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ON THE SET OF OPERATORS HAVING A GIVEN SIMPLY CONNECTED SPECTRAL SET

by

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ABSTRACT. We study some properties of the set of operators having a given compact simply connected set as a spectral set. Such set of operators is arcwise connected, and closed in the uniform (norm) topology if the spectral set is convex. Also we study the "tensor product" of such two sets of operators.

RESUMEN. Se estudian algunas propiedades de la familia de operadores que poseen un conjunto dado, compacto y simplemente conexo, como uno de sus conjuntos espectrales. Estos operadores forman un conjunto conexo por arcos el cual es cerrado en la topología de la norma si el conjunto espectral es conexo. También se estudia un "productos tensorial" entre tales familias de operadores.

Consider an infinite dimensional complex Hilbert space H, and let L(H) denote the algebra of all operators on H. If $A \in L(H)$, then the *spectrum* $\sigma(A)$ of A is defined to be the set of all complex numbers λ for which $A-\lambda$ is not invertible. A subset X of the complex plane $\mathbb{C}$ is said to be a *spectral set* of an operator A, if $\sigma(A) \subset X$, and for every analytic function φ on X, we have

$$\|\varphi(A)\| \leqslant \|\varphi\|_{\infty} .$$

We denote by $O(X)$ the set of all operators in $L(H)$ having X as a spectral set. In this paper we study some properties of $O(X)$.

LEMMA 1. *If* $X = \bar{D}$, *the closed unit disc, then* $O(\bar{D})$ *is a convex subset of* $L(H)$.

Proof: By von Neumann's theorem [6], $\bar{D}$ is a spectral set of an operator $A \in L(H)$ if, and only if $\|A\| \leqslant 1$, and thus $O(\bar{D})$ is the set of all contractions on H. Let $A, B \in O(\bar{D})$ then $\|A\| \leqslant 1$, $\|B\| \leqslant 1$, and for $t \in [0,1]$,

$$\begin{aligned}\|tA+(1-t)B\| &\leqslant t\|A\| + (1-t)\|B\| \\ &\leqslant t+(1-t) = 1.\end{aligned}$$

This implies that $tA+(1-t)B \in O(\bar{D})$, for every $t \in [0,1]$, thus $O(\bar{D})$ is a convex subset of $L(H)$. ∎

COROLLARY 1. $O(\bar{D})$ *is arcwise connected.*

THEOREM 1. *If* X *is a compact simply connected subset of* $\mathbb{C}$, *then* $O(X)$ *is arcwise connected.*

Proof. Let $T, S \in O(X)$, by von Neumann's theorem [6] there are operators $A, B \in O(\bar{D})$, such that

$T = \varphi(A)$ and $S = \varphi(B)$, for some analytic homeomorphism $\varphi:\bar{D} \to X$, which exists by the Riemann mapping theorem (see [1] page 221). Since, by Corollary 1, $O(\bar{D})$ is arcwise connected, there is $\theta:[0,1] \to O(\bar{D})$, a continuous function such that $\theta(0) = A$ and $\theta(1) = B$. Now consider $\varphi\circ\theta:[0,1] \to O(X)$, since and θ are continuos, $\varphi\circ\theta$ is continuos and $\varphi\circ\theta(0) = \varphi(A) = T$, $\varphi\circ\theta(1) = \varphi(B) = S$. This implies that $O(X)$ is arcwise connected. ∎

For $T \in L(H)$, the *numerical range* $W(T)$ of T is defined by

$$W(T) = \{\lambda \in \mathbb{C} : \lambda = (Te,e),\ \text{for } e \in H,\ \|e\| = 1\}.$$

To prove $O(X)$ is closed in the norm topology for a convex set X, we need the following two lemmas.

LEMMA 2. *If* $(T_n) \subset L(H)$ *converges weakly to* $T \in L(H)$, *and if* Y *is a compact subset of* $\mathbb{C}$ *such that* $\overline{W(T_n)} \subset Y$ *for every* n, *then* $\overline{W(T)} \subset Y$.

Proof: If $\lambda \in \overline{W(T)}$, then for every $\varepsilon > 0$ there is a unit vector $e \in H$ such that $|((T-\lambda)e,e)| < \varepsilon$, which implies

$$|((T_n-\lambda)e,e)| \leqslant |((T_n-T)e,e)| + |((T-\lambda)e,e)| < 2\varepsilon .$$

Thus we have $d(\lambda,\overline{W(T_n)}) < 2\varepsilon$ for every $n > N$, so that $d(\lambda,Y) < 2\varepsilon$, where d is the distance function. Since Y is compact and ε is arbitrary, $\lambda \in Y$, or $\overline{W(T)} \subset Y$. ∎

LEMMA 3. *If* $(T_n) \subset L(H)$ *converges (in norm) to* $T \subset L(H)$, *then* $\varphi(T_n)$ *converges (in norm) to* $\varphi(T)$, *for any analytic function* φ.

Proof: First we prove that T_n^k converges to T^k for every positive integer k. For k = 2 we have

$$\|T_n^2-T^2\| \leq \|T_n-T\|\|T_n+T\|$$

$$\leq \frac{\varepsilon}{2M}2M = \varepsilon ,$$

where $M = \max\{\|T_n+T\|:n = 1,2,\ldots,\}$. Assuming the induction step for k we obtain, for k+1,

$$\|T_n^{k+1}-T^{k+1}\| \leq \|T_n^{k+1}-T_n^kT\| + \|T_n^kT-T^{k+1}\|$$

$$\leq \|T_n^k\|\|T_n-T\| + \|T\|\|T_n^k-T^k\| .$$

Let $M = \max\{\|T_n^k\|, \|T\|:n = 1,2,\ldots\}$, then, we obtain

$$\|T_n^{k+1}-T^{k+1}\| \leq M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} = \varepsilon .$$

This implies that T_n^k converges to T^k, for every positive integer k, and thus for any polynomial P_m we have that $P_m(T_n)$ converges to $P_m(T)$, and taking the limit on m we obtain for any analytic function φ that $\varphi(T_n)$ converges to $\varphi(T)$. ∎

Now, we prove the following:

THEOREM 2. *If* X *is a compact convex subset of* $\mathbb{C}$, *then* O(X) *is uniformly closed*, i.e. *closed in the norm operator topology.*

Proof: Let $(T_n) \subset O(X)$ such that (T_n) converges in norm to $T \in L(H)$. Since X is a convex spectral set of T_n for every n, we have by Williams theorem which says that $\overline{W(A)}$ is the intersection of all convex spectral sets of A, see Williams [9]:

$$\overline{W(T_n)} \subset X, \quad n = 1,2,\ldots .$$

Lemma 2, implies that $\overline{W(T)} \subset X$. In [4] it is shown that $\overline{W(T)}$ contains $\sigma(T)$ for any operator $T \in L(H)$, thus $\sigma(T) \subset X$. To end the proof of the theorem we have to show that

$$\|\varphi(T)\| \leqslant \|\varphi\|_\infty$$

for any function φ, analytic on X. By Lemma 3, $\varphi(T_n)$ converges (in norm) to $\varphi(T)$, so that

$$\|\varphi(T)\| \leqslant \|\varphi(T) - (T_n)\| + \|\varphi(T_n)\|.$$

Since X is a spectral set of T_n, $n = 1,2,\ldots$, we have $\|\varphi(T_n)\| \leqslant \|\varphi\|_\infty$, $n = 1,2,\ldots$. This implies that

$$\|\varphi(T)\| \leqslant \varepsilon + \|\varphi\|_\infty ,$$

and since ε is arbitrary, the theorem is proved. ■

Let X and Y be subsets of $\mathbb{C}$, $O(X) \subset L(H_1)$ and $O(Y) \subset L(H_2)$, where H_1 and H_2 are two Hilbert spaces. If $H_1 \otimes H_2$ denotes the tensor product of H_1 and H_2 then we define the "*tensor product*" $O(X) \otimes O(Y)$ to be

$O(X) \otimes O(Y) = \{T \otimes S \in H_1 \otimes H_2$ such that $T \in O(X)$ and $S \in O(Y)\}$.

If $XY = \{xy : x \in X,\ y \in Y\}$, then one can ask which is the relation between $O(X) \otimes O(Y)$ and $O(XY)$. For this we need a proposition and two lemmas. The proposition comes from [7].

PROPOSITION. *If* X *is a compact simply connected spectral set of an operator* $A \in L(H)$, *then there is a normal operator* $N \in L(K)$ *such that* $\sigma(N) \subset \partial X$ (*the boundary of* X) *and for any analytic function on* X, *we have*

$$P_H \varphi(N) P_H = \varphi(A) P_H , \qquad (*)$$

where P_H *is the projection of the Hilbert space* K *onto* H $(H \subset K)$. *Any operator satisfying* (*) *is said to be a normal dilation of* A.

LEMMA 4. *If* N_1 *and* N_2 *are normal operators, then the tensor product* $N_1 \otimes N_2$ *is normal.*

<u>*Proof*</u>: consider,

$$\begin{aligned}
(N_1 \otimes N_2)^*(N_1 \otimes N_2) - (N_1 \otimes N_2)(N_1 \otimes N_2)^* \\
&= (N_1^* \otimes N_2^*)(N_1 \otimes N_2) - (N_1 \otimes N_2)(N_1^* \otimes N_2^*) \\
&= (N_1^* N_1) \otimes (N_2^* N_2) - (N_1 N_1^*) \otimes (N_2^* N_2) \\
&\quad + (N_1 N_1^*) \otimes (N_2^* N_2) - (N_1 N_1^*) \otimes (N_2 N_2^*).
\end{aligned}$$

Thus we have,

$$\begin{aligned}
(N_1 \otimes N_2)^*(N_1 \otimes N_2) - (N_1 \otimes N_2)(N_1 \otimes N_2)^* \\
&= (N_1^* N_1 - N_1 N_1^*) \otimes N_2^* N_2 + N_1 N_1^* \otimes (N_2^* N_2 - N_2 N_2^*) ,
\end{aligned}$$

this equality proves the lemma. ∎

LEMMA 5. *If* N_1 *and* N_2 *are normal dilations of* T *and* S *respectively, then* $N_1 \otimes N_2$ *is a normal dilation of* $T \otimes S$.

<u>*Proof*</u>. If N_1 and N_2 are normal dilations of T

and S then we have

$$P_{H_1}\varphi_1(N_1)P_{H_1} = \varphi_1(T)P_{H_1}$$

$$P_{H_2}\varphi_2(N_2)P_{H_2} = \varphi_2(S)P_{H_2} \ ,$$

for any analytic functions φ_1 and φ_2 on neighborhoods of $\sigma(N_1)$ and $\sigma(N_2)$, respectively, where $P_{H_1}:K_1 \to H_1$ and $P_{H_2}:K_2 \to H_2$ are aprojections. Now, by lemma 4, $N_1 \otimes N_2$ is normal and for any analytic function φ on a neighborhood of $\sigma(N_1 \otimes N_2)$, we have

$$P_{H_1\otimes H_2}\varphi(N_1\otimes N_2)P_{H_1\otimes H_2} = P_{H_1\otimes H_2}\varphi(N_1)\otimes\varphi(N_2)P_{H_1\otimes H_2}$$

$$= P_{H_1}\varphi(N_1)P_{H_1}\otimes P_{H_2}\varphi(N_2)P_{H_2}$$

$$= (T)P_{H_1}\otimes\varphi(S)P_{H_2} = \varphi(T\otimes S)P_{H_1\otimes H_2} \ .$$

This implies that $N_1 \otimes N_2$ is a normal dilation of $T \otimes S$. ■

Now we come to our third main result.

THEOREM 3. *If* X, Y *are simply connected subsets of* $\mathbb{C}$, *then* $O(X)\otimes O(Y) \subset O(XY)$.

Proof. Let $T\otimes S \in O(X)\otimes O(Y)$, then X is a spectral set of T and Y is a spectral set of S. By the proposition above there exist normal dilations N_1 and N_2 of T and S, respectively, such that $\sigma(N_1) \subset \partial X$ and $\sigma(N_2) \subset \partial Y$. Lemma 5 implies that $N_1 \otimes N_2$ is a normal dilation of $T\otimes S$; i.e., $P_H\varphi(N_1\otimes N_2)P_H = \varphi(T\otimes S)P_H$, for any analytic function φ and $P_H:K \to H$, where $H = H_1 \otimes H_2 \subset K = K_1 \otimes K_2$.

Since $\sigma(N_1 \otimes N_2) = \sigma(N_1)\sigma(N_2)$, see [2], we have $\sigma(N_1 \otimes N_2) \subset XY$, and since $N_1 \otimes N_2$ is normal, XY is a spectral set of $N_1 \otimes N_2$, see [4]. Thus for any analytic function φ on XY, we have

$$\|(N_1 \otimes N_2)\| \leqslant \|\varphi\|_\infty .$$

This implies that

$$\|\varphi(T \otimes S)\| = \|P_H \varphi(N_1 \otimes N_2)\|$$

$$\leqslant \|\varphi(N_1 \otimes N_2)\| \leqslant \|\varphi\|_\infty ,$$

and since $\sigma(T \otimes S) = \sigma(T)\sigma(S) \subset XY$, we conclude that $T \otimes S \in O(XY)$, i.e. $O(X) \otimes O(Y) \subset O(XY)$. ∎

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