Revista Colombiana de Matemáticas
Vol. XV (1981) págs. 113 - 146

CONDITIONS FOR A REALIZATION FUNCTOR TO COMMUTE WITH FINITE PRODUCTS

by

Carlos RUIZ SALGUERO and
Roberto RUIZ S.

ABSTRACT. If Δ is the category whose objects are the sets $\{0,1,\ldots,n\}$ with weakly monotonic functions as morphisms, Δ^o is the opposite category, $\Delta^o S$ is the category of simplicial sets (covariant functors $\Delta^o \to$ *Sets*), and $Y:\Delta \to Top$ is the functor which sends $\{0,1,\ldots,n\}$ to the topological $\Delta(n)$ (standard n-simplex in $\mathbb{R}^{n+1}$), then it is well known the existence of a pair of adjoint functors: the *geometric realization* $R_Y:\Delta^o S \to Top$ and the *singular complex functor* $S_Y:Top \to \Delta^o S$, both built by systematic manipulation of the $\Delta(n)$'s. The purpose of this paper is: (1) to construct generalizations of R_Y and S_Y for any functor $Y:\delta \to A$, where A is arbitrary and A is a category having coproducts and pushouts, (2) to show that any pair of adjoint functors $R:\delta^o S \to A$ and $S:A \to \delta^o S$ comes from such constructions, and (3) as a result, to find conditions for R to commute with finite products.

RESUMEN. Sea Δ la categoría de conjuntos $[n] = \{0,1,2,\ldots,n\}$ con funciones monó

tonas como morfismos, Δ^oS la categoría de conjuntos simpliciales (funtores contravariantes $\Delta \to$ *Sets*) y $Y:\Delta \to$ *Top* el funtor que envía [n] a $\Delta(n)$, el n-simplejo canónico de $\mathbb{R}^{n+1}$, entonces es bien sabido como construír un par de funtores adjuntos asociados a Y: la *realización geométrica* $R_Y:\Delta^oS \to$ *Top* y el *funtor singular* S_Y:*Top* $\to \Delta^oS$. El objeto de este artículo es: (1) construír generalizaciones de R_Y y S_Y para cualquier funtor $Y:\delta \to A$ donde δ es una categoría arbitraria y A tiene coproductos y sumas amalgamadas ("pushouts"), (2) mostrar que cualquier par de funtores adjuntos $R:\delta^oS \to A$ y $S:A \to \delta^oS$ proviene de tal construcción para algún Y, y (3), usando lo anterior, hallar condiciones para que R conmute con productos finitos.

Introducción. In Algebraic Topology it is well known the existence of a pair of adjoint functors: the *geometric realization* and the *singular functor*, both built by systematic manipulations of the topological simplexes $\Delta(n)$.

In [4] it was shown that actually each functor $Y:\Delta \to$ *Top* induces adjoint functors S_Y:*Top* $\to \Delta^oS$ and $R_Y:\Delta^oS \to$ *Top* (Δ^oS = simplicial sets) and that if R_Y commutes with finite products, then many aspects of the clasical Algebraic Topology can be developed for the new "model" Y. A similar procedure was developed for the category of simplicial sets instead of *Top*.

More generally we are interested in the posibility of a theory parallel to the generalization in [4], developed when the model Y is taken from a category δ to a category A with products and pushhouts.

Examples of that kind of changes can be found in (a) the subdivision functor at the simplicial level due to KAN [9], (b) the realization used by SEGAL in [6] in which Δ has been simplified, (c) the n-skeletons of simplicial sets in which the category Δ has been "truncated", (d) Milnor's realization $||:\Delta^{o}S \to Top$ which does not commutes with finite products, something undesirable in many circunstances ([1] Chap.III,§3), and which has forced topologists to replace *Top* by categories with better properties (cf[5]), for example the category K of *Kelley* spaces, (e) significantly different of the category *Top* (but keeping Δ) are the category *Cat* of small categories and the category Δ^{o}-*Gr* of simplicial groups.

As for the contents of this article, suppose given a **functor** $Y:\delta \to A$. We call *singular functor associated to* Y to the functor $S_Y:A \to \delta^{o}S$ ($\delta^{o}S$ = category of contravariant functors $\delta \to Set$) defined by $S_Y(A) = A(Y(\ ),A)$, and by composition on the arrows. Under some conditions (for example when A is a co-complete category) S_Y admits a left adjoint that we denote by $R_Y:\delta^{o}S \to A$.

Models being crucial in our study, since it is possible to obtain information on R_Y and S_Y from the properties of Y, it is natural to ask whether or not a pair of adjoint functors (R,S), $R:\delta^{o} \to S$ and $S:A \to \delta^{o}S$, is defined by a model $Y:\delta \to A$. We answer this affirmatively: *every adjoint pair is defined by a model* (prop.1.10). Thus the functors mentioned above (except possibly Segal's [6]) and also the functor *Nerv*:*Cat* → $\Delta^o S$ are defined by models. For the last one it is its left adjoint which is considered as a realization.

In fact we show that the study of models $Y \in SA$ and the study of adjoint pairs $(R,S):\delta^o S \to A$ are equivalent from the categorical point of view (Theorem 1.6): *If* $PA(\delta^o S, A)$ *denotes the category of adjoint pairs* (R,S) *then the "restriction" functor* $(R,S) \rightsquigarrow R \circ \delta$ *from* $PA(\delta^o S, A)$ *into* δA *is an equivalence of categories* (here $\delta:\delta \to \delta^o S$ denotes the canonical inclusion functor).

Once this is done it is possible to provide information on the realization functor R_Y (and the singular functor S_Y) based upon the properties of model Y. We do it as far as commutativity of R_Y with products is concerned by showing that: $R:\delta^o S \to A$ commutes with finite products if it commutes with products of the form $\delta(x)\times\delta(y)$ and if the functor $(—)\times R(X)$ admits a right adjoint, for each X in $\delta^o S$.

For the case $A = \delta^o S$ it is known that if $E_x:\delta^o S \to S$ denotes the evaluation functor $F \rightsquigarrow F(x)$, R commutes with a certain type of limit if and only if $E_x \circ R$ does for each x in δ. In general, R_Y commutes with a certain type of limit if and only if $R_{E_x Y}$ does for each x in δ. That reduces the problem of commutativity of $R:\delta^o S \to \delta^o S$ with products to the more simple case of functors R_Z for $Z:\delta \to S$. We used in [7] the results obtained in [2] ir order to characteriza $R_Z:\delta^o S \to S$ as a *Milnor-type* realization through optimal pairs (x,y) with $x \in X(n)$, $y \in Z(n)$, where x is non degenerated and y is an interior point. That is possible if Z admits no co-simplicial points and its interior points are stable under codegeneracies. Then there are necessary and sufficient conditions on $Z:\Delta \to \mathcal{S}$ (resp. $Z:\Delta \to \Delta^{o}\mathcal{S}$) in order that R_Z commutes with finite products.

As an application we show that Kan's first subdivision does not commute with finite products.

TABLE OF CONTENTS.

§0. <u>Natural transformations between adjoint pairs</u>.

We recall in this paragraph the existence of a bijection $\lambda \leadsto \bar{\lambda}$ from *Trans* (F,G) onto *Trans* $(\bar{G},\bar{F})$, where *Trans* stands for natural transformations and $F,G:A \to B$ are left adjoints of $\bar{F},\bar{G}:B \to A$, respectively. For $\lambda : F \to G$ and B an objet of $\mathcal{B}$, the morphism $\bar{\lambda}_B : \bar{G}(B) \to \bar{F}(B)$ if defined with the help of the composition

$$0.1 \quad (\bar{G}(B), \bar{G}(B)) \simeq \mathcal{B}(G\bar{G}(B), B) \xrightarrow{\lambda^*_{\bar{G}(B)}} \mathcal{B}(F\bar{G}(B), B)$$
$$\simeq \mathcal{A}(\bar{G}(B), \bar{F}(B))$$

where the two isomorphisms are given by adjointness and $\lambda^*_{\bar{G}(B)}$ maps $u \rightsquigarrow u \circ \lambda_{\bar{G}(B)}$. The desired morphism $\bar{\lambda}_B$ is then the image of the identity of $\bar{G}(B)$ by the map 0.1. If we denote by $\varphi^F : id_{\mathcal{A}} \to \bar{F}F$ and $\psi^F : F\bar{F} \to id_{\mathcal{B}}$ the natural transformations defining adjointness, then the morphism $\bar{\lambda}_B$ is explicity given by

$$0.2 \quad \bar{\lambda}_B : \bar{G}(B) \xrightarrow[\varphi^F_{\bar{G}(B)}]{} \bar{F}F\bar{G}(B) \xrightarrow[\bar{F}(\lambda_{\bar{G}(B)})]{} \bar{F}G\bar{G}(B)$$
$$\xrightarrow[\bar{F}(\psi^G_B)]{} \bar{F}(B).$$

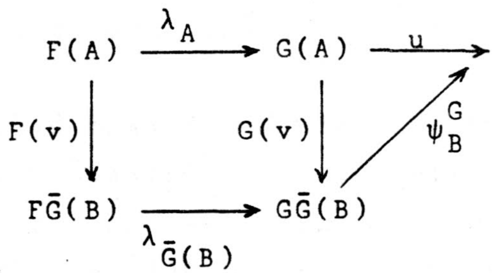
There is another way to define $\bar{\lambda}_B$, by its behavior with respect to each arrow $A \to \bar{G}(B)$ of the category $\mathcal{A}$. It is sumarized in the following theorem:

0.3. **THEOREM.** *The natural transformation* $\bar{\lambda} : \bar{G} \to \bar{F}$ *defined by* (0.2) *is the only one such that for each* $A \in \mathcal{A}$ *and each* $A \in \mathcal{B}$ *the following diagram commutes, where ad = isomorphism of adjointness:*

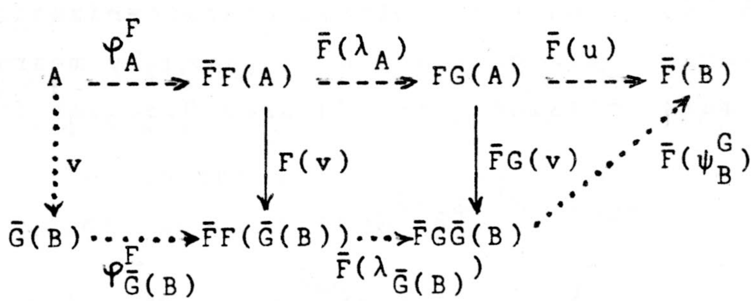
$$\begin{array}{ccc}
\mathcal{B}(G(A), B) & \xrightarrow[\sim]{ad^G} & \mathcal{A}(A, \bar{G}(B)) \\
{\scriptstyle (\ )\cdot\lambda_A} \downarrow & & \downarrow {\scriptstyle \bar{\lambda}_B \cdot (\ )} \\
\mathcal{B}(F(A), B) & \xrightarrow[ad^F]{\sim} & \mathcal{A}(A, \bar{F}(B))
\end{array}$$

i.e. for each $\mu : G(A) \to B$ *one has the equation* $ad^F(u_0\lambda_A) = \bar{\lambda}_B \circ ad^G(u)$.

Proof: D.M. Kan [10] proved the existence and uniqueness of a natural transformation $\bar{\lambda}$ which makes the above diagram commutative. What we want to show here is that with the definition we gave of $\bar{\lambda}$ the above diagram commutes. Let v denote ad(u), where $u : G(A) \to B$, and consider the following diagram



The square commutes since λ is a natural transformation. As for the triangle, $\psi^G_B \circ G(v) : G(A) \to B$ is the map which corresponds by adjointness to v. But ad(v) = u, since ad is an isomorphism, thus the triangle commutes. Next we map this diagram by $\bar{F}$ and extend the left part by the naturalness of $\varphi^F : id_A \to \bar{F}F$, to get the following commutative diagram



The composition of dotted arrows is $\bar{\lambda}_B \circ v = \bar{\lambda}_B \circ ad(u)$, while from the composition of the broken arrows one gets $\bar{F}(u\circ\lambda_A)\circ\varphi_A^F$ which equals $ad(u\circ\lambda_A)$. That ends the proof of 0.3. ■

0.4. **NOTE.** From the unicity of $\bar{\lambda}$ it follows that if $F \overset{\lambda}{\rightarrow} G \overset{u}{\rightarrow} H$ are natural transformations and if $\bar{H}$ is a right adjoint of H, then $\overline{\mu\circ\lambda} = \bar{\lambda}\circ\bar{\mu}$; and also if 1_G denotes the identical natural transformation $G \rightarrow G$ then $\bar{1}_G = 1_{\bar{G}}$. Consecuently, λ is an isomorphism if and only if $\bar{\lambda}$ is one too. In fact:

0.5. **THEOREM.** *The map* $\lambda \rightsquigarrow \bar{\lambda}$ *is a bijection between the sets Trans*(F,G) *and Trans*($\bar{G}$,$\bar{F}$).

Proof. The map is injective since of $\bar{\lambda} = \bar{\mu}$ then for each A in A, B in B and $u:G(A) \rightarrow B$ it holds that $\bar{\lambda}_B\circ ad(u) = \bar{\mu}_B\circ ad(u)$ and from Theorem 0.3 it follows that $ad(u\circ\lambda_A) = ad(u\circ\mu_A)$. Since ad is a bijection, then $u\circ\lambda_A = u\circ\mu_A$. Taking $B = G(A)$ and $u = id_{\bar{G}(A)}$ one gets that $\lambda_A = \mu_A$ for each A in A. That the map is onto can be proved using the opposite categories and applying the existence theorem for λ instead of $\bar{\lambda}$. ■

0.6. **NOTE.** Another helpful characterization of the relation between λ and $\bar{\lambda}$ is given by means of the formulae obtained from Theorem 0.3., as follows

$$\bar{\lambda}_B = ad^F(\psi_B^G \circ \lambda_{\bar{G}(B)})$$

$$ad^F(\lambda_A) = \bar{\lambda}_{G(A)}\circ\varphi_A^G \ .$$

It is also convenient to notice that the natural transformation $\bar{\lambda}:\bar{G} \to \bar{F}$ is the only one having the following property with respect to λ: in order for a diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\ \beta\ } & B \\ {\scriptstyle \lambda_A}\downarrow & \nearrow{\scriptstyle \alpha} & \\ G(A) & & \end{array}$$

to commute it is necessary and sufficient that the diagram

$$\begin{array}{ccc} & \xrightarrow{ad(\beta)} & \bar{F}(B) \\ A & & \uparrow{\scriptstyle \bar{\lambda}_B} \\ & \xrightarrow{ad(\alpha)} & \bar{G}(B) \end{array}$$

commutes. In fact using again the formula of Theorem 0.3, we get $\bar{\lambda}_B \circ ad(\alpha) = ad(\alpha \circ \lambda_A) = ad(\beta)$.

§1. <u>On the equivalence of the category</u> $\delta\mathcal{A}$, <u>of models over</u> $\mathcal{A}$, <u>and the category</u> $P\mathcal{A}(\delta^o\mathcal{S},\mathcal{A})$, <u>of adjoint pairs</u>. Let δ and $\mathcal{A}$ be two categories. We denote by $\mathcal{S}$ and $\delta^o\mathcal{S}$ the categories of sets and presheaves of δ (contravariant functors $\delta \to \mathcal{S}$ and natural transformations) respectively; δ is embedded in $\delta^o\mathcal{S}$ by means of a fully faithful functor $h:\delta \to \delta^o\mathcal{S}$ which to each object X assigns the representable functor $h(X) = \delta(-,X)$. We recall the lemma of Yoneda and Grothendieck.

1.1. LEMMA. *The set of natural transformations*

$\blacklozenge(X) \to F$, *where* $F \in \delta^{o}S$, *is in a one to one correspondence with the set* $F(X)$. *More precisely the map* $\lambda \rightsquigarrow \lambda_X(id_X)$ *defines a one to one and onto function:* $Trans(\blacklozenge(X),F) \stackrel{\to}{\leftarrow} F(X)$.

Taking $F = \blacklozenge(Y)$, the following chain of isomorphisms $\delta^{o}S(\blacklozenge(X),\blacklozenge(Y)) = Trans(\blacklozenge(X),\blacklozenge(Y)) \simeq \blacklozenge(Y)(X) = \delta(X,Y)$ shows that $\blacklozenge$ is fully-faithful.

Now we establish a useful notation for adjoint functors: $R:\mathcal{B} \to \mathcal{A}$ *is left adjoint to* $S:\mathcal{A} \to \mathcal{B}$ will be written (R,S) *is an adjoint pair* or more simply $(R,S):\mathcal{B} \to \mathcal{A}$ *is adjoint*. In what follows we will also need to mention explicitly the isomorphism of adjointness. It will be agreed that the statement (R,S,α) *is adjoint* means that $\alpha:\mathcal{A}(R(\ ),-) \to \mathcal{B}(-,S(\ ))$ is a natural isomorphism.

1.2. **DEFINITION.** Given two adjoint pairs $(R,S,\alpha):\mathcal{B} \to \mathcal{A}$ and $(R_1,S_1,\beta):\mathcal{C} \to \mathcal{B}$, we define the composition $(R,S,\alpha)\circ(R_1,S_1,\beta)$ as the adjoint pair (RR_1,S_1S,γ) where $\gamma:\mathcal{A}(RR_1(\ ),-) \to \mathcal{C}(-,S_1S(\ ))$ is given, for each X in $\mathcal{C}$ and each Y in $\mathcal{A}$, by $\gamma^{X,Y} = \beta^{X,SY}\circ\alpha^{R_1X,Y}$. We will write the transformation γ as $\beta\circ\alpha$.

1.3. **NOTE.** Given an adjoint pair $(R,S,\alpha):\mathcal{B} \to \mathcal{A}$ we will denote by $\psi^{\alpha}:RS \to 1_{\mathcal{A}}$ and $\varphi^{\alpha}:1_{\mathcal{B}} \to SR$ the natural transformations defined by (and defining) α. With this notation the formulae of Note 0.6., become, for each natural transformation $\lambda:R \to R'$ and X in $\mathcal{B}$ and Y in $\mathcal{A}$:

$$\bar{\lambda}_Y = \alpha(\psi_Y^{\alpha'} \circ \lambda_{S'(Y)})$$

and

$$\alpha(\lambda_X) = \bar{\lambda}_{R'(X)} \circ \varphi_X^{\alpha'} .$$

As a consequence of these formulae we are able to define natural transformations between adjoint pairs giving only λ or, equally well $\bar{\lambda}$.

1.4. DEFINITION. (a) By a natural transformation $\lambda:(R,S,\alpha) \to (R',S',\alpha')$ between adjoint pairs, we mean a natural transformation $\lambda:R \to R'$. We compose them in the obvious way. (b) $PA(B,A)$ will denote the category whose objets are adjoint pairs $(R,S,\alpha):B \to A$ and whose morphisms are natural transformations between them.

Now we will exhibit a natural functor $PA(\delta^o S,A) \overset{M}{\to} \delta$ and then we will face the questions of under which condition on the category A the functor M admits adjoint or better even is an equivalence of categories. The reason why we are interested in this question is that the objets of δA can be visualized as a generalization of the acyclic models of algebraic topology. In fact, some of these objets will be used in order to develope "homotopy theories" on A. It is the subject of "Fully Simplicial Categories" by the authors (to appear).

M is defined in the following way; if $(R,S,\alpha):\delta^o S \to A$ is an adjoint pair we take $M(R,S,\alpha) = R\circ\delta$. If $\lambda:(R,S,\alpha) \to (R_1,S_1,\alpha_1)$ is a natural transformation, then $M(\lambda): R\circ \mathfrak{s} \to R_1 \circ \mathfrak{s}$ is taken to be $\lambda_{\mathfrak{s}(X)}: R(\mathfrak{s}(X)) \to R_1(\mathfrak{s}(X))$ for each X in δ.

1.5. With no conditions on the category $\mathcal{A}$, neither on δ, there exists, associated with each $Y:\delta \to \mathcal{A}$ a "singular" functor $S_Y:\mathcal{A} \to \delta^o\mathcal{S}$, which under some conditions admits a left adjoint. S_Y is given by the formulae: (a) $S_Y(A):\delta^o \to \mathcal{S}$ is the functor $S_Y(A)(?) = \mathcal{A}(Y(?),A)$; (b) if $f:A \to A'$ is a morphism of $\mathcal{A}$, then $S_Y(f)$ is the natural transformation $\mathcal{A}(Y(?),A) \to \mathcal{A}(Y(?),A')$ induced by composition with f.

The procedure just described actually defines a contravariant functor

$$S_{(?)}:\delta\mathcal{A} \to \mathrm{Func}(\mathcal{A},\delta^o\mathcal{S}), \qquad Y \rightsquigarrow S_Y.$$

If we assume that for each S_Y there exist R_Y and α_Y such (R_Y,S_Y,α_Y) is an adjoint pair then one has defined a functor $\delta\mathcal{A} \xrightarrow{P} \mathcal{PA}(\delta^o\mathcal{S},A)$, $Y \rightsquigarrow (R_Y,S_Y,\alpha_Y)$... More precisely if in the category $\mathcal{A}$ every S_Y admits a left adjoint in the standard sense, for example when $\mathcal{A}$ is co-complete, then let $\theta:\mathrm{Ob}\delta\mathcal{A} \to \mathrm{Ob}\mathcal{PA}(\delta^o\mathcal{S},A)$ be a map such that $\theta(Y) = (R,S_Y,\alpha)$, which exists because of the axiom of choice. We will denote (R_Y,S_Y,α_Y) by $\theta(Y)$. The functor $P = P^\theta:\delta\mathcal{A} \to \mathcal{PA}(\delta^o\mathcal{S},A)$ is then defined in the following way: $P^\theta(Y) = \theta(Y)$. If $f:Y \to Z$ is a morphism in $\delta\mathcal{A}$ then $S_f:S_Z \to S_Y$, we take $P^\theta(f)$ to be the natural transformation $\lambda:\theta(Y) \to \theta(Z)$ such that $\bar{\lambda} = S_f$. Recall that there exist a formula which relates λ

and $\bar{\lambda}$ when the isomorphism of adjointness have been given, which is actually what θ is good for (see Note 1.3 and definition 1.4 (a)). That formula applied to our case is

$$\alpha_Y(\lambda_X) = \bar{\lambda}_{R_Z(X)} \circ \varphi_X^{\alpha_Z},$$

where X is an object of $\delta^o S$. When $\bar{\lambda}$ is S_f then $\lambda = P^\theta(f)$. From Note 0.4 it follows that for each θ, P^θ is in fact a covariant functor.

1.6. THEOREM. *There exists a natural isomorphism u between* $M \circ P^\theta$ *and the identity functor of* δA. *Also there exists a natural isomorphism v between* $P^\theta \circ M$ *and the identity functor of* $PA(\delta^o S, A)$. *Consequently when θ exists, the category* $PA(\delta^o S, A)$ *and* A *are equivalent.*

<u>*Proof*</u>. In order to define u we need to define a natural isomorphism $u = \{u_Y\}_{Y \ \delta A}$, where $u_Y : R_Y \circ \delta \to Y$. Recall that for u_Y to be an isomorphisms it is necessary and sufficient that for each x in δ, $u_{Y,x} : R_Y(\delta(x)) \to Y(x)$ is an isomorphism in A. We take $u_{Y,x}$ as the unique morphism in A whose image by the following chain of ismomorphisms is the identity of Y(x):

$$A(R_Y(\delta(x)), Y(x)) \xrightarrow{\alpha_Y} \delta^o S(\delta(x), S_Y(Y(x))) \xrightarrow{y.g.}$$
$$S_Y(Y(x))(x) \overset{def}{=\!=} A(Y(x), Y(x))$$

where α_Y is the isomorphism of adjointness of $\theta(Y) = (R_Y, S_Y, \alpha_Y)$ and y.g. stands for the isomorphism of Yoneda-Grothendieck of Lemma 1.1.

Equivalently $u_{Y,x}$ can be defined as the unique morphism such that for each A of $\mathcal{A}$, the map $k \rightsquigarrow k \circ u_{Y,x}$ is the isomorphism inverse to the composition β_Y (or $\beta_Y^{x,A}$):

$$\mathcal{A}(R_Y\boldsymbol{\delta}(x),A) \xrightarrow{\alpha_Y} \delta^{o}S(\boldsymbol{\delta}(x),S_Y(A)) \xrightarrow{y.g.}$$

$$S_Y(A)(x) \overset{def}{=\!=} \mathcal{A}(Y(x),A).$$

In what follows, if $f:Y \to Z$ is a morphism in $\delta\mathcal{A}$, we will use the notation $R_f:R_Y \to R_Z$ for the natural transformation on the realization which defines, and is defined by, $P^{\theta}(f)$. We need to prove that for such an $f:Y \to Z$ the following diagram commutes:

$$\begin{array}{ccc} R_Y \circ \boldsymbol{\delta} & \xrightarrow{\ u_Y\ } & Y \\ {\scriptstyle (R_f)_{\boldsymbol{\delta}}}\downarrow & & \downarrow{\scriptstyle f} \\ R_Z \circ \boldsymbol{\delta} & \xrightarrow[\ u_Z\ ]{} & Z \end{array}$$

this is equivalent to the commutativity, for each object x in δ and each A in $\mathcal{A}$, of the following diagram

$$\begin{array}{ccc} \mathcal{A}(R_Y\boldsymbol{\delta}(x),A) & \xleftarrow{\ \mathcal{A}(u_{Y,x},A) = \beta_Y^{-1}\ } & \mathcal{A}(Y(x),A) \\ {\scriptstyle \mathcal{A}(R_{f,\boldsymbol{\delta}(x)},A)}\uparrow & & \uparrow{\scriptstyle \mathcal{A}(f_x,A)} \\ \mathcal{A}(R_Z\boldsymbol{\delta}(x),A) & \xleftarrow[\ \mathcal{A}(u_{Z,x},A) = \beta_Z^{-1}\ ]{} & \mathcal{A}(Z(x),A) \end{array}$$

consequently, it is enough to prove the commutativity of

$$\begin{array}{ccccc}
 & & \xrightarrow{\qquad\qquad\qquad\beta_Y\qquad\qquad\qquad} & & \\
A(R_Y\delta(x),A) & \xrightarrow{\alpha_Y} & \delta^{\circ}S(\delta(x),S_Y(A)) & \xrightarrow{y.g.} & S_Y(A)(x)=A(Y(x),A) \\
\Big\uparrow A(R_{f,\delta(x)},A) & & \Big\uparrow \delta^{\circ}S(\delta(x),S_{f,A}) & & S_{f,A,x}\Big\uparrow = A(f_x,A)\Big\uparrow \\
A(R_Z\delta(x),A) & \xrightarrow[\alpha_Z]{} & \delta^{\circ}S(\delta(x),S_Z(A)) & \xrightarrow[y.g.]{} & S_Z(A)(x)=A(Z(x),A)
\end{array}$$

From left to right the squares commute, the first one by Theorem 0.3 and the second by naturality of y.g.

We now define the natural isomorphism $\nu:P^{\theta}\circ M\to 1$. Let $(R,S,\gamma):\delta^{\circ}S\to A$ be an adjoint pair. Then $M(R,S,\gamma) = R\circ\delta$, and therefore $P^{\theta}M(R,S,\gamma) = \theta(R\circ\delta)$, which we can write as $\theta(R\circ\delta) = (R_{R\delta},S_{R\delta},\alpha_{R\delta})$. We need to exhibit a natural transformation between $R_{S\delta}$ and R. But from Theorem 0.3, since γ and $\alpha_{R\delta}$ are given, it suffies to give a natural transformation $\bar{\nu}_{(R,S,\gamma)} = \bar{\nu}:S\to S_{R\delta}$. We take

$$\bar{\nu}_{A,x}:S(A)(x)\to S_{R\delta}(A)(x)$$

to be the map inverse of the composite isomorphism

$$\beta_{(\gamma)}:R_{R\delta}(A)(x)=A(R(\delta(x)),A)\xrightarrow{\gamma}\delta^{\circ}S(\delta(x),S(A))\xrightarrow{y.g.}S(A)(x)\,.$$

That $\bar{\nu}$ is natural follows from the commutativity of the diagram

1.7.
$$\begin{array}{ccc}
S_{R'\delta} & \xrightarrow{S_{\mu}} & S_{R\delta} \\
\bar{\nu}_{(R',S',\gamma')}\Big\uparrow & & \Big\uparrow\bar{\nu}_{(R,S,\gamma)} \\
S' & \xrightarrow[\bar{\lambda}]{} & S
\end{array}$$

where $\lambda : R \to R'$ is a natural transformation, $\bar{\lambda}$ was clarified in Note 1.3, $\mu : R \circ \delta \to R' \circ \delta$ is the restriction of λ to δ, and (R', S', γ'), (R, S, γ) are adjoint pairs.

This commutativity is equivalent to that of the following diagram for each $A \in \mathcal{A}$ and $x \in \delta$:

$$
\begin{array}{ccccc}
S_{R'\delta}(A)(x) & \xrightarrow{\quad S_\mu \quad} & S_{R\delta}(A)(x) & & \\
\text{def} \downarrow (=) & & (=) \downarrow \text{def} & & (1) \\
\mathcal{A}(R'\delta(x), A) & \xrightarrow[(?)\circ\lambda_{\delta(x)}]{(?)\circ\mu_x \;=} & \mathcal{A}(R\delta(x), A) & & \\
\gamma' \downarrow & & \downarrow \gamma & & (2) \\
\delta^{o}S(\delta(x), S'(A)) & \xrightarrow{\quad \bar{\lambda}_A \circ (?) \quad} & \delta^{o}S(\delta(x), S(A)) & & \\
\text{y.g.} \downarrow & & \downarrow \text{y.g.} & & (3) \\
S'(A)(x) & \xrightarrow[\quad \bar{\lambda}_{A,x} \quad]{} & S(A)(x) & &
\end{array}
$$

Square (1) commutes by definition of S_μ. Square (2) commutes because given γ, γ', and λ, then $\bar{\lambda}$ is, by Theorem 0.3, the unique natural transformation which leaves diagram of the kind (2) commutative. Finaly, square (3) commutes by definition of the isomorphism of Yoneda-Grothendieck. That ends the proof of Theorem 1.6. ∎

Let's assume that we are given a natural transformation $\lambda : R \to R'$.

We want to give explicitly the diagram of realizations corresponding to the diagram 1.7; if we denote by Y and Y' the restrictions $R \circ \delta$ and $R' \circ \delta$, by $\mu : Y \to Y'$ the restriction of λ, and $\upsilon = \upsilon_{(R,S,\gamma)}$, $\upsilon' = \upsilon_{(R',S',\gamma')}$, then the dual diagram of 1.7 is

1.8.
$$\begin{array}{ccc} R_{Y'} & \xleftarrow{R_\mu} & R_Y \\ \downarrow{\scriptstyle \upsilon'} & & \downarrow{\scriptstyle \upsilon} \\ R' & \xleftarrow[\lambda]{} & R \end{array}$$

Notice that since in this diagram υ and υ' are isomorphism (because $\bar{\upsilon}'$ and $\bar{\upsilon}$ are isomorphisms) then

1.9. COROLARY. (a) *In order for* R_μ *to be an isomorphism it is necessary and sufficient that* λ *be an isomorphism. Moreover if for an* X, λ_X *is an isomorphism then so is* $(R_\mu)_X$, *and conversely.*
(b) *Let* $Y, Y' \in \delta A$ *and* $f : Y \to Y'$ *be a natural transformation. In order for* f *to be an isomorphism it is necessary and sufficient that* R_f *(respectively* S_f*) be an isomorphism.*

EVERY ADJOINT PAIR IS DEFINED BY A MODEL. We formalize the idea of "modeling" an adjoint pair, as follows:

1.10. PROPOSITION. *For any adjoint pair* $(R,S) : \delta^o S \to A$ *there exist* $Y \in \delta A$ *and an isomorphism* $\Pi : S \xrightarrow{\sim} S_Y$. *Conversely, given* $Z \in \delta A$, *if* S_Z *admits a left adjoint* R *then there exists an isomorphism* $R \circ \delta \xrightarrow{\sim} Z$.

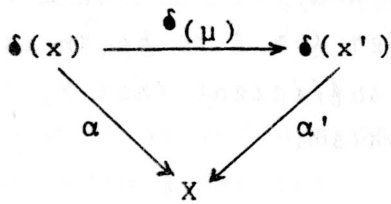
Proof. In the proof of Theorem 1.6, and independently of the existence of θ, the isomorphism

$R\circ\spadesuit \tilde{\to} Z$ is implicity given. ■

§2. <u>When</u> R, <u>from an adjoint pair</u> $(R,S):\delta^{o}S \to A$, <u>commutes with finite products</u>? Let us recall the following

2.1. **LEMMA.** *Every object of $\delta^{o}S$ is the inductive limit of representable objects, that is to say objets of the form $\spadesuit(x)$ with $x \in \delta$.*

To explain the meaning of this lemma, let $X \in \delta^{o}S$ and let δ/X denote the category whose objets are pairs (x,α) where $x \in \delta$ and $\alpha \in X(x)$. A morphism $\mu:(x,\alpha) \to (x',\alpha')$ is a morphism $\mu:x \to x'$ in δ such that $X(\mu)(\alpha') = \alpha$, in other words μ is such that the following diagram commutes:



(recall that $X(x) \simeq \delta^{o}S(\spadesuit(x),X)$, thus $\alpha \in X(x)$ represents a transformation $\spadesuit(x) \to X$ denoted also by α; this also justifies the notation δ/X for this category, known as the category of objets of $\spadesuit(\delta)$ over X).

Let $F = F_X$ denote the "source functor" $\delta/X \to \delta^{o}S$ given by the formulae $F(x,\alpha) = \spadesuit(x)$, and for $\mu:(x,\alpha) \to (x',\alpha')$, $F(\mu) = \spadesuit(\mu):\spadesuit(x) \to \spadesuit(x')$.

We define a natural transformation $\lambda:F_X \to c_X$ (where $c_X:\delta/X \to \delta^{o}S$ is the constant functor of value X) by $\lambda_{(x,\alpha)} = \alpha$, which induces a morphism $\lambda' : \varinjlim F_X \to X$ as a consequence of the natural bijection $\mathrm{Hom}(F_X, c_X) \simeq \mathrm{Hom}(\varinjlim F_X, X)$.

What lemma 2.1 *says is that* $\varinjlim F_X$ *exists in* $\delta^{\circ}S$ *and that* λ' *is an isomorphism, for every object* $X \in \delta^{\circ}S$. Moreover the isomorphism is natural. In fact if $f : X \to Y$ is a morphism of $\delta^{\circ}S$, it induces a functor $\tilde{f} : \delta/X \to \delta/X$, $(x,\alpha) \leadsto (x, f\circ\alpha)$, which itself induces a morphism $f' : \varinjlim F_X \to \varinjlim F_Y$ such that the following diagram commutes

$$\begin{array}{ccc} \varinjlim F_X & \xrightarrow{f'} & \varinjlim F_Y \\ \Big\downarrow {\scriptstyle \lambda'_X} & & \Big\downarrow {\scriptstyle \lambda'_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

The proof of Lemma 2.1 can be taken almost word by word from the one in [1] for the case $\delta = \Delta$.

2.2. PROPOSITION. *Let* $U, V : \delta^{\circ}S \to A$ *be functors which commute with inductive limits and let* $\gamma : U \to V$ *be a natural transformation. Then, in order for* γ *to be an isomorphism it is necessary and sufficient that for each object* $x \in \delta$ *the arrow* $\gamma_{\boldsymbol{\delta}(x)} : U\boldsymbol{\delta}(x) \to V\boldsymbol{\delta}(x)$ *be an isomorphism.*

<u>*Proof*</u>. It is sufficient to consider the diagram of the pag. 132, where $\gamma * F_X$ stands fot the Godement's composition of the natural transformations $\mathrm{id}F_X : F_X \to F_X$ and $\gamma : U \to V$. The top square commutes since it does so for each one of the structural arrows involved in $\varinjlim U\circ F_X$. The natural transfor

$$\begin{array}{ccc}
\varinjlim(U\circ F_X) & \xrightarrow{\varinjlim(\gamma^* F_X)} & \varinjlim(V\circ F_X) \\
\simeq \Big| & & \Big| \simeq \\
U(\varinjlim F_X) & \xrightarrow[\gamma_{\varinjlim F_X}]{} & V(\varinjlim F_X) \\
U(\lambda'_X) \simeq \Big\downarrow & & \Big\downarrow \simeq V(\lambda'_X) \\
U(X) & \xrightarrow[\gamma_X]{} & V(X)
\end{array}$$

mation $\gamma^* F_X$ is an isomorphis since it can be factored as

$$\begin{array}{ccccc}
 & \delta & & & \\
{}^{\text{source}}\nearrow & & \searrow^{\boldsymbol{\delta}} & & \\
\delta/X & \xrightarrow[F_X]{} & \delta^\circ S & \underset{V}{\overset{U}{\rightrightarrows}}\ {\scriptstyle\downarrow\gamma} & A
\end{array}$$

(here "source" $(x,\alpha) = x$). By hypothesis the restriction of γ to δ is an isomorphism and, *a fortiori*, when restricted to δ/X. That ends the proof of proposition 2.2.

2.3. THEOREM. *Let* A *be a category with finite products, and* $R:\delta^\circ S \to A$ *a functor which satisfies the following two conditions for a given* X:

(a) $R(?\times X)$ *and* $R(?)\times R(X)$ *commute with inductive limits.*

(b) *the natural transformation* $R(?\times X) \to R(?)\times R(X)$ *induced by the projections is an isomorphism when restricted to elements of the form* $\boldsymbol{\delta}(x)$ *with* $x\in\delta$. *Then, the arrow* $R(Y\times X) \to R(Y)\times R(X)$, *induced by the* *projections, is an isomorphism, for every* $Y \in \delta^{o}S$.

The proof is an inmediate consequence of the previous proposition.

2.4. **THEOREM.** *Let* A *be a category with finite products, and* $R:\delta^{o}S \to A$ *be a covariant functor which admits a rigth adjoint and such that for each* $X \in \delta^{o}S$ *the functor* $(-)\times R(X)$ *commutes with inductive limits. If for each* $x,y \in \delta$ *the canonical arrow* $R(\mathfrak{s}(x)\times\mathfrak{s}(y)) \to R\mathfrak{s}(x)\times R\mathfrak{s}(y)$ *is an isomorphism, then* R *commutes with finite products.*

<u>*Proof*</u>. The natural transformation $R(?\times\mathfrak{s}(y)) \to R(?)\times R(\mathfrak{s}(y))$ is an isomorphism. In fact it is a particular case of the previous theorem when $X = \mathfrak{s}(y)$. Since R admits right adjoint and by hypothesis $(-)\times R(X)$ commutes with inductive limits, the composite $R(?)\times R(X)$ also commutes with inductive limits. Since the functor $(-)\times V:\delta^{o}S \to \delta^{o}S$ admits a right adjoint then $R(-\times V):\delta^{o}S \to A$ also commutes with inductive limits. Thus for each $R \in \delta^{o}S$, and each $y \in \delta$ the canonical arrow $R(Y\times\mathfrak{s}(y)) \to R(Y)\times R(\mathfrak{s}(y))$ is an isomorphism. Considering Y fixed, it follows that $R(Y\times ?) \to R(Y)\times R(?)$ is a natural isomorphism on the representable objects $\mathfrak{s}(y)$. We apply theorem 2.3 to conclude that for each X, $R(Y\times X) \to R(Y)\times R(X)$ is an isomorphism. That ends the proof.

2.5. **COROLLARY.** *Let* $(R,S):\delta^{o}S \to A$ *be an adjoint pair. If for each* $X \in \delta^{o}S$, *the functor* $(-)\times R(X):A \to A$ *admits a right adjoint, then* R *commutes with finite products if and only if* R *commutes with finite prod* *ucts of representable objects, i.e.:*

$$R(\delta(x)\times\delta(y)) \rightleftarrows R(\delta(x))\times R(\delta(y)).$$

For example when A is a category of presheaves, then the functor $(-)\times Z$ $(Z \in A)$ admits a right adjoint, and the corolary applies. In particular in the categories of sets and simplicial sets. Also the corollary applies when A is the category of groups, the category of Kelley spaces, and the category *Cat* of small categories, see §4.4.

This last example permits us to generalize the proof in §2 of [3], to give the following.

2.6. PROPOSITION. *Let* (R,S) *as in corollary* 2.5. *be defined by a model* $Y:\delta \to A$ *such that* $S\circ Y \tilde{\to} \delta$ *and* $R\circ S \tilde{\to} id_A$. *Then* R *commutes with finite products.*

Proof. It follows from the natural isomorphisms $R(\delta(x)\times\delta(y)) \simeq R(S\cdot Y(x)\times S\cdot Y(y)) \simeq R(S(Y(x)\times Y(y))) \simeq RS(Y(x)\times Y(y)) \simeq Y(x)\times Y(y) \simeq R\delta(x)\times R\delta(y)$. ■

An example of this situation is the pair (G,N) where $N:Cat \to \Delta^o S$ is the nerve functor. These results on functors commuting with finite products generalize that of the realization functor $|\,|:\Delta^o S \to Kelly$ as presented in [1].

ANOTHER VERSION OF PROPOSITION. 2.2. Let $(R_i,S_i):\delta^o S \to A$ $(i = 1,2)$ be two adjoint pairs, and $\lambda:R_1 \to R_2$ a natural transformation. With the notation of §1, we establish the following workable version of proposition 2.2:

2.7. THEOREM. *The following statents are equivalent:*

(i) λ *is an isomorphism.*

(ii) $\bar{\lambda}$ *is an isomorphism.*

(iii) $S_{\lambda'}$ *is an isomorphism* ($\lambda' = \lambda * \delta$ = *restriction of* λ *to representable objets*).

(iv) $R_{\lambda'}$ *is an isomorphism.*

(v) λ' *is an isomorphism.*

Proof. (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv) are obvious; (ii) $\Leftrightarrow$ (iii) follows from 1.7; (i) $\Leftrightarrow$ (v) is obvious; (v) $\Leftrightarrow$ (iii) since $S_{\lambda'}$ is the image of λ' by the functor $S:\delta A \to \mathrm{Fun}(A,\delta^o S)$. ∎

2.8. NOTE. If for each $X \in \delta^o S$ the functor $(-)\times R(X):A \to A$ admits a right adjoint $(\ )^{R(X)}:A \to A$ then we can consider the following pairs of adjoint functors:

$$\delta^o S \xrightarrow{(-)\times X} \delta^o S \xrightarrow{R} A \xrightarrow{S} \delta^o S \xrightarrow{(\ )^X} \delta^o S$$

$$\delta^o S \xrightarrow{R} A \xrightarrow{(-)\times R(x)} A \xrightarrow{(\ )^{R(X)}} A \xrightarrow{S} \delta^o S .$$

In order to prove that the two functors $\delta^o S \to A$ given by the compositions of the left parts are isomorphic it is sufficient (and necessary) to prove that the functor $A \to \delta^o S$, given by composition on the right, are isomorphic. That happens, and can be seen easily, when $R(\delta(x)\times\delta(y)) \simeq R\delta(x)\times R\delta(y)$. However if what is wanted is to show that the canonical map $R(X\times Y) \to T(X)\times R(Y)$ associated with the projections is an isomorphism, then the machinery of this paragraph is needed.

§3. Decomposition of realizations in $\delta^o S$ in terms of realization in S.

Let's consider a realization functor $R_Y:\delta^o S \to \delta^o S$ associated with a model $Y:\delta \to \delta^o S$. Since realizations of this kind can be a little complex, we will show that they can be decomposed in realizations $\delta^o S \to S$ associated with models $\delta \to S$. It will follow that a realization of the first kind commutes with a limit if and only if each one of the realization of the second kind in its decomposition commutes with that limit. Since we have shown that the behaviour of the limits of a realization depends of the behaviour on models, the work of deciding whether or not R_Y commutes with products can be highly simplified. This is the method we will use to show that Kan's first sub-division does not commute with products.

3.1. **LEMMA.** *Let* $E:\delta^o S \to S$ *and* $Y:\delta \to \delta^o S$ *be functors. If* E *admits a right adjoint then there exists a natural isomorphism* $e = e_{E,Y}:S_Y \circ S_{E\delta} \simeq S_{EY}$. *Also* $E\circ R_Y \simeq R_{EY}$ *(Notations of* 1.5).

Proof. Let C be a set and $x \in \delta$. Then $(S_{EY}(C))(x) = S(EY(x),C)$, Since E has a right adjoint then by Theorem 1.6, $E \simeq R_{E\delta}$ and its right adjoint is $S_{E\delta}$. Then

$$(S_{EY}(C))(x) \simeq \delta^{o}S(Y(x), S_{E\bullet}(C)) = (S_Y(S_{E\bullet}(C)))(x).$$

By composition of adjoint pairs the last statement of the lemma follows. ∎

An important particular case of 3.1 is that of the "evaluation" functor $E_x:\delta^{o}S \to S$, where x runs over δ, given by $E_x(F) = F(x)$. On the arrows it is defined in the obvious way. These functors admit right (and left) adjoints, namely $S_{E_x\bullet}$. More explicitly this functor is given on each $C \in S$ and each $t \in \delta$ by $(S_{E_x\bullet}(C))(t) = S(\delta(x,t), C)$. The formula in 3.1 becomes $R_{E_xY} \simeq E_x \circ R_Y$ for each x in δ. Is is clear from this last formula that if R_Y commutes with a limit so does R_{E_xY}. Conversely and thanks to the properties induced in $\delta^{o}S$ by the evaluation functors we have:

3.2. **PROPOSITION.** *For each* $F:\delta \to \delta^{o}S$ *the following statements are equivalent*: (a) *the natural map* $\alpha:\varprojlim(R_Y \circ F) \to R_Y(\varprojlim F)$ *is an isomorphism*; (b) *for every* $x \in \delta$, *the natural map* $\varprojlim(R_{E_xY} \circ F) \to R_{E_xY}(\varprojlim F)$ *is an isomorphism.*

<u>*Proof*</u>. It is to be shown that if F stands for R_Y then the canonical morphism $F(\varprojlim U) \to \varprojlim FU$ is an ismorphism in $\delta^{o}S$. But that is so if and only if for every $x \in \delta$, the associated map $E_xF:\varprojlim U \to E_x \varprojlim FU$ is an isomorphis. But since E_xF (by hypothesis) and E_x commute with $\varprojlim$ then the conclusion follows. ∎

3.3. NOTE. The formula $R_{E_x Y} \simeq E_x \circ R_Y$ provides us (with the help of 1.6) of a mechanism which allows us to determine one realization, within the family of all the functors adjoint to S_Y, by fixing the realizations at the set theoretical level.

3.4. COROLLARY. $R_Y : \Delta^{\circ}S \to \Delta^{\circ}S$ *commutes with finite products if and only if, for every integers* $p,n,m \geqslant 0$, *the canonical map*

$$R_{E_p Y}(\Delta[n]\times\Delta[m]) \to Y(n)_p \times Y(m)_p$$

is a biyection.

§4. Conditions on models $Y:\Delta \to S$ for R_Y to commute with finite products. A point y of a model Y is *interior* if for each monomorphism ω of Δ, if y belongs to Im(Y(ω)) then ω is an identity. Recall the following propierties on α model Y, defined in [2].

MO.1. Y *does not have co-simplicial sub-sets with only one point in each dimension.*

MO.2. *If* y *is an interior point of* Y *then so is* Y(σ)(y) *for each epimorphism* σ *of* Δ *(whenever defined).* In other words, Y is stable for interior points by co-degenerancies.

In [2] we proved that MO.1 in equivalent to *every* y ∈ Y" *admits a unique decomposition* $y = Y(\partial)(y')$, *where* ∂ *is a monomorphism of* Δ *and* y' *is an interior point* . A co-simplicial set with this property was called an Eilenberg-Zilber type cosimplicial set.

Given a simplicial set X and a co-simplicial set Y we will say that a pair $(x,y) \in \coprod_n X(n) \times Y(n)$ is *optimal* if x is non-degenerate and y is an interior point.

Conditions MO.1 and MO.2 on a model $Y:\Delta \to S$ allow us to determine optimal representative on the equivalence classes of Milnor's relation [7]. With their help we now specify the condition on Y in order that R_Y commutes with finite products.

We know that R_Y commutes with finite products if and only if for each $n,m \geq 0$ the canonical map $R_Y(\Delta[n]\times\Delta[m]) \to R_Y(\Delta[n])\times R_Y(\Delta[m])$ is bijective. Since furthermore $R_Y(\Delta[n])$ is canonically isomorphic to Y(n) the composite

$\varphi = \varphi^{n,m}: R_Y(\Delta[n]\times\Delta[m]) \to Y(n)\times Y(m)$ is the one induced to the quotient by the maps

$(\alpha,\beta,Z) \rightsquigarrow (Y(\alpha)(Z),Y(\beta)(Z))$ where $\alpha:[r] \to [n]$, $\beta:[r] \to [m]$, $Z \in Y(r)$ (for $r \geq 0$).

The properties MO.1 and MO.2 on the model Y imply that in each equivalence class $[(\alpha,\beta,z)]$ there exists one and only one optimal representative. Therefore φ is inyective if and only if for any pair of optimal representative (α,β,z) and (α',β',z') if $\varphi(\alpha,\beta,Z) = \varphi(\alpha',\beta',Z')$ then $\alpha = \alpha'$, $\beta = \beta'$ and $Z = Z'$.

This fact can be translated in terms of the model Y in the following way: first recall that a pair $[m] \overset{\beta}{\leftarrow} [r] \overset{\alpha}{\rightarrow} [n]$ is by definition *non degenerate* if whenever there exist an epimorphism $\sigma:[r] \to [p]$ and arrows $\alpha':[p] \to [n]$, $\beta':[p] \to [m]$

such that $\alpha'\sigma = \alpha$ and $\beta'\sigma = \beta$, *then* σ = identity. We then define condition MO.3 for the model Y as follows:

MO.3. *For every pair of non degenerated diagram* $[m] \overset{\beta}{\leftarrow} [r] \overset{\alpha}{\rightarrow} [n]$ *and* $[m] \overset{\beta'}{\leftarrow} [r'] \overset{\alpha'}{\rightarrow} [n]$, *if there exist* $Z \in Y(r)$ *and* $Z' \in Y(r')$ *such that* $Y(\alpha)(Z) = Y(\alpha')(Z')$ *and* $Y(\beta)(Z) = Y(\beta')(Z')$ *then* $\alpha = \alpha'$, $\beta = \beta'$ *and* $Z = Z'$.

Equivalent let $Y(\alpha,\beta): Y(r) \rightarrow Y(n) \times Y(m)$ denote the map induced by a nondegenerated diagram $[m] \overset{\beta}{\leftarrow} [r] \overset{\alpha}{\rightarrow} [n]$, then Y satisfies MO.3 when the images by the $Y(\alpha,\beta)$'s of the set of interior points are pairwise disjoint. With this definition we have:

4.1. **PROPOSITION.** *Suppose that* Y *satisfies* MO.1 *and* MO.2. *In order that* $\varphi^{m,n}$ *be injective for every* n *and* m *it is necessary and sufficient that* Y *satisfies* MO.3.

The necessary and sufficient condition on Y for $\varphi^{n,m}$ to be onto is also evident:

MO.4. *For each* n *and* m, *and* $(u,v) \in Y(n) \times Y(m)$ *there exist* $p \geq 0$, $\omega \in Y(p)$ *and a diagram* $[n] \overset{\alpha}{\leftarrow} [p] \overset{\beta}{\rightarrow} [m]$ *such that* $Y(\alpha)(\omega) = u$ *and* $Y(\beta)(\omega) = v$.

4.2. **THEOREM.** *If a model* $Y: \Delta \rightarrow S$ *satisfies* MO.1, *and* MO.2, *then in order that* $R_Y: \Delta^o S \rightarrow S$ *commutes with finite products it is necessary and* *sufficient that* Y *satisfies* MO.3 *and* MO.4.

4.3. COROLLARY. *Let* $Y:\Delta \to \Delta^{o}S$ *be a covariant functor, if for each* n, $E_n \circ Y$ *satisfies* MO.1 *and* MO.2 *then a necessary and sufficient condition in order that* $R_Y:\Delta^{o}S \to \Delta^{o}S$ *commutes with finite products is that for each* n, $E_n \circ Y$ *has properties* MO.3 *and* MO.4.

§5. Kan's first sub-division does not commute with products. Other examples. We first give an alternative definition of Kan's first sub-division $Sd:\Delta^{o}S \to \Delta^{o}S$ as follows. Since Sd is a realization functor we first give the model which defines it and denote it by $\Delta':\Delta \to \Delta^{o}S$; Δ' associates to each $[n]$ the nerve $NP_o([n])$ of $P_o([n])$, the category associated with the ordered set of non-empty subsets of $[n]$ (Cf. [1]). More explicity.

$$\Delta'[n]_p = \{(A_0,\dots,A_p) \mid \phi \neq A_0 \subseteq A_1,\dots \subseteq A_2 \subseteq [n]\}$$

and if $\omega:[p] \to [q]$ then $\Delta'[n]_\omega = \omega^*:\Delta'[n]_q \to \Delta'[n]_p$ is given by $\omega^*(A_0,\dots,A_q) = (B_0,\dots,B_p)$ where $B_i = A_{\omega(i)}$. The definition of Δ' is completed associating to $\delta:[n] \to [m]$ the arrow $\delta_*:\Delta'[n] \to \Delta'[m]$ which in each p is given by $\delta_*(A_0,\dots,A_p) = (\delta(A_0),\dots,\delta(A_p))$.

Notice that $Sd \simeq R_{\Delta'}$. On the other hand Δ' satisfies in each simplicial degree MO.1 and MO.2 (Cf [2]). Thus is order to show that $R_{\Delta'}$ does not commute with finite products we need to show that

$E_p \circ \Delta'$ fails for some p to have either MO.3 or MO.4. We will show that actually $E_p \circ \Delta'$ fails to have MO.3 for each p. We cannot assure anything about MO.4 but we have the impression that it holds.

5.1. PROPOSITION. *For each* $p \geq 0$, *the model* $\Delta'_p = E_p \circ \Delta'$ *does not satisfy* MO.3.

Proof. (a) A point in dimension 2 of $Y = \Delta'_0$ is a non empty sub-set A_0 of $[2]$. It is interior if it is $[2]$ itself. We consider the diagrams $[1] \overset{\alpha}{\leftarrow} [2] \overset{\beta}{\rightarrow} [1]$, $[1] \overset{\alpha'}{\leftarrow} [2] \overset{\beta'}{\rightarrow} [1]$ where $\alpha = \sigma^o = \beta'$, $\beta = \sigma^o = \alpha'$. Notice that $(\alpha,\beta) \neq (\alpha',\beta')$, both are non degenerate, and $([2])$ is an interior point in dimension 2 of $\Delta'_0 = Y$. Now we take, with the notations of MO.3, $Z = Z'$, $([2]) \in Y(2)$ (that is to say $r = r' = 2$). It is clear that $Y(\alpha)(Z) = ([1]) = Y(\alpha')(Z')$ and $Y(\beta)(Z) = ([1]) = Y(\beta')(Z')$. However $\alpha \neq \alpha'$ and $\beta \neq \beta'$.

(b) More generally we take for $Y = \Delta'_p$, $m=n > p+1$ and consider the diagram $[n] \overset{\alpha}{\leftarrow} [n+1] \overset{\beta}{\rightarrow} [n]$ and $[n] \overset{\alpha'}{\leftarrow} [n+1] \overset{\beta'}{\rightarrow} [n]$ where $\alpha = \sigma^n = \beta'$ and $\beta = \sigma^{n-1} = \alpha'$. Next we consider $A_0 = \{0\}$, $A_1 = \{0,1\}, \ldots,$ $A_{p-1} = \{(0,1,\ldots,p-1\}$, $A_p = [n+1]$ and $Z = Z' = (A_0,\ldots,A_p) \in Y(n+1)$. It is clear that $Y(\alpha)(Z) = Y(\alpha')(Z')$ and $Y(\beta)(Z) = Y(\beta')(Z')$. However $\alpha \neq \alpha'$.

5.2. OTHER EXAMPLES. When $Y:\Delta \to \text{Top}$, the set theoretical part can be faced at the light of §4 because the underlying set of $R_Y(K)$ is precisely $R_{\underline{Y}}(K)$ where $\underline{Y}$ is the underlying cosimplicial set of the cosimplicial space Y. That is the case when Y is the cosimplicial space of the topological Δ^n's. It is well known that the map

$$|\Delta[n]\times\Delta[m]| \to \Delta^n\times\Delta^m$$
$$[\alpha,\beta,t] \rightsquigarrow (\alpha^*(t),\beta^*(t))$$

is a set theoretical isomorfism. In the proof of this is implicit that of MO.3 and MO.4, it can be found in [8] and [12].

This generalizes to any case in which the *underlying set functor* has a left adjoint. For example when R is the left adjoint of *Nerv*:*Cat* $\to \Delta^o S$, whose model is the cosimplicial category of the categories associated to finite ordered sets (the category associated to $[n] = \{0,1,...,n\}$ is Y^n with set of objects $[n]$ and a morphism $p \to q$ whenever $p \leq q$). In this case Y^0 has a point and Y^1 has two, both of which belong to Y^1, therefore MO.1 holds. On the other hand since the only interior point is $0 \in Y^0$, then MO.2 and MO.3 are obvious, so is MO.4. Therefore, at the set theoretical level, R_Y (the left adjoint of *Nerv*) commutes with finite products. The categorical part can be completed easily from the isomorphism

$$R_Y(\Delta[m]\times\Delta[n]) \cong Y^m\times Y^n$$

which is obvious, using 2.5.

When in a model $Y:\Delta \to A$, Y^0 is not the final object of A, then, except for trivial situations, R_Y does not commute with finite products. This is so because

$$R_Y(\Delta[n]\times\Delta[0]) \cong R_Y(\Delta[m]) \cong Y^n$$

while

$$R_Y(\Delta[n])\times R_Y(\Delta[0]) \simeq Y^n\times Y^0 .$$

This is the case with the realization of models of form RC(Y) (the right cut of Y) given by $(RC(Y))^n = Y^{n+1}$, with the same faces and degeneracies as Y, except for the last face and degeneracy of each level which are omitted.

Back to the case of the topological Δ^n's, since $RC(Y)^0 = Y^1 = \Delta^1$ then R_Z (with Z = RC(Y)) does not commute with finite products. It is clear however that it has a co-simplicial point namely $n \rightsquigarrow p^n$, $p^n = \{(0,0,\ldots,1)\} \subset R^{n+2}$.

If we consider Z-P (it happens to be a cosimplicial space again) it has no cosimplicial points. Let $(Z-P)^0 = \Delta^1 - P^0$. Then R_{Z-P} does not commutes with finite products. Here MO.2 and MO.3 also hold but MO.4 fails. As shown in [11], its realization functor (resp. singular functor), however, is homotopically equivalent to the Geometric Realization (resp. usual Singular Functor). It is shown there also the fact that Z is, as far as it concerns to realization, homotopically null.

**

BIBLIOGRAFIA

[1] P.Gabriel and M.Zisman: *Calculus of Fractions and Homotopy Theory*, Springer-Verlag, 1967.

[2] C.Ruiz S. and R.Ruiz: *Remarks about the Eilenberg-Zilber type decomposition in co-simplicial sets*, Rev.Col.Mat. XII, 3-4 (1978) 61-82.

[3] C.Ruiz S., R.Ruiz and S.Frías: *On Kan's Condition*, Rev.Col.de Mat., VI (1972) 1-16.

[4] R.Ruiz: Change of models in Top and $\Delta^{o}S$. Doctoral Thesis. Temple Univ., 1978.

[5] N.E.Steenrod: *A convenient category of topological spaces*, Michigan Math. J. 14 (1967) 133-152.

[6] G.Segal: *Categories and cohomology theories*. Preprint.

[7] C.Ruiz S. and R.Ruiz: *Characterization of the set theoretical realization in the non Euclidean case*, Proceedings of the Am. Math.Soc. (to appear).

[8] J.Milnor: *The geometric realization of a semi-simplicial complex*, Annals of Math. 65 (1957).

[9] D.M.Kan: *On C.C.S. complexes*, Am.Jour of Math. 79 (1957) 449-476.

[10] D.M.Kan: *Adjoint Functors*, Trans. of the Am-Math. Soc. 87 (1958) 357-362.

[11] R.Ruiz: *Subdivisiones relativas sobre pre-haces y homologías simpliciales isomorfas*, Rev. Col.de Mat., XIV (1980) 17-42.

[12] May Peter: *Simplicial objects in algebraic Topology*, Van Nostrand.

[13] D.Quillen: *Homotopical Algebra*, Lecture Notes 43 Springer Verlag, 1967.

* *

Departamento de Matemáticas

Universidad Nacional de Colombia

Apartado Aéreo 2509

Bogotá, D.E. COLOMBIA

Departamento de Matemáticas
Universidad del Valle
Apartado aéreo 2188
C A L I, COLOMBIA

(Recibido en diciembre de 1979, la versión revisada en septiembre de 1981).

* * *