,J.P., *Nonlinear perturbations of a linear elliptic problem near its first eigenvalue*, J. Differential Equations 30 (1978), 1-19.

[10] Fucik,S., *Nonlinear equations with noninvertible linear part*, Czech. Math. J., 24 (1974), 467-495.

[11] Fucik,S., *Remarks on some nonlinear boundary value problems*, Comment. Math. Univ. Carolinae 17 (1976), 721-730.

[12] Fucik,S., and Krbec, M., *Boundary value problem with bounded nonlinearity and general null-space of the linear part*, to appear, Math.Z.

[13] Hess, P. *Nonlinear pertubations of linear elliptic and parabolic problems at resonance: existence of multiple solutions*, preprint, 1979.
[14] Kazdan, J.L. and Kramer, R.J. *Invariant criteria for existence of solutions to second-order quasilinear elliptic equations*, Comm. Pure Appl. Math. 31 (1978), 619-645.
[15] Kazdan J.L. and Warner, F.W., *Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math. 28 (1975), 567-597.
[16] Landesman, E.M. and Lazer, A.C., *Nonlinear perturbations of linear elliptic equations at resonance*, J. Math. Mech. 19 (1970), 609-623.
[17] Podolak, E., *On the range of operator equations with an asymptotically nonlinear term*, Indiana Univ. Math. J. 25 (1976), 1127-1137.
[18] Amann, H., *Saddle points and multiple solutions of differential equations*, Math. Z. (1979), 127-166.
[19] Ambrosetti, A. and Prodi, G., *On the inversion of some differentiable mapping with singularities between Banach spaces*, Annali Mat. Pura Appl., 93 (1972), 231-247.

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