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CONGRUENCES IN REGULAR CATEGORIES

by

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RESUMEN. Se investiga la composición de congruencias en categorias regulares y se demuestra, entre otros resultados, que la *condición de Lawvere* (toda relación de equivalencia es una congruencia) es equivalente en tales categorias a cualquiera de las siguientes propiedades: (i) la compuesta de congruencias que conmutan es una congruencia, (ii) un morfismo regular con congruencia r envía toda congruencia que conmuta con r a una congruencia en la imagen, (iii) cualquier par de morfismos regulares con congruencias que conmutan y tienen intersección trivial posee un "pushout" que es simultáneamente un "pullback". Con lo anterior es posible caracterizar las categorías regulares en las que la compuesta de congruencias es siempre una congruencia, generalizando así hechos bien conocidos del Algebra Universal.

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§1. <u>Introducción</u>. The purpose of this article is to examine, in a regular category (in Grillet's terminology, [6]) the nature of the condition: *every equivalence relation is a congruence*. This latter is often called *Lawvere's condition* because it is one of the conditions he used to characterize an algebraic category [8]. It is known [1,page 4] that even a complete and cocomplete regular category need not satisfy Lawvere's condition and, hence, this condition cannot be replaced by any conditions which simply assert the existence of limits or colimits. However, Lawvere's condition is an easy theorem in any variety or finite variety of algebras. It also holds in abelian categories.

The main purpose here is to give three statements each equivalent to Lawvere's condition in a regular category. Two of these are algebraic in nature and for varieties they are well-known theorems of universal algebra [5], [12]. For example *if two congruences commute then their composition is their join*. The third statement asserts that a certain pushout exists and is also a pullback, the most categorical of the three statements (see Theorem 17). On the way to this theorem we investigate some properties of the composition of relations and congruences (sections 3) and of the joins of congruences in regular categories (section 4).

A preliminary version of this paper was announced in [2] but there the results were proved only for regular categories having coequalizers of any pair of morphisms (regular categories need only have coequalizers of kernel pairs). Some of the results of [2] were reproduced by T.H. Fay in [3]. In the present version the results are proved for arbitrary regular categories.

§2. **Preliminaries**. We assume the reader knows the basic elementary concepts of category theory: limits, colimits, kernel and cokernel pairs, equalizers and coequalizers, pullbacks, pushouts, images, etc. as presented in [10] or [11].

An *onto* morphism is a coequalizer of some pair of morphisms, these are called *regular* morphisms in [1] and [6]. Throughout, C will be a *regular* category, that is: (C1) *C has finite limits*; (C2) *every morphism* f *of* C *has a factorization* f = mq, m *monic,* q *onto (regular decomposition)*; (C3) *if* fg' = gf' *is a pullback then* f *onto implies* f' *onto*.

If (e,e') is a kernel pair in C, say the Kernel pair of f, and f = mq is the regular decomposition of f then q is the coequalizer of the pair (e,e'). Hence, *kernel pairs have coequalizer*. This property is used in [1] as one of the axioms for regular categories.

Some properties of onto morphisms in regular categories are [6, p.133-134]:

LEMMA 1. *If* f *and* g *are onto, so is* fg *(if defined). If* fg *is onto, so is* f. f *is an isomorphism if, and only if, it is monic and onto. If* f *and* g *are onto so is* f×g.

The following notation will be used throughout. In diagrams, ↣ will be monic and ↠ will be onto. Always π will represent a projection of a product to a factor, with a subscript to indicate which projection. If $f:A \to B$ and $g:C \to D$ then $f \times g$ is the induced morphism $A \times B \to C \times B$, while if $f:A \to B$ and $g:A \to C$, then $<f,g>$ is the induced morphism $A \to B \times C$. The *diagonal* morphism $\Delta_A : A \to A \times A$ is given by $\Delta_A = <1_A, 1_A>$. Equalizers and coequalizers of the pair f,g will be denoted, respectively, Equ(f,g) and Coequ(f,g).

DEFINITION. (a) If $f:A \to B$ is a morphism in C *a congruence of* f is defined as Cong f = Equ($f\pi_1$,$f\pi_2$) where π_1, π_2 are the projections $A \times A \to A$.

(b) If s is a subobjet of A×A, a *quotient of* s is defined as Quot s = Coequ($\pi_1 s$,$\pi_2 s$).

Congruences always exist in a regular category but quotient do not necessarily exist; however, if s = Congr f then ($\pi_1 s$,$\pi_2 s$) is a kernel pair of f and so it has a coequalizer. Hence,

LEMMA 2. *Any congruence has a quotient, unique up to isomorphism.*

Of course, every quotient is onto and every onto morphism is the quotient of its congruence.

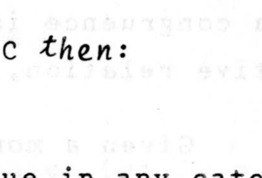
Similarly, any congruence is the congruence of its quotient.

Since we have pullbacks the intersection of subobjects is well defined and we have easily:

LEMMA 3. *If* $f:A \to B$ *and* $g:A \to C$ *then:* $(\mathrm{Congr}\, f) \cap (\mathrm{Congr}\, g) = \mathrm{Congr}\langle f,g\rangle$.

The following lemma which is true in any category is also used below. Its proof is straightforward.

LEMMA 4. *Consider the diagram*



where $r = \mathrm{Equ}(u,v)$, *then the square can be completed to a pullback if and only if* $s = \mathrm{Equ}(ug,vg)$. *And dually for coequalizers and pushouts.*

§3. <u>Composition of relations</u>. A *relation* is, of course, a subobject of a product $A\times B$. Following Grillet [6] or, equivalenty, following the procedure in sets we get the following definition.

DEFINITION. Let $r:R \to A\times B$ and $s:S \to B\times C$ be relations and let $u = \mathrm{Equ}(\pi_B r\pi_R, \pi_B s\pi_S) : U \to R\times S \rightrightarrows B$, then $r\circ s$ is defined as $\mathrm{Im}(\langle\pi_A r\pi_R u, \pi_C s\pi_S u\rangle)$, a relation contained in $A\times C$.

This composition is examined in detail by Grillet who remarks, in particular, that it is associative.

A congruence $c:C \to A\times A$ is a relation. It is easily checked that $\Delta_A \subset c$ and if $\tau : A\times A \to A\times A$ is the twisting morphism, $\tau = \langle \pi_2, \pi_1 \rangle$, then there exists an isomorphism $k : C \to C$ so that $c = ck$ and $k^{-1} = k$. Also [6, p. 155] $c \circ c \subset c$. In other words, a congruence is a reflexive, symmetric and transitive relation, in short an *equivalence relation*.

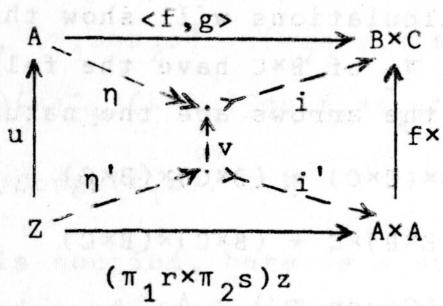
Given a morphism f and a monic i so that fi is defined, then the regular factorization $i'f' = fi$ defines the *direct image of* i *by* f, denoted $f_*(i)$. Also, if j is monic, the pullback $fj' = jf'$ defines an *inverse image* j' *of* j *by* f, denoted $f^{-*}(j)$. The next lemma gives an alternate definition of the composition of two congruences.

LEMMA 5. *Let* $f : A \to B$ *and* $g : A \to C$ *then* $(\text{Congr } f) \circ (\text{Congr } g) \simeq (f\times g)^{-*}(\text{Im}\langle f,g \rangle)$.

Proof. Note that $\langle f,g \rangle = (f\times g)\Delta_A$. Using the definition of a congruence one sees that (1) is a pullback where $r = \text{Congr } f$, $s = \text{Congr } g$.

$$
(1)\quad
\begin{array}{ccc}
A\times A & \xrightarrow{\ f\times g\ } & B\times C \\
\uparrow{\scriptstyle \pi_2 r\times \pi_1 s} & & \uparrow{\scriptstyle f\times g} \\
R\times S & \xrightarrow[\ \pi_1 r\times \pi_2 s\ ]{} & A\times A
\end{array}
\qquad\qquad
(2)\quad
\begin{array}{ccc}
A & \xrightarrow{\ \Delta_A\ } & A\times A \\
\uparrow{\scriptstyle u} & & \uparrow{\scriptstyle \pi_2 r\times \pi_1 s} \\
Z & \xrightarrow[\ z\ ]{} & R\times S
\end{array}
$$

Put $z = \text{Equ}(\pi_2 r \pi_R, \pi_1 s \pi_S)$. By Lemma 4, (2) is a pullback. Also, by definition, $r \circ s = \text{Im}\langle \pi_1 r \pi_R z,\ \pi_2 s \pi_S z \rangle = (\pi_1 r\times \pi_2 s)_*(z)$, since z is monic. Composing (1) and (2) we get the pullback

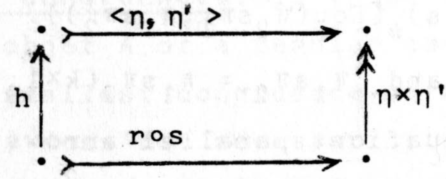


Now factor $\langle f,g\rangle = i$, η onto, i monic. Let $iv = (f\times g)i'$ be a pullback, then for some η', $v\eta' = \eta u$ is also a pullback. By C3, η' is onto and by the uniqueness of the regular decomposition $i' = (f\times g)^{-*}(i) = ros$. ■

To show the usefullness of the above characterization of the composition of congruences we prove the next two propositions.

COROLLARY 6. *Let* r *and* s *be congruences in* A *with* $r\cap s \simeq \Delta_A$ *and* $ros \simeq 1_{A\times A}$. *If* η = Quot r *and* η' = Quot s *then* $\langle\eta,\eta'\rangle: A \to A/R\times A/S$ *is an isomorphism where* A/S *is the codomain of* η *and* A/S *that of* η'.

Proof. By Lemma 3, $\langle\eta,\eta'\rangle$ is monic and so $\mathrm{Im}\langle\eta,\eta'\rangle = \langle\eta,\eta'\rangle$. Hence ros is defined by the pullback



By Lemma 1, $\eta\times\eta'$ is onto. By hypothesis ros is an isomorphism so $\langle\eta,\eta'\rangle h = (\eta\times\eta')(ros)$ is onto. Thus, $\langle\eta,\eta'\rangle$ is onto and monic. ■

A few calculations will show that the projections π_B and π_C of $B\times C$ have the following properties, where the arrows are the natural ones:

$$\mathrm{Congr}\ \pi_B = B\times(C\times C) \to (B\times C)\times(B\times C)$$

$$\mathrm{Congr}\ \pi_C = (B\times B)\times C \to (B\times C)\times(B\times C)$$

$$(\mathrm{Congr}\ \pi_B)\cap(\mathrm{Congr}\ \pi_C) \simeq \Delta_{B\times C}$$

$$(\mathrm{Congr}\ \pi_B)\circ(\mathrm{Congr}\ \pi_C) \cong 1_{(B\times C)\times(B\times C)}$$

$$\cong (\mathrm{Congr}\ \pi_C)\circ(\mathrm{Congr}\ \pi_B).$$

This and Corollary 6 give a characterization of the product in regular categories:

COROLLARY 7. A *is a product of two objects iff there exist two congruences in* A *with trivial intersection and composition.*

We next give a useful characterization of $s\circ s$ when s is a symmetric relation in $A\times A$.

LEMMA 8. *Let* s *be a symmetric subobjetc of* $A\times A$, *then* $s\circ s = (\pi_2 s\times\pi_2 s)_*(\mathrm{Congr}\ \pi_1 s)$.

Proof. Recall that there is an isomorphism $k = k^{-1}: S \to S$ so that $s = sk$ where τ is the twisting morphism on $A\times A$. Now $s\circ s = (\pi_1 s\times\pi_2 s)_*(\mathrm{Equ}(\pi_2 s\pi_1, \pi_1 s\pi_2))$. But $\pi_2 s\pi_1 = \pi_1 s\pi_1(k\times 1_S)$ and $\pi_1 s\pi_2 = \pi_1 s\pi_2(k\times 1_S)$. We have the following situation: parallel arrows $\overset{a}{\underset{b}{\rightrightarrows}}$ where there is an isomorphism c with $c^2 = 1$ and $a'c = a$, $b'c = b$ for some a', b'. By using twice the fact that $c^2 = 1$ we get that $c\mathrm{Equ}(a',b') \cong \mathrm{Equ}(a,b)$.

In particular, $(\pi_1 s\times\pi_2 s)\mathrm{Equ}(\pi_2 s\pi_1,\pi_1 s\pi_2)$

$= (\pi_2 s\times\pi_2 s)(k\times 1_S)(k\times 1_S)\mathrm{Equ}(\pi_1 s\pi_1,\pi_1 s\pi_2)$

$= (\pi_2 s\times\pi_2 s)\mathrm{Congr}(\pi_1 s)$. ∎

To end this section, here is a summary of some of the properties of composition used below.

LEMMA 9. *Let* s, t *and* u *be subobjets of* $A\times A$.

(1) $s\circ\Delta \cong s$.

(2) *If* $t\subset u$ *then* $s\circ t\subset s\circ u$.

(3) *If* $\Delta\subset t$ *then* $s\subset s\circ t$.

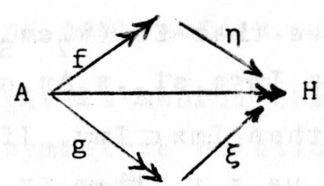
(4) *If* $s\subset u$ *and* $t\subset u$ *where* u *is transitive then* $s\circ t\subset u$.

Proof. (1) Observe that $\mathrm{Equ}(\pi_2 s\pi_S,\pi_1\Delta\pi_A) = \langle 1_S,\pi_2 s\rangle$, hence $s\circ\Delta \cong \mathrm{Im}\langle\pi_1 s1_S,\pi_2\Delta\pi_2 s\rangle = s$.
(2) Note, if $y\ell = x$ then $\mathrm{Im}x\subset\mathrm{Im}y$. If $\mu = \mathrm{Equ}(\pi_2 s\pi_S,\pi_1 t\pi_T)$ and $\mu a = t$, then $\langle\pi_S\mu,a\pi_T\mu\rangle$ equalizes $(\pi_2 s\pi_S,\pi_1 u\pi_U)$ and the result follows. To prove (3) use (2) which gives $s\cong s\circ\Delta\subset s\circ t$, and for (4) use (2) twice. ∎

§4. Joins of congruences. If r and s are congruences in an objet A of a regular category, the *join* $r\vee s$ (a smallest congruence of A containing both r and s) does not necessarily exist. From Lemma 9, it follows that $r\circ s$ is a congruence if and only if $r\circ s = r\vee s$, and so in this case the join exists. Unfortunately the converse is not valid, the join $r \vee s$ may exist without $r \circ s$ being a congruence (obviously $r \circ s \subset r \vee s$ in this case). In this section we characterize those pairs of congruences for which $r \vee s$ exists in a regular category, and those for which $r \vee s = r \circ s$.

PROPOSITION 10. *Let* r *and* s *be congruences in* A *with quotients* f *and* g, *respectively. Then* $r \vee s$ *exists in* A *if and only if* f *and* g *have a pushout* $\eta f = \xi g$ (*in which case* $r \vee s$ = Congr ηf).

<u>*Proof*</u>. If $r \vee s$ exists, let h = Quot($r \vee s$). Since $r \subset r \vee s$ then μ coequalizes $(\pi_1 r, \pi_2 r)$ and the same happens with $(\pi_1 s, \pi_2 s)$. Hence, there are morphisms η, ξ such that $h = \eta f = \xi g$:



(3)

Now, if $\alpha_1 f = \alpha_2 g = \beta$ then Congr β contains Congr f and Congr g, and so Congr h $= r \vee s \subset$ Congr β . This implies the existence of $\delta : H \to B$ such that $\beta = \delta h$. Therefore,

$$\alpha_1 f = \delta \eta f \ , \quad \alpha_2 g = \delta \xi g \ ,$$

and because f and g are onto,

$$\alpha_1 = \delta \eta \qquad \alpha_2 = \delta \xi$$

showing that diagram (3) is a pushout. Conversely, given a pushout like (3), it is straightforward to prove that Congr ηf = Congr ξg is $r \vee s$. ∎

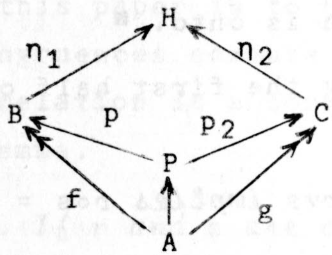
COROLLARY 11. *The following are equivalent in a regular category:*

(1) *Any pair of onto morphisms with domain* A *has a pushout.*

(2) *Any pair of congruences in* A *has a join (the class of congruences in* A *forms a lattice).*

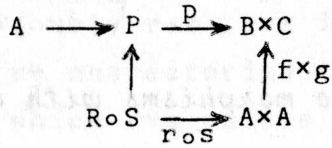
From last corollary, in a regular category having coequalizers of any pair of morphisms the class of congruences of any object forms a "big" lettice, because by Lemma 4 the coequalizers provide pushouts of diagrams $\begin{smallmatrix}\uparrow & \\ \cdot & \rightarrow\end{smallmatrix}$.

PROPOSITION 12. *Let* r *and* s *be congruences in* A *for which* r∨s *exists, with quotients* f *and* g, *respectively. Consider the following diagram where* $\eta_1 f = \eta_2 g$ *is a pushout and* $\eta_1 p_1 = \eta_2 p_2$ *is a pullback:*



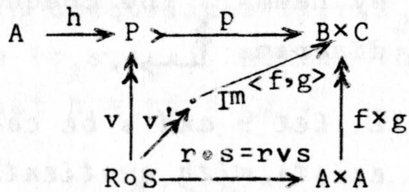
then, r∘s = r∨s *if and only if the induced morphism* A → P *is onto.*

Proof. Suppose h:A → P is onto and let p = $\langle p_1, p_2\rangle$:P → B×C. Then p = Equ($\eta_1\pi_B$,$\eta_2\pi_C$) and is monic. Hence ph = ⟨f,g⟩ has image p. We now use the characterization of r∘s of Lemma 5. Since p = Equ($\eta_1\pi_B$,$\eta_2\pi_C$) we get that r∘s is as shown in the following pullback diagram.



Now, $r\vee s$ = Congr $\eta_1 f$ = Congr $\eta_2 r$ = $\mathrm{Equ}(\eta_1 f\pi_1, \eta_2 g\pi_2)$ = $\mathrm{Equ}(\eta_1\pi_B(f\times g), \eta_2\pi_C(f\times g))$. Hence $(f\times g)(r\vee s)$ factors through p giving $r\vee s \subset r\circ s$. The other inclusion is in Lemma 9. Similarly $r\vee s \cong s\circ r$.

Suppose now that $r\circ s \cong r\vee s$. Then we have two pullbacks with p as before.



Since f g is onto, so are v and v' and the uniqueness of the regular decomposition gives that $p \cong \mathrm{Im}\langle f,g\rangle$. Hence h is onto. ■

Note that, using the first half of the proof, we get

LEMMA 3. $r\circ s = r\vee s$ *implies* $r\circ s = s\circ r$.

§5. Commuting congruences. We first examine the situation of commuting congruences and prove for regular categories the well-known theorem of universal algebra that if two congruences in A commute then the quotient for one sends the other to a congruence. Then we prove our main theorems.

DEFINITION. Let r and s be congruences in A, then r and s *commute* if $r \circ s \cong s \circ r$. The congruences are said to *commute strongly* (or are strongly commuting) if $r \circ s = s \circ r = r \vee s$.

After the final lemma of last section the following conditions are equivalent for a pair of congruences r and s:

(i) r and s commute strongly.

(ii) $r \circ s = r \vee s$.

(iii) $r \circ s$ is a congruence.

If r and s commute then it is easily shown that $r \circ s$ is an equivalence relation. This means that in any regular category satisfying Lawvere's condition (for example in an algebraic variety) $r \circ s$ is a congruence and so it commutes strongly. One of the purposes of this paper is to show the converse: if commuting congruences commute strongly then any equivalence relation is a congruence. First we need a technical lemma.

LEMMA 14. *If* r *and* s *are congruences in* A *and* $\eta : A \to B$ *then* $(\eta\times\eta)_*(r \circ s) \subset (\eta\times\eta)_*(r) \circ (\eta\times\eta)_*(s)$.

<u>*Proof*</u>. Let $(\eta\times\eta)_*(r) = \bar{r} : \bar{R} \to B\times B$ and $(\eta\times\eta)_*(s) = \bar{s} : \bar{S} \to B\times B$ where $\bar{r}a = (\eta\times\eta)r$ and $\bar{s}b = (\eta\times\eta)s$. Put $\mu = \mathrm{Equ}(\pi_2 r \pi_R, \pi_1 s \pi_S)$ and $\nu = \mathrm{Equ}(\pi_2 \bar{r} \pi_{\bar{R}}, \pi_1 \bar{s} \pi_{\bar{S}})$. We have $\pi_2 \bar{r}(a\pi_R \mu) = \eta\pi_2 r \pi_R \mu = \eta\pi_1 s \pi_S \mu = \pi_1 \bar{s}(b\pi_S \mu)$ so $\langle a\pi_R \mu, b\pi_S \mu\rangle = \nu d$ for some d. Now $\bar{r} \circ \bar{s} = \mathrm{Im}\langle \pi_1 \bar{r} \pi_{\bar{R}} \nu, \pi_2 \bar{s} \pi_{\bar{S}} \nu\rangle$, while

$$(\eta\times\eta)_*(r\circ s) = \mathrm{Im}((\eta\times\eta)\mathrm{Im}\langle\pi_1 r\pi_R\mu,\pi_2 s\pi_S\mu\rangle)$$
$$= \mathrm{Im}((\eta\times\eta)\langle\pi_1 r\pi_R\mu,\pi_2 s\pi_S\mu\rangle.$$

Calculate

$$(\eta\times\eta)\langle\pi_1 r\pi_R\mu,\pi_2 s\pi_S\mu\rangle = \langle\pi_1\bar{r}a\pi_R\mu,\pi_2 s\bar{b}\pi_S\mu\rangle$$
$$= (\pi_1\bar{r}\times\pi_2\bar{s})\nu d = \langle\pi_1\bar{r}\pi_{\bar{R}}\nu,\pi_2\bar{s}\pi_{\bar{S}}\nu\rangle d.$$

This gives the result. ∎

PROPOSITION 15. *Let* r *be a congruence in* A *with quotient* η. *Then, if* s *is a congruence in* A *and* r *and* s *commute strongly then* $(\eta\times\eta)_*(s)$ *is a congruence.*

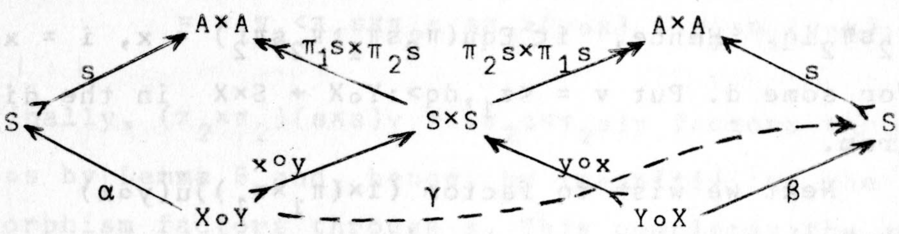
Proof. As above write $\bar{s}$ for $(\eta\times\eta)_*(s)$. Since $r\circ s$ = r s is a congruence containing r, by [4, p. 158] we have that $\overline{ros}$ is a congruence. Also if $\eta:A\to B$, $\bar{r} = \Delta_B$, but $s\subset ros$ so by Lemma 14 $\bar{s}\subset\overline{ros}$ $\subset \bar{r}o\bar{s} \cong \Delta_B o\bar{s} \cong \bar{s}$. Hence $\bar{s}\cong\overline{ros}$ is a congruence. ∎

We now come to a more intricate argument which is the key to the main result.

PROPOSITION 16. *Let* $s:S\to A\times A$ *be an equivalence relation and let* $x = \mathrm{Congr}\,\pi_2 s$, $y = \mathrm{Congr}\,\pi_1 s$. *Then* x *and* y *commute.*

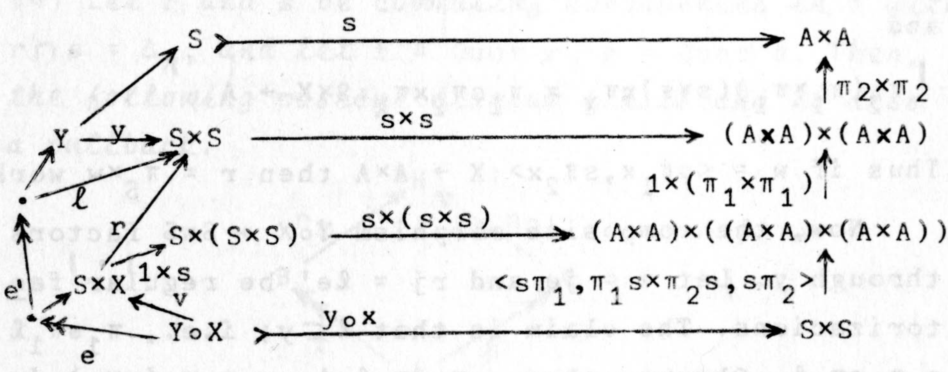
Proof. The proof which follows is modelled directly on that for sets except that the use of symmetry is concentrated at the beginning and transitivity at the end. We have that $xoy = (\pi_1 s\times\pi_2 s)^{-*}$ $(\mathrm{Im}\langle\pi_1 s,\pi_2 s\rangle)$ and $yox = (\pi_2 s\times\pi_1 s)^{-*}(\mathrm{Im}\langle\pi_2 s,\pi_1 s\rangle)$ by Lemma 5. Also $\mathrm{Im}\langle\pi_1 s,\pi_2 s\rangle = s$ while $\mathrm{Im}\langle\pi_2 s,\pi_1 s\rangle$

$= sk$ where k is the isomorphism giving symmetry. Since $s \cong sk$ we have the following diagram, a pair of pullbacks:



The aim is to construct a morphism $\gamma : X \circ Y \to S$ so that $(\pi_2 s \times \pi_1 s)(x \circ y) = s\gamma$. By the universal property of a pullback, $x \circ y$ will factor through $y \circ x$ giving one inclusion. The other inclusion will follow similarly.

<u>The construction of</u> γ. The general scheme of the construction is given in the diagram below. Note that the composition of the morphisms in the vertical column at the right is $\pi_2 s \times \pi_1 s$. The morphisms v and r must be found.



Put $u = \langle s\pi_1, \pi_1 s \times \pi_2 s, s\pi_2 \rangle$. Then $\pi_1 u(y \circ x) = s\pi_1(y \circ x)$ and $z_1 = \pi_1(y \circ x)$ is onto since $\Delta_S \subset y \circ x$. Next $\pi_2 u(y \circ x) = (\pi_1 s \times \pi_2 s)(y \circ x) = s\alpha$ and $\pi_3 u(y \circ x) =$

$s\pi_2(y \circ x)$. Define $z_2 : Y \circ X \to S \times S$ by $z_2 = \langle \alpha, \pi_2(y \circ x) \rangle$. It will be shown that $\mathrm{Im} z_2 \subset x$. Factor $z_2 = iq$, q onto, i monic. Now

$\pi_2 s \pi_1 iq = \pi_2 s\alpha = \pi_2(\pi_1 s \times \pi_2 s)(y \circ x) = \pi_2 s \pi_2 (y \circ x) = \pi_2 s \pi_2 iq$. Hence, $i \subset \mathrm{Equ}(\pi_2 s \pi_1, \pi_2 s \pi_2) = x$, $i = xd$ for some d. Put $v = \langle z_1, dq \rangle : Y \circ X \to S \times X$ in the diagram.

Next we wish to factor $(1 \times (\pi_1 \times \pi_1))u(y \circ x)$ through y.

The problem is to find $r : S \times X \to S \times S$ so that

$$(s \times s)r = [1 \times (\pi_1 \times \pi_2)](s \times (s \times s))(1 \times x)$$
$$= [1 \times (\pi_1 \times \pi_1)](s \times (s \times s)x).$$

Applying π_1 of $(A \times A) \times (A \times A)$ we get $s\pi_S$. Applying π_2 yields:

$$(\pi_1 \times \pi_1)\pi_{(A \times A)^2}[s \times (s \times s)x] = (\pi_1 \times \pi_1)(s \times s)x\pi_X : S \times X \to A \times A.$$

Now apply the projections of $A \times A$:

$$\pi_1(\pi_1 \times \pi_1)(s \times s)x\pi_X = \pi_1 s \pi_1 x \pi_X : S \times X \to A$$

and

$$\pi_2(\pi_1 \times \pi_1)(s \times s)x\pi_X = \pi_1 s \pi_2 x \pi_X : S \times X \to A.$$

Thus if $w = \langle s\pi_1 x, s\pi_2 x \rangle : X \to A \times A$ then $r = \pi_S \times w$ works.

Now, the composite morphism $Y \circ X \to S \times S$ factors through y. Let $v = je$ and $rj = \ell e'$ be regular factorizacions. The claim is that $\ell \subset y$; i.e., $\pi_1 s \pi_1 \ell = \pi_1 s \pi_2 \ell$. Observe that $\pi_1 s \pi_1 \ell e' e = \pi_1 \pi_1 (s \times s) rje = \pi_1 \pi_1 [1 \times (\pi_1 \times \pi_1)] u(y \circ x) = \pi_1 \pi_{A \times A} u(y \circ x) = \pi_1 s \pi_1 (y \circ x)$, while

$$\pi_1 s\pi_2 \ell e'e = \pi_1\pi_2(s\times s)rje = \pi_1\pi_2[1\times(\pi_1\times\pi_1)]u(y\circ x)$$

$$= \pi_1(\pi_1\times\pi_1)\pi_{(A\times A)^2}u(y\circ x) = \pi_1\pi_1\pi_{(A\times A)^2}u(y\circ x)$$

$$= \pi_1\pi_1<\pi_1 s\times\pi_2 s, s\pi_2>(y\circ x) = \pi_1 s\pi_1(y\circ x).$$

Finally, $(\pi_2\times\pi_2)(s\times s)y = (\pi_2 s\times\pi_2 s)y$ factors through $s\circ s$ by Lemma 8 and, hence, by transitivity, the morphism factors through s. This completes the construction and the proof. ∎

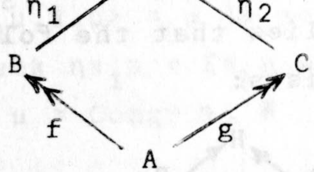
THEOREM 17. *The following are equivalent in a regular category.*

(1) *Every equivalence relation is a congruence.*

(2) *If* r *and* s *are congruences in* A *which commute then they strongly commute.*

(3) *If* r *and* s *are commuting congruences in* A *and* η = Quot r *then* $(\eta\times\eta)_*(s)$ *is a congruence.*

(4) *Let* r *and* s *be commuting congruences in* A *with* $r\cap s = \Delta_A$, *and let* f = Quot r, g = Quot s. *Then, the following pushout diagram exists and is also a pullback.*



Proof. The following implications will be proved: (1) ⇒ (2) ⇒ (3) ⇒ (1) and (2) ⇒ (4) ⇒ (1).

(1) ⇒ (2). Let r and s be commuting congruences in A. Since $\Delta_A \subset r \subset r \circ s$, $r \circ s$ is reflexive. To check the symmetry let k be the isomorphism such that $\tau r = rk$ and k' the isomorphism such that $\tau s = sk'$. Note that $\tau(r \circ s) \cong sk' \circ rk \cong s \circ r$.

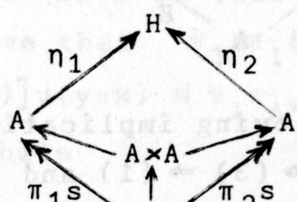
Now use the associativity of the composition of relations, $(r \circ s) \circ (s \circ r) \cong r \circ s$. Hence $s \circ r \cong r \circ s$ is an equivalence relation.

(2) ⇒ (3). This is Proposition 15.

(3) ⇒ (1). Let s be an equivalence relation in A with $x = \mathrm{Congr}\,\pi_1 s$, $y = \mathrm{Congr}\,\pi_2 s$. By Proposition 16, $x \circ y = y \circ x$ so $s \circ s = (\pi_2 s \times \pi_2 s)_*(x)$ is a congruence. But, $s \circ s \cong s$.

(2) ⇒ (4). Let r and s be commuting congruences then (2) implies $r \circ s = r \vee s$ and by Proposition 10 the pushout exists. By Proposition 12 the induced map $h: A \to P$ in the pullback of η_1 and η_2 is onto. But $\langle f,g \rangle$ factors through h and so $\mathrm{Congr}\, h \subseteq \mathrm{Congr}\, \langle f,g \rangle = (\mathrm{Congr}\, f) \cap (\mathrm{Congr}\, g) = \Delta_A$, implying that h is also monic and so an isomorphism.

(4) ⇒ (1). Let s be an equivalence relation. Note that $\Delta_A \subset s$, $\pi_1 s$ and $\pi_2 s$ are onto, $x = \mathrm{Congr}\,\pi_1 s$ and $y = \mathrm{Congr}\,\pi_2 s$ commute by Proposition 16, and $x \cap y = \mathrm{Congr} \langle \pi_1 s, \pi_2 s \rangle = \mathrm{Congr}\, s = \Delta_S$ because s is monic. Then (4) implies that the following pushout-pullback diagram exists:



Since $\Delta_A \subset s$ then $1_A = \pi_1 s\delta = \pi_2 s\delta$ for certain δ and so

$$\eta_1 = \eta_1\pi_1 s\delta = \eta_2\pi_2 s\delta = \eta_2.$$

Therefore, $s = \mathrm{Equ}(\eta\pi_1, \eta\pi_2) = \mathrm{Congr}\,\eta_1$. ■

COROLLARY 18. *The following are equivalent in a regular category.*

(1) *Every reflexive relation is a congruence.*

(2) *Any pair of congruences commute strongly.*

(3) *Any pair of onto morphisms* f, g *with common domain have a pushout. And any such pushout is a pullback whenever* <f,g> *is monic.*

(4) *Any pair of onto morphisms* f *and* g *with* <f,g> *monic is a pullback of some pair of morphisms.*

<u>*Proof*</u>. (1) ⇒ (2). Composite of congruences is reflexive.

(2) ⇒ (3). If for any pair of congruences we have $r\circ s = r\vee s$, then Proposition 10 applied to the congruences of f and g gives the pushout. Now, if <f,g> is monic, Theorem 17-(3) applies.

(3) ⇒ (4). Trivial.

(4) ⇒ (1). If μ is reflexive then $\pi_1\mu$ and $\pi_2\mu$ are onto, and $\langle\pi_1\mu, \pi_2\mu\rangle = \mu$ is monic, then we must have a pullback $\eta\pi_1\mu = \xi\pi_2\mu$ where $\eta = \xi$ because $\Delta_A \subset \mu$. Hence $\mu = \mathrm{Congr}\,\eta$. ■

Theorem 17 underlines how "algebraic" is the condition that equivalence relations be congruences. The categories of Corollary 18 would seem to be worth detailed study. Next corollary is clear from

Last one it also follows from Theorem 1, page 174, in [4].

COROLLARY 19. *Let C be a regular category satisfying Lawvere's condition. Then, in each object every pair of congruences commutes iff every reflexive relation is a congruence.*

* *

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