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# LOCAL LIMIT THEOREMS FOR THE CRITICAL GALTON-WATSON PROCESS WITH IMMIGRATION

by

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ABSTRACT. This paper considers a critical Galton-Watson branching process with immigration, in which the aperiodic offspring distribution $\{p_0, p_1, \ldots\}$ satisfies a $j^2 \log j$-condition (i.e. $\sum_{j=1}^{\infty} p_j j^2 \log j < \infty$) and the immigration distribution a $j \log j$-condition. The asymptotic behavior of the n-step transition probabilities $P_n(i,j)$ ad $n \to \infty$, $j \to \infty$, and $i/n$ and $j/n$ remain bounded, is established. As an application of this result the asymptotic behavior of the invariant measure of the process is obtained.

RESUMEN. Este artículo considera un proceso de ramificación crítico de Galton-Watson con inmigración, en el cual la distribución aperiódica de nacimientos $\{p_0, p_1, \ldots\}$ satisface una condición de tipo $j^2 \log j$, $\sum_{j=1}^{\infty} p_j j^2 \log j < \infty$, y la distribución de inmigración satisface una condición de tipo $j \log j$. Se establece el comportamiento asintótico de las probabilidades de

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transición en el paso n cuando $n \to \infty$, $j \to \infty$, y $i/n$, $j/n$ permanecen acotadas. Como aplicación, se obtiene el comportamiento asintótico de la medida invariante del proceso.

§1. Introduction. Let $\{X_n^{(i)}\}$ be a Galton-Watson branching process allowing immigration (GWI) initiated by i ancestors ($X_0^{(i)} \equiv i$), in which branching occurs in accordance with the offspring distribution $\{p_0, p_1, \ldots\}$, and which, at each generation, is augmented by an independent immigration component with probability measure $\{q_0, q_1, \ldots\}$. In terms of the respective probability generating functions (p.g.f.'s) f and h its n-step transition probabilities $P_n(i,j)$ are given by (see e.g. Athreya & Ney (1972) p.263):

$$g_n^{(i)}(s) = \sum_{k=0}^{\infty} P_n(i,k)s^k = (f_n(s))^i \prod_{r=0}^{n-1} h(f_r(s)),$$

$$|s| \leq 1, \quad i,n \in \mathbb{N}_0 ,$$

where $f_0(s) = s$, $f_n(s) = f(f_{n-1}(s))$, $n \in \mathbb{N}$, and $\prod_{r=0}^{-1} (\cdot)_r = 1$ ($\mathbb{N}$ denotes the positive and $\mathbb{N}_0$ the nonnegative integers).

The GWI $\{X_n^{(i)}\}$ may be written as

$$X_n^{(i)} = X_n^{(0)} + Z_n^{(i)} , \qquad i,n \in \mathbb{N}_0$$

where $X_n^{(0)}$ and $Z_n^{(i)}$ are independent and $\{Z_n^{(i)}\}$ constitutes an ordinary Galton-Watson process (GW) initiated by i parent particles with p.g.f. f and whose n-step transition probabilities $Q_n(i,j)$ have the representation

$$(f_n(s))^i = \sum_{j=0}^{\infty} Q_n(i,j)s^j, \qquad i,n \in \mathbb{N}_0, \qquad |s| \leq 1.$$

In this paper *we are concerned only with the critical aperiodic situation*, that is when

(1.1) $f'(1-) = \sum_{k=1}^{\infty} kp_k = 1$ and $\text{g.c.d.}\{k \in \mathbb{N} | p_k > 0\} = 1$,

and *assume throughout* that

(1.2) $p_1 < 1$

(1.3) $0 < q_0 < 1$

$$f''(1-) = \sum_{k=2}^{\infty} k(k-1)p_k < \infty$$

and

$$h'(1-) = \sum_{k=1}^{\infty} kq_k < \infty$$

Furthermore, let $\alpha = \frac{1}{2}f''(1-)$ and $\gamma = \frac{1}{\alpha}h'(1-)$.

Under the additional hypotheses

(1.4) $\sum_{k=2}^{\infty} p_k k^2 \log k < \infty$

and

(1.5) $\sum_{k=2}^{\infty} q_k k \log k < \infty$

we shall obtain a fairly complete description of the asymtotic behavior of the transition probabilities $P_n(i,j)$ as

(1.6) $n \to \infty$, $j \to \infty$, $j/n$ remains bounded ($i \in \mathbb{N}_0$ fixed)

and

(1.7) $n \to \infty$, $i \to \infty$, $j \to \infty$ $i/n$ and $j/n$ remain bounded.

Referring to these situations, which are examined in sections 2 and 3 respectively, we introduce the following notations. Let us agree that $\lim\limits_{(j,n)}$ or $\widetilde{(j,n)}$

and

$$\lim_{(i,j,n)} \quad \text{or} \quad \widetilde{(i,j,n)}$$

will indicate that we evaluate a limit or assert asymptotic equivalence under conditions (1.6) and (1.7), respectively. If it is convenient to specify the upper bound C for j/n in (1.6) and the common upper bound D for i/n and j/n in (1.7), we will write more distinctly (j,n,C) and (i,j,n,D) instead of (j,n) and (i,j,n) respectively.

For future reference we summarize some well-known results. Here and in the sequel $M_1, M_2, \ldots, M, M', M'', \ldots, A, C, L, \ldots$ denote suitable positive constants.

**THEOREM A.** *If* (1.4) *holds, then*

(1.8) $$\sup_{j\geq 1} Q_n(i,j) \leq \frac{M_1 . i}{n^2}, \quad i \in \mathbb{N},$$

(1.9) $$Q_n(i,j) \underset{(j,n)}{\sim} i(\alpha n)^{-2} \exp\{-j/\alpha n\}, \quad i \in \mathbb{N} \text{ fixed.}$$

Theorem A is due to Kesten, Ney and Spitzer (1966). Their local exponential law (1.9) corresponds to Yaglom's famous global exponential law, while our main result (2.1) is the local version of the limit theorem

$$\lim_{n\to\infty} E(\exp\{-\theta(\alpha n)^{-1} X_n^{(i)}\}) = (1+\theta)^{-\gamma}, \quad i \in \mathbb{N}_0$$

which was obtained independently by Foster (1969), Pakes (1971) and Seneta (1970). Due to Pakes (1972) is the existence proof of the following limit function

(1.10) $$\vartheta(s) = \lim_{n\to\infty} n^{\gamma} g_n^{(i)}(s), \quad |s| < 1, \quad i \in \mathbb{N}_0.$$

His precise result is

THEOREM B. *If conditions* (1.4) *and* (1.5) *are fulfilled, then the convergence in* (1.10) *is uniform over compact subsets of the open unit disc and the limit function* $\vartheta$ *satisfies the functional equation* $\vartheta(s) = h(s)\,\vartheta(f(s))$, $|s| < 1$.

As a consequence, the coefficients $\theta_j$ of the power series representation of $\vartheta$,

$$(1.11) \quad \vartheta(s) = \sum_{k=0}^{\infty} \theta_k s^k, \quad |s| < 1,$$

are given by

$$(1.12) \quad \theta_j = \lim_{n\to\infty} n^{\gamma} P_n(i,j), \quad i,j \in \mathbb{N}_0$$

and satisfy

$$0 \leq \theta_j = \sum_{k=0}^{\infty} \theta_k P(k,j), \quad j \in \mathbb{N}_0 .$$

Henceforth, we refer to $\{\theta_j \mid j \in \mathbb{N}_0\}$ as the unique (because of (1.12)) stationary measure of the GWI $\{X_n^{(i)}\}$, whose existence and uniqueness, up to a constant multiple, is already ensured by (1.1)-(1.3) (Seneta (1969)).

The local limit theorems of this paper prove to be useful in quite different contexts. In Bühler & Mellein (1980) they provide answers to questions raised in connection with the distribution of generations in critical GWI's (Schützhold (1975)) and the so-called quasi-competition of independent processes (Bühler (1967,1978)). In Mellein (1979,1981) they are used when deriving limiting diffusions for conditioned critical GW and GWI. Finally they permit a detailed study of the asymptotic behavior of the Green function of the GWI (Mellein (1979)).

To give an example of the results gotten we state the following theorem which is a direct consequence of Theorem 2.1.

THEOREM 1.1. *If* (1.4) *and* (1.5) *hold, then* $\lim_{j\to\infty} \theta_j j^{1-\gamma} = (\alpha^\gamma \Gamma(\gamma))^{-1}$ .

## §2. A local limit theorem for the GWI — fixed initial population.

For fixed i and j, we saw in (1.12) that $P_n(i,j) \sim \theta_j n^{-\gamma}$ as $n \to \infty$. Now we want to let $j \to \infty$ also.

THEOREM 2.1. *If* (1.4) *and* (1.5) *hold, then, for fixed* $i \in \mathbb{N}_0$ ,

(2.1) $$\lim_{(j,n)} n\left(\frac{j}{n}\right)^{1-\gamma} \exp\{\frac{j}{\alpha n}\}\alpha^\gamma \Gamma(\gamma) P_n(i,j) = 1.$$

Before entering into details we give a brief outline of the proof. We shall show that the p.g.f. $g_n^{(0)}$ of $\{P_n(0,j) \mid j \in \mathbb{N}_0\}$ may be represented as

( 2.2) $$g_n^{(0)}(s) = \left[\frac{1+(N-1)\alpha(1-s)}{1+(n-1)\alpha(1-s)}\right]^\gamma \Lambda(n,N,s), \qquad n > N,$$

with N a positive integer depending on the p.g.f.'s f and h, and $\Lambda(n,N,s)$ a power series in s which converges absolutely in s = 1. Just as in Kesten, Ney & Spitzer (1966) a comparison of coefficients in (2.2) then reveals the (j,n)-asymptotic behavior of $P_n(0,j)$. Using the easily verified relation

(2.3) $$P_n(i,j) = \sum_{k=0}^{j} P_n(0,k)Q_n(i,j-k) , \qquad i,j \in \mathbb{N}_0$$

we finally will find that

(2.4) $P_n(i,j) \underset{(j,n)}{\sim} P_n(0,j)$, i fixed. ∎

We start by introducing the following notations. Let $g_n \equiv g_n^{(0)}$, D be the unit disc, and define for a complex power series a, $a(z) = \sum_{k=0}^{\infty} a_k z^k$, which converge absolutely in 1, a $\ell_1$ norm $\|a\| \equiv \|a(z)\| = \sum_{k=0}^{\infty} |a_k|$ which obviously satisfies

$$(2.5) \quad \sup_{z \in D} |a(z)| \leq \|a\| \quad \text{and} \quad \|a\| \cdot \|b\| \geq \|ab\| .$$

A first step to arrive at the product representation (2.2) is

**LEMMA 2.1.** *Assume* (1.5), *then there is a positive integer* N *such that, for* $s \in D$ *and all* $n > N$ ,

$$g_n(s) = g_N(s) \prod_{k=N}^{n-1} \{1-h'(1)(1-f_k(s))\} \prod_{k=N}^{n-1} (1+a_k(s)) ,$$

*where* $a_N(s), a_{N+1}(s), \ldots,$ *are certain power series, satisfying* $\sum_{k=N}^{\infty} \|a_k(s)\| < \infty$.

*Proof.* (i) *Choice of* N. Remembering that the right-hand side of

$$\|h'(1)(1-f_k(s))\| = 2h'(1)(1-f_k(0))$$

tends to zero as $k \to \infty$, we may choose $N_1 \in \mathbb{N}$ in such a way that

$$(2.6) \quad \|h'(1)(1-f_k(s))\| \leq \frac{1}{2} \quad \text{for all} \quad k \geq N_1 .$$

Now let $N_2 \in \mathbb{N}$ such that

$$\frac{2h'(1)}{1+N_2\alpha} \leq \frac{1}{2}$$

and put $N = \max(N_1, N_2, 4)$, *fixed for the remainder of this section.*

(ii) *Definition of* $a_N(s), a_{N+1}(s), \ldots$ . It is easily seen that

$$h(s) = 1+h'(1)(s-1)+\delta(s), \quad s \in D,$$

where

$$\delta(s) = (s-1) \sum_{r=2}^{\infty} (q_r \sum_{m=1}^{r-1} (s^m-1)).$$

We use this relation to obtain, for $n > N$,

$$g_n(s) = \prod_{k=0}^{n-1} h(f_k(s)) = g_N(s) \prod_{k=N}^{n-1} h(f_k(s))$$

$$= g_N(s) \prod_{k=N}^{n-1} \{1-h'(1)(1-f_k(s))+\delta(f_k(s))\}$$

$$= g_N(s) \prod_{k=N}^{n-1} \{1-h'(1)(1-f_k(s))\} \prod_{k=N}^{n-1} (1+a_k(s)) ,$$

with $a_k(s)$ defined by

$$a_k(s) = \frac{\delta(f_k(s))}{1-h'(1)(1-f_k(s))}, \qquad k = N, N+1, \ldots ,$$

which is justified for all $s \in D$ by (2.5) and (2.6).

(iii) *Bounding the* $a_k$. We first state two inequalities. The obvious one

$$(2.7) \qquad \|f_k^m(s)-1\| \leq 2, \quad k,m \in \mathbb{N}$$

and

$$(2.8) \qquad \|f_k^m(s)-1\| \leq \frac{m}{k}M, \quad k,m \in \mathbb{N},$$

the latter one following from

$$\|f_k^m(s)-1\| = \|(f_k(s)-1) \sum_{j=0}^{m-1} f_k^j(s)\| \leq m\|f_k(s)-1\| = 2m|1-f_k(0)|$$

Now we turn to the estimation of $a_k$. Observing that

$$a_k(s) = \delta(f_k(s)) \sum_{j=0}^{\infty} \{h'(1)(1-f_k(s))\}^j , \qquad k \geqslant N,$$

then (2.5)-(2.8) yield

$$\frac{1}{2}\sum_{k=N}^{\infty} \| a_k(s) \| \leqslant \frac{1}{2}\sum_{k=N}^{\infty} \| \delta(f_k(s)) \| \sum_{j=0}^{\infty} 2^{-j}$$

$$= \sum_{k=N}^{\infty} \| \delta(f_k(s)) \| \leqslant \sum_{k=1}^{\infty} \| f_k(s)-1 \| \sum_{r=2}^{\infty} (q_r \sum_{m=1}^{r-1} \| f_k^m(s)-1 \|)$$

$$= \sum_{r=2}^{\infty} q_r \sum_{m=1}^{r-1} \sum_{k=1}^{\infty} \| f_k(s)-1 \| \, \| f_k^m(s)-1 \|$$

$$\leqslant \sum_{r=2}^{\infty} q_r \sum_{m=1}^{r-1} \{2 \sum_{k=1}^{m} \| f_k(s)-1 \| + \sum_{k=m}^{\infty} \frac{mM^2}{k^2} \}$$

which is easily seen to be convergent using once more (2.8) and (1.5). Q.E.D.

Up to a slight modification, the following lemma is Kesten, Ney & Spitzer's (1966,p.528) Lemma 8. It is worth noting that this statement is the only part of their proof of the local limit theorem (1.9) where the hypothesis (1.4) enters explicitly.

**LEMMA 2.2.** *If* (1.4) *holds, then*

$$\sum_{k=1}^{\infty} \| 1-f_k(s) - \frac{1}{k\alpha+(1-s)^{-1}} \| < \infty .$$

**LEMMA 2.3.** *Suppose that* (1.4) *and* (1.5) *hold. Then, for all* $n > N$ *and* $s \in D$,

$$(2.9) \qquad g_n(s) = g_N(s) \prod_{k=N}^{n-1} (1- \frac{h'(1)}{k\alpha+(1-s)^{-1}}) \prod_{k=N}^{n-1} (1+b_k(s)) ,$$

*and*

$$(2.10) \qquad \prod_{k=N}^{n-1} \{1- \frac{h'(1)}{k\alpha+(1-s)^{-1}}\} = \{\frac{1+(N-1)\alpha(1-s)}{1+(n-1)\alpha(1-s)}\}^{\gamma} \prod_{k=N}^{n-1} \{1+c_k(s)\}$$

*where* $b_N(s), b_{N+1}(s), \ldots, c_N(s), c_{N+1}(s), \ldots$ *are certain power series with*

$$\sum_{k=N}^{\infty} \|b_k(s)\| < \infty, \quad \sum_{k=N}^{\infty} \|c_k(s)\| < \infty .$$

*Proof.* For $k \geqslant N$, we define $b_k(s)$ as error term by

$$\{1-h'(1)(1-f_k(s))\}(1+a_k(s)) = \left\{1-\frac{h'(1)}{k\alpha+(1-s)^{-1}}\right\}(1+b_k(s)),$$

where the $a_k(s)$ are the power series as defined in Lemma 2.1. Now observe that

$$(2.11) \quad \left\|\frac{1}{k\alpha+(1-s)^{-1}}\right\| = \left\|\frac{1}{1+k\alpha}-\frac{1}{(1+k\alpha)^2}\sum_{j=1}^{\infty}\left\{\frac{k}{1+k\alpha}\right\}^{j-1} s^j\right\| = \frac{2}{1+k\alpha}$$

to conclude that, in view of the choice of N, that

$$\left\|\left\{1-\frac{h'(1)}{k\alpha+(1-s)^{-1}}\right\}^{-1}\right\| \leqslant \sum_{r=0}^{\infty}\left\|\frac{h'(1)}{k\alpha+(1-s)^{-1}}\right\|^r \leqslant \sum_{r=0}^{\infty} 2^{-r} = 2, \quad k \geqslant N.$$

Therefore, with the aid of (2.6), we may estimate the norm of

$$b_k(s)$$

$$= \left\{1-\frac{h'(1)}{k\alpha+(1-s)^{-1}}\right\}^{-1}\left\{\{1-h'(1)(1-f_k(s))\}a_k(s)+\frac{h'(1)}{k\alpha+(1-s)^{-1}} - h'(1)(1-f_k(s))\right\}, \quad \text{for all } k \geqslant N, \text{ by}$$

$$\|b_k(s)\| \leqslant 2\left\{2\|a_k(s)\|+h'(1)\left\|\frac{1}{k\alpha+(1-s)^{-1}}-(1-f_k(s))\right\|\right\},$$

and (2.9) follows from Lemma's 2.1 and 2.2. Now notice that $h'(1) = \alpha\gamma$ to check that

$$c_k(s) = \frac{1-\frac{\alpha\gamma(1-s)}{1+k\alpha(1-s)} - \{1-\frac{\alpha(1-s)}{1+k\alpha(1-s)}\}^\gamma}{\{\frac{1+(k-1)\alpha(1-s)}{1+k\alpha(1-s)}\}^\gamma}, \quad k = N, N+1, \ldots$$

satisfy (2.10). Using (2.11) we find that

$$\left\| 1-\frac{\alpha\gamma(1-s)}{1+k\alpha(1-s)} - \{1-\frac{(1-s)\alpha}{1+k\alpha(1-s)}\}^\gamma \right\| = \left\| \sum_{j=2}^{\infty} \binom{\gamma}{j}\{-\frac{\alpha(1-s)}{1+k\alpha(1-s)}\}^j \right\|$$

$$= \left\| \{\frac{\alpha(1-s)}{1+k\alpha(1-s)}\}^2 \sum_{j=0}^{\infty} \binom{\gamma}{j+2}\{-\frac{\alpha(1-s)}{1+k\alpha(1-s)}\}^j \right\|$$

$$\leqslant \{\frac{2\alpha}{1+k\alpha}\}^2 \Gamma(\gamma+1) \sum_{j=0}^{\infty} \{\frac{2\alpha}{1+k\alpha}\}^j \leqslant 8\Gamma(\gamma+1)\alpha^2(1+k\alpha)^{-2}, \quad k \geqslant N.$$

But this together with

$$\left\| \{\frac{1+(k-1)\alpha(1-s)}{1+k\alpha(1-s)}\}^\gamma \right\| \geqslant \{\frac{1+(k-1)\alpha}{1+k\alpha}\}^\gamma > 2^{-\gamma} > 0, \quad k \geqslant N,$$

completes the proof. Q.E.D.

We are now in a position to validify the product representation (2.2).

**LEMMA 2.4.** *Assume that* (1.4) *and* (1.5) *hold. Then there is a constant* M *and a bounded power series* $\Lambda(n,N,s)$, *with* $\|\Lambda(n,N,s)\| < M < \infty$ *for all* $n > N$, *which satisfies* (2.2) *for all* $s \in D$.

*Proof*. Let $b_k(s)$, $c_k(s)$, $k \geqslant N$, the power series as defined in Lemma 2.3 and set

$$\Lambda(n,N,s) = g_N(s) \prod_{k=N}^{n-1} (1+b_k(s))(1+c_k(s)).$$

The result follows from the cited lemma. Q.E.D.

To detect the asymptotic behavior of $P_n(0,j)$ by comparing coefficients in (2.2) we need some more results.

LEMMA 2.5. *Let* $\vartheta(s)$ *be the g.f.* (1.11) *of the stationary measure of the* GWI *and* $\Lambda(n,N,s)$ *as defined in Lemma* 2.4 *and put*

$$\sum_{j=0}^{\infty} \psi_{n,N,j} s^j \equiv \psi(n,N,s) = \Lambda(n,N,s)\,\{1+(N-1)\alpha(1-s)\}^{\gamma}.$$

*If* (1.4) *and* (1.5) *hold, then*

$$\text{(2.12)} \qquad \lim_{n\to\infty} \psi(n,N,s) = \vartheta(s)\{\alpha(1-s)\}^{\gamma}, \qquad |s| < 1,$$

*the convergence being uniform over compact subsets of the open unit disc. Furthermore,*

$$\text{(2.13)} \qquad \vartheta(s) \sim \{\alpha(1-s)\}^{-\gamma} \quad \textit{as} \quad s \to 1 \quad \textit{with} \quad |s| < 1,$$

$$\text{(2.14)} \qquad \sum_{k=0}^{\infty} |\psi_{n,N,k}| < M' < \infty, \qquad n > N, \quad M' \textit{ appropriately}$$

*chosen,*

$$\text{(2.15)} \qquad \sum_{k=0}^{\infty} k\psi_{n,N,k} = h'(1), \qquad n > N,$$

$$\text{(2.16)} \qquad \lim_{n\to\infty} \psi_{n,N,k} =: \psi_k, \qquad k \in \mathbb{N}_0, \quad n > N,$$

$$\text{(2.17)} \qquad \sum_{k=0}^{\infty} \psi_k = 1$$

*and*

$$\text{(2.18)} \qquad \sum_{k=0}^{\infty} |\psi_k| < \infty .$$

<u>*Proof*</u>. To conclude (2.12) we write $\psi(n,N,s)$ as

$$\psi(n,N,s) = n^{\gamma} g_n(s) n^{-\gamma}\{1+(n-1)\alpha(1-s)\}^{\gamma}$$

and apply Theorem B. By (2.2), $\psi(n,N,1) = 1$ for all $n > N$, so that use of Lemma 2.4 and (2.12) yields (2.13). (2.14) follows from (2.5), Lemma 2.4 and the obvious fact that $\|(1+(N-1)\alpha(1-s))^{\gamma}\| < \infty$. (2.16) is a consequence (see e.g. Rényi (1966) p.121) of Cauchy's generalized integration formula and the uniform convergence in (2.12). (2.17) and (2.18) are immediate from (2.14) and (2.16). By differentiating

$$g_n(s) = \psi(n,N,s)(1+(n-1)\alpha(1-s))^{-\gamma}, \quad n > N,$$

we get

$$g_n'(s) - \psi(n,N,s)\alpha\gamma(n-1)(1+(n-1)\alpha(1-s))^{-\gamma-1}$$

$$= \psi'(n,N,s)(1+(n-1)\alpha(1-s))^{-\gamma}, \quad n > N, \quad |s| < 1.$$

But $g_n'(1) = nh'(1) < \infty$ so that the formerrelation guarantees the existence of $\psi'(n,N,1)$. More precisely it gives $\psi'(n,N,1) = nh'(1) - (n-1)\alpha\gamma = h'(1)$, proving (2.15). Q.E.D.

**REMARK 2.1.** In the proof of Theorem 2 of Pakes (1972, p. 282), which is stated with an error, appears (after the obvious correction) the relation (2.13) with $s\nearrow 1$.

**LEMMA 2.6.** *Let* $\beta_{k,j} = \binom{\gamma+j-k-1}{j-k}$, $k\in\mathbb{N}_0$, $j\in\mathbb{N}$, $k \leq j$. *Then*

(2.19) $0 \leq \beta_{\cdot,j}$ *increases* / *decreases* *on* $\{0,1,\ldots,j\}$ *if* $\gamma < 1$ / $\gamma \geq 1$,

(2.20) $\beta_{k,j} \sim j^{\gamma-1}(\Gamma(\gamma))^{-1}$ *as* $j \to \infty$, $k\in\mathbb{N}_0$,

(2.21) $\sum_{k=0}^{j}\beta_{k,j} \sim j^{\gamma}(\Gamma(\gamma+1))^{-1}$ *as* $j \to \infty$.

*Proof*. Observe that

$$(1-s)^{-\gamma} = \sum_{k=0}^{\infty}(-1)^k\binom{-\gamma}{k}s^k = \sum_{k=0}^{\infty}\binom{\gamma+k-1}{k}s^k, \quad |s| < 1.$$

Hence (2.20) and (2.21) follow from (to get (2.20) one has to use (2.19) which is easily checked) the Tauberian theorem for power series (see e.g. Feller (1971)). Q.E.D.

We are now prepared to complete the proof of Theorem 2.1.

**PROOF OF THEOREM 2.1. FOR** $i = 0$. We use the notations introduced in the previous lemmas. A comparison of coefficients in

$$g_n(s) = \psi(n,N,s)\left\{\frac{1}{1+(n-1)\alpha(1-s)}\right\}^{\gamma} \qquad n > N$$

gives

$$(2.22)\quad P_n(0,j) = \sum_{k=0}^{j} \psi_{n,N,k}\left\{\frac{1}{1+(n-1)\alpha}\right\}^{\gamma}\binom{\gamma+j-k-1}{j-k}\left\{\frac{(n-1)\alpha}{1+(n-1)\alpha}\right\}^{j-k},$$

$$n > N, \quad j \in \mathbb{N}_0$$

or

$$P_n(0,j)\alpha^{\gamma}\Gamma(\gamma)\exp\left\{\frac{j}{\alpha n}\right\}n\left(\frac{j}{n}\right)^{1-\gamma} = \sum_{k=0}^{j} d_{k,j,n,N}, \qquad n > N,$$

where $\qquad d_{k,j,n,N} = a_n b_{k,j} c_{k,j,n} \psi_{n,N,k}$

with

$$a_n = \left\{\frac{n\alpha}{1+(n-1)\alpha}\right\}^{\gamma},$$

$$b_{k,j} = \binom{\gamma+j-k-1}{j-k}\Gamma(\gamma) j^{1-\gamma}$$

and

$$c_{k,j,n} = \exp\left\{\frac{j}{\alpha n}\right\}\left\{\frac{(n-1)\alpha}{1+(n-1)\alpha}\right\}^{j-k}.$$

It is easily seen that, for fixed $k \in \mathbb{N}_0$,

$$\lim_{n\to\infty} a_n = \lim_{(j,n)} c_{k,j,n} = \lim_{j\to\infty} b_{k,j} = 1,$$

which in turn implies

$$(2.23) \qquad \lim_{(j,n)} d_{k,j,n,N} = \psi_k , \qquad k \in \mathbb{N}_0 .$$

Now let $\varepsilon > 0$ and $C < \infty$. We will show that there exists a constant $L = L(\varepsilon, C)$ such that

$$(2.24) \qquad \lim_{(j,n,C)} \sum_{k=L}^{j} |d_{k,j,n,N}| < \varepsilon .$$

But then, in view of (2.23) and (2.18), we deduce for sufficiently large $L' \geqslant L$

$$\lim_{(j,n,C)} \Big| \sum_{k=0}^{j} d_{k,j,n,N} - \sum_{k=0}^{\infty} \psi_k \Big|$$

$$\leqslant \lim_{(j,n,C)} \Big| \sum_{k=0}^{L'-1} (d_{k,j,n,N} - \psi_k) \Big| + \lim_{(j,n,C)} \Big| \sum_{k=L'}^{j} d_{k,j,n,N} - \sum_{k=L'}^{\infty} \psi_k \Big|$$

$$\leqslant \lim_{(j,n,C)} \sum_{k=L'}^{j} |d_{k,j,n,N}| + \sum_{k=L'}^{\infty} |\psi_k| < 2\varepsilon ,$$

which, by (2.17), is the desired result.

To prove (2.24) we argue as follows, beginning with the

*Case* $\gamma \geqslant 1$. From (2.19) and (2.20) it is seen that there is a constant $L_1$ such that

$$|b_{k,j}| \leqslant L_1 , \qquad 0 \leqslant k \leqslant j.$$

The existence of a constant $L_2 = L_2(C)$ such that

$$|c_{k,j,n}| \leqslant L_2, \qquad 0 \leqslant k \leqslant j, \quad j/n \leqslant C$$

is obvious. Thus, by (2.14),

$$\lim_{(j,n,C)} \sum_{k=L}^{j} |d_{k,j,n,N}| \leqslant L_1 L_2 \lim_{n\to\infty} \sum_{k=L}^{\infty} |\psi_{n,N,k}| = L_1 L_2 \sum_{k=L}^{\infty} |\psi_k| .$$

Henceforth (2.18) permits to choose L satisfying (2.24).

*Case* $\gamma < 1$. We notice that $b_{[j/2],j}$ is bounded (it converges to $2^{1-\gamma}$ as $j \to \infty$; $[x]$ means the largets integer $\leqslant x$) and we may therefore deduce from (2.19) the existence of a constant $L_3$ such that

$$|b_{k,j}| \leqslant L_3 , \qquad 0 \leqslant k \leqslant j/2.$$

Proceeding as in the previous case we find that there is a constant $L = L(\varepsilon, C)$ with

$$(2.25) \qquad \lim_{(j,n,C)} \sum_{k=L}^{[j/2]} |d_{k,j,n,N}| < \varepsilon .$$

To estimate

$$\sum_{k=[j/2]}^{j} |d_{k,j,n,N}|$$

we observe that there is a constant $L_4 = L_4(C)$ such that

$$|a_n c_{k,j,n}| \leqslant L_4 , \quad 0 \leqslant k \leqslant j, \quad j/n \leqslant C.$$

Setting

$$\mathcal{J}_{n,N,j} = \max\{|\psi_{n,N,k}| \,|\, k \in \{[j/2],\ldots,j\}\} ,$$

we therefore get $(j/n \leqslant C)$

$$\sum_{k=[j/2]}^{j} |d_{k,j,n,N}| \leqslant L_4 \mathcal{J}_{n,N,j} \sum_{k=[j/2]}^{j} |b_{k,j}|$$

$$\leqslant L_4' j \mathcal{J}_{n,N,j} , \quad \text{by (2.21).}$$

Finally, to assure that $\lim_{(j,n,C)} j\mathcal{J}_{n,N,j} = 0$ , which in combination with (2.25) proves (2.24), it suffices to show that $\lim_{k\to\infty} k\psi_{n,N,k} = 0$ , uniformly for $n > N$. But the latter relation is a consequence of (2.15). This completes the proof of Theorem 2.1 for $i = 0$.

**PROOF OF THEOREM 2.1. FOR** $i \geq 1$. We first examine the case $\gamma > 1$. Using (1.12) and the previously found $(j,n)$-behavior of $P_n(0,j)$ we obtain for an integervalued function $k(j)$

$$\frac{P_n(0,k(j))}{P_n(0,j)} \leq A \qquad \text{for all sufficiently large } j,n \text{ with } j/n \leq A' \text{ and } k(j) \leq j, \quad A = A(A'),$$

and hence for all sufficiently large $j,n$:

$$\begin{aligned}\sum_{k=0}^{j-1} P_n(0,k)Q_n(i,j-k) &\leq AjP_n(0,j)\sup_{r\geq 1} Q_n(i,r) \\ &\leq AM_1 ijn^{-2}P_n(0,j) \quad \text{by (1.8)} \\ &\leq A''jn^{-2}P_n(0,j).\end{aligned}$$

In the *case* $\gamma \leq 1$ we observe that $\sum_{k=0}^{j-1} P_n(0,k) \leq 1$ and get similarly

$$\sum_{k=0}^{j-1} P_n(0,k)Q_n(i,j-k) \leq A'''n^{-2}.$$

These estimates together with (2.3) and $\lim_{n\to\infty} Q_n(i,0) = 1$ prove (2.4) and hence the theorem. Q.E.D.

§3. <u>Local limit theorems – large initial population</u>. The objective of this section is to study the $(i,j,n)$-behavior of $P_n(i,j)$. The analysis consists of a careful examination of the relation (2.3). Theorem 2.1 and Theorem 3.1, the latter one giving the (i,j,n)-behavior of $Q_n(i,j)$, will be applied.

**3.1. THE GW-CASE.** Kesten, Ney & Spitzer (1966) have been concerned with $Q_n(i,j)$ when all three variables are large. In their description of the asymptotic behavior of $Q_n(i,j)$ they impose no restriction on i. We use their result, which we restate as Lemma 3.1, to develop the (i,j,n)-behavior of $Q_n(i,j)$. The more restrictive assumption on i results in a more manageable formula.

To fascilitate the statements of the results, we introduce the following notations, where as usual $I_\rho$ denotes the Bessel function of order $\rho$ with imaginary argument.

$$A_n(i,j) = (\alpha n)^{-1}\exp\{-\frac{i+j}{\alpha n}\}\sqrt{i/j}\,I_1(\frac{2}{\alpha n}\sqrt{ij}), \qquad i,j,n\in\mathbb{N}\ .$$

$$B_n(i,r) = \binom{i}{r}(1-f_n(0))^r(f_n(0))^{i-r}, \qquad i,n\in\mathbb{N},\ r\in\mathbb{N}_0,\ i\geq r.$$

$$C_n(i,j,r) = B_n(i,r)\exp\{-\frac{j}{\alpha n}\}\frac{j^{r-1}}{(r-1)!}(\alpha n)^{-r}, \qquad j,r,n\in\mathbb{N}.$$

$$J_K = \{(i,j,n)\in\mathbb{N}^3\,|\,\frac{i}{n}\leq K,\frac{j}{n}\leq K\}, \qquad 0<K<\infty.$$

**LEMMA 3.1.** *Let* $R\geq 1$. *If* (1.4) *holds, then*

$$(3.2)\qquad Q_n(i,j) = \sum_{r=1}^{R} C_n(i,j,r) + 0_j(\frac{1}{n})\min(1,\frac{1}{n}) + G_n(i,R)\ ,$$

*with*

$$G_n(i,R) = 0(\frac{1}{n\sqrt{R}}\sum_{r=R+1}^{i} B_n(i,r))\ ,$$

*as* j *and* n *behave as in* (1.6) *and* i *varies arbitrarily except for the restriction* $i\geq R$.

**THEOREM 3.1.** *Let* $0<K<\infty$. *If condition* (1.4) *is fulfilled, then*

$$Q_n(i,j) \underset{(i,j,n,K)}{\frown} A_n(i,j).$$

*Proof*. Let $\varepsilon > 0$ be given. We first compare $A_n(i,j)$ and the last term of the right-hand side in (3.2).

(i) Define $\delta = 1 + \frac{1}{\alpha}$ and observe that, with $N_1$ sufficiently large, for all $n \geqslant N_1$ and $i \geqslant R \geqslant 1$

$$\sum_{r=R+1}^{i} B_n(i,r) = 1 - \sum_{r=0}^{R} B_n(i,r)$$

$$\leqslant 1 - B_n(i,0) \;=\; 1 - (f_n(0))^i$$

$$\leqslant 1 - (1-\frac{\delta}{n})^i \;\leqslant\; \delta\frac{i}{n} .$$

Under the further assumptions $(i,j,n) \in J_K$, $n \geqslant N_1$, $i \geqslant R$, we find the estimates

$$\sum_{r=R+1}^{i} B_n(i,r) \leqslant \delta\frac{i}{n} \leqslant \delta\alpha\exp\left\{\frac{2K}{\alpha} - \frac{i+j}{\alpha n}\right\}\frac{\sqrt{ij}}{\alpha n}\sqrt{i/j}$$

$$\leqslant \alpha n(1+\alpha)\exp\left\{\frac{2K}{\alpha}\right\}A_n(i,j) . \tag{3.3}$$

Next observe that Lemma 3.1. provides constants $N_2$ and $M$ such that

$$\frac{|G_n(i,R)|\,n\sqrt{R}}{\sum_{r=R+1}^{i} B_n(i,r)} \leqslant M, \qquad n \geqslant N_2.$$

This leads, together with (3.3), to

$$\frac{|G_n(i,R)|}{A_n(i,j)} \leqslant M\alpha(1+\alpha)\exp\left\{\frac{2K}{\alpha}\right\}\frac{1}{\sqrt{R}} ,$$

for all $R \geqslant 1$ and $(i,j,n) \in J_K$ with $n \geqslant N' = \max(N_1,N_2)$ and $i \geqslant R$.

Now we choose $R''$ such that

$$\varepsilon\sqrt{R''} \geqslant M\alpha(1+\alpha)\exp\{\frac{2K}{\alpha}\} ,$$

and turn to the comparison of $A_n(i,j)$ and the first summand in (3.2).

(ii) We will prove: There are constants $R' = R'(\varepsilon) > R''$ and $N_3 = N_3(\varepsilon)$ such that

$$(3.4)\qquad 1-\varepsilon \leqslant \frac{\sum_{r=1}^{R'} C_n(i,j,r)}{A_n(i,j)} \leqslant 1+\varepsilon , \quad (i,j,n)\in J_K \text{ with } i,n \geqslant N_3 ,$$

proceding as follows. It is not difficult to see that

$$B_n(i,r) \underset{(i,n,K)}{\sim} \frac{1}{r!}\{\frac{i}{\alpha n}\}^r \exp\{-\frac{i}{\alpha n}\} , \qquad r\in\mathbb{N}_0 ,$$

and inspection of (3.1) shows that there exists a $R' > R''$ such that

$$(3.5)\qquad 1-\frac{\varepsilon}{2} \leqslant \frac{1}{I_1(2x)}\sum_{k=0}^{R'-1}\frac{1}{k!(k+1)!}x^{2k+1} \leqslant 1, \qquad x\in\left|0,\frac{K}{\alpha}\right|$$

Combining these facts one finds that there is a $N_3 = N_3(\varepsilon)$ such that, for all $(i,j,n)\in J_K$ with $i,n \geqslant N_3$ and all $r\in\{1,2,\ldots,R'\}$ ,

$$(3.6)\qquad (1-\frac{\varepsilon}{2})D_n(i,j,r) \leqslant C_n(i,j,r) \leqslant (1+\varepsilon)D_n(i,j,r) ,$$

where

$$D_n(i,j,r) = (\alpha n r!(r-1)!)^{-1}\{\frac{i}{\alpha n}\}^r\{\frac{j}{\alpha n}\}^{r-1}\exp\{-\frac{i+j}{\alpha n}\} , \qquad r = 1,2,\ldots,R'$$

sum up to

$$(3.7)\qquad \sum_{r=1}^{R'} D_n(i,j,r) = \frac{1}{\alpha n}\sqrt{i/j}\exp\{-\frac{i+j}{\alpha n}\}\sum_{k=0}^{R'-1}(k!(k+1)!)^{-1}\{\frac{\sqrt{ij}}{\alpha n}\}^{2k+1}$$

Now use (3.7) and the left-hand inequality in (3.6) to verify

$$\frac{\sum_{r=1}^{R'} C_n(i,j,r)}{A_n(i,j)} \geqslant \frac{(1-\varepsilon/2)}{I_1(2\sqrt{ij}/\alpha n)} \sum_{k=0}^{R'-1} (k!(k+1)!)^{-1}\left\{\frac{\sqrt{ij}}{\alpha n}\right\}^{2k+1} ,$$

for all $(i,j,n)\in J_K$ with $i,n \geqslant N_3$. This, together with the first estimate in (3.5), yields the first part of (3.4). The second inequality of (3.4) is obtained similarly, using the upper bounds in (3.5) and (3.6).

(iii) For $(i,j,n)\in J_K$ we have

$$\min(1,\tfrac{i}{n})/A_n(i,j) \leqslant \alpha^2 n\exp\{\tfrac{2K}{\alpha}\}\min(1,\tfrac{i}{n})\tfrac{n}{i}$$

$$\leqslant \alpha^2 n\exp\{\tfrac{2K}{\alpha}\} .$$

Consequently, there exists a $N_4$ so that

$$|\text{middle summand in (3.2)}|/A_n(i,j) \leqslant \varepsilon ,$$

for all $(i,j,n)\in J_K$ with $n \geqslant N_4$ and j sufficiently large.

Finally, in view of the choice of R", (i), (ii), (iii) and Lemma 3.1 it is clear, that

$$1-3\varepsilon \leqslant Q_n(i,j)/A_n(i,j) \leqslant 1+3\varepsilon ,$$

for all $(i,j,n)\in J_K$ with $n > \max(N',N_3,N_4)$, $i \geqslant \max(R',N_3)$ and j sufficiently large. This completes the proof. Q.E.D.

**REMARK 3.1.** The assumption $i \to \infty$ in Theorem 3.1 may be dropped. This follows from (1.9) and the behavior of the Bessel function at 0+. More precisely, when $j,n \to \infty$, $j/n \leqslant K$, $\frac{i}{j} \to 0$ (i.e. especially for i fixed), then the asymptotic behavior of $A_n(i,j)$,

$$A_n(i,j) = \frac{1}{\alpha n}\exp\{-\frac{i+j}{\alpha n}\}\sqrt{i/j}(\frac{1}{\alpha n}\sqrt{ij} + O(\sqrt{j/i})\ )$$

$$\underset{\frac{i}{j}\to 0,\ (j,n)}{\sim} (\alpha n)^{-2} i\exp\{-\frac{j}{\alpha n}\}\ ,$$

and, by (1.9), that of $Q_n(i,j)$ coincide. This proves the remark.

3.2. THE GWI-CASE. We introduce

$$H_n(i,j) = \frac{1}{\alpha n}\{\frac{j}{i}\}^{(\gamma-1)/2} \exp\{-\frac{i+j}{\alpha n}\} I_{\gamma-1}(\frac{2}{\alpha n}\sqrt{ij})$$

to state the local limit theorem for the GWI as follows.

THEOREM 3.2. *Let* $0 < K < \infty$. *If* (1.4) *and* (1.5) *hold, then*

(3.8) $$P_n(i,j) \underset{(i,j,n,K)}{\sim} H_n(i,j)$$

Defining $R_n(i,j,k) = P_n(0,k)Q_n(i,j-k)$, $0 \leqslant k \leqslant j$, we note that equation (2.3) may be reproduced in the form

(3.9) $$P_n(i,j) = \sum_{k=0}^{j} R_n(i,j,k)\ .$$

For $0 < \varepsilon < \frac{1}{2}$ we decompose the sum in (3.9) into the three summands

$$S_n(i,j,\varepsilon) = R_n(i,j,j) + \sum_{k=\varepsilon j}^{(1-\varepsilon)j} R_n(i,j,k)$$

$$T_n(i,j,\varepsilon) = \sum_{k=0}^{\varepsilon j} R_n(i,j,k)\ ,$$

$$U_n(i,j,\varepsilon) = \sum_{k=(1-\varepsilon)j}^{j-1} R_n(i,j,k)\ ,$$

which we will compare with $H_n(i,j)$ (For simplicity we drop the square brackets, but $\varepsilon j$ etc. will always be understood to mean $[\varepsilon j]$ etc.).

LEMMA 3.2. *If the conditions of Theorem* 3.2. *are fulfilled, then there are constants* M *and* M' *such that, for all* $0 < \varepsilon < \frac{1}{2}$ *and* $(i,j,n) \in J_K$ ,

(a) $U_n(i,j,\varepsilon)/H_n(i,j) \leq M\varepsilon$

(b) $T_n(i,j,\varepsilon)/H_n(i,j) \leq M'\varepsilon^{\gamma}$

*Proof*. An application of (1.8) and Theorem 2.1. provides the estimate $U_n(i,j,\varepsilon) \leq M'K\frac{\varepsilon}{n}(j/n)^{\gamma}$ for all $(i,j,n) \in J_K$, while for all $i,j,n \in \mathbb{N}$ $nH_n(i,j) \geq M''(j/n)^{\gamma-1}$. Put $M = M'K^2/M''$ to complete the proof of part (a). The proof of part (b) is similar. Q.E.D.

*Proof of Theorem* 3.2. It follows from Theorem 2.1. that, for all $(i,j,n) \in J_K$ and each $0 < \varepsilon < \frac{1}{2}$, uniformly for $k \in \{[\varepsilon j], [\varepsilon j]+1, \ldots, [(1-\varepsilon)j], j\}$ ,

$$P_n(0,k) \underset{(i,j,n)}{=} (1+0(1))\left[\frac{1}{\alpha^{\gamma}\Gamma(\gamma)n^{\gamma}} k^{\gamma-1}\exp\{-\frac{k}{\alpha n}\}\right]$$

and from Theorem 3.1, that

$$Q_n(i,j-k) \underset{(i,j,n)}{=} (1+0(1))\left[\frac{1}{\alpha n}\exp\{-\frac{1+j-k}{\alpha n}\}I_1(\frac{2}{\alpha n}\sqrt{i(j-k)})\sqrt{\frac{i}{j-k}}\right]$$

Combining these facts and

$$Q_n(i,0) \underset{(i,n)}{=} (1+0(1))\exp\{-\frac{i}{\alpha n}\}$$

one obtains

$$S_n(i,j,\varepsilon) \underset{(i,j,n)}{=} \frac{1}{n}(1+0(1))\left[\frac{1}{\alpha^{\gamma}\Gamma(\gamma)}\{\frac{j}{n}\}^{\gamma-1} \exp\{-\frac{i+j}{\alpha n}\} + \right.$$

$$+ \sum_{k=\varepsilon j}^{(1-\varepsilon)j} \frac{1}{\Gamma(\gamma)\alpha^{\gamma+1}n^{\gamma}} k^{\gamma-1}\sqrt{\frac{i}{j-k}}\, I_1\left(\frac{2}{\alpha n}\sqrt{i(j-k)}\right)\exp\left\{-\frac{i+j}{\alpha n}\right\}\Bigg]$$

Now by a tedious but straightforward analysis it is seen that for all $R \in \mathbb{N}$

$$S_n(i,j,\varepsilon) \underset{(i,j,n)}{=}$$

$$\frac{1}{n}(1+0(1))\left[\frac{1}{\alpha}(\sqrt{j/i})^{\gamma-1}\exp\left\{-\frac{i+j}{\alpha n}\right\}\sum_{r=0}^{R+1}\frac{\lambda(r-1,\varepsilon)}{r!\Gamma(\gamma+r)}\left\{\frac{\sqrt{ij}}{\alpha n}\right\}^{2r+\gamma-1}\right.$$

$$\left. + \frac{1}{\alpha^{\gamma+1}\Gamma(\gamma)n^{\gamma}}\exp\left\{-\frac{i+j}{\alpha n}\right\}V_n(i,j,R,\varepsilon)\right] ,$$

where

$$V_n(i,j,R,\varepsilon) = \sum_{r=R+1}^{\infty}\frac{i^{r+1}}{r!(r+1)!}\left\{\frac{1}{\alpha n}\right\}^{2r+1}\sum_{k=\varepsilon j}^{(1-\varepsilon)j} k^{\gamma-1}(j-k)^r$$

and where for $0 \leqslant \varepsilon < \frac{1}{2}$, $\lambda$ is such that

$$\int_{\varepsilon}^{1-\varepsilon} x^{\gamma-1}(1-x)^k dx = \lambda(k,\varepsilon)\frac{\Gamma(\gamma)\Gamma(k+1)}{\Gamma(\gamma+k+1)} , \qquad k = 0,1,\ldots .$$

$$\lambda(-1,\varepsilon) \equiv 1 .$$

This permits a comparison of $S_n(i,j,\varepsilon)$ and $H_n(i,j)$.

(i) $$\frac{1}{\alpha^{\gamma+1}\Gamma(\gamma)n^{\gamma}}\exp\left\{-\frac{i+j}{\alpha n}\right\}V_n(i,j,R,\varepsilon)/nH_n(i,j)$$

$$\leqslant M'''K\sum_{k=R+1}^{\infty}\frac{1}{k!(k+1)!}\left(\frac{K}{\alpha}\right)^{2k+1} \qquad \text{for all } (i,j,n) \in J_K.$$

(ii) $$(\sqrt{j/i})^{\gamma-1}\exp\left\{-\frac{i+j}{\alpha n}\right\}\sum_{r=0}^{R}\frac{\lambda(r-1,\varepsilon)}{r!\Gamma(\gamma+r)}\cdot\left\{\frac{1}{\alpha n}\sqrt{ij}\right\}^{2r+\gamma-1}/\alpha n H_n(i,j)$$

$$= \frac{1}{I_{\gamma-1}(\frac{2}{\alpha n}\sqrt{ij})} \sum_{r=0}^{R} \frac{\lambda(r-1,\varepsilon)}{r!\Gamma(\gamma+r)} \{\frac{1}{\alpha n}\sqrt{ij}\}^{2r+\gamma-1}.$$

Thus, by (3.9) and Lemma 3.2, the theorem follows. Q.E.D.

Considering the behavior of $I_{\gamma-1}(z)$ as $z \to 0+$, one finds with the aid of Theorem 2.1 that Remark 3.1 has an analogue.

**REMARK 3.2.** The conclusion of Theorem 3.2 is valid without the assumption $i \to \infty$. When $j,n \to \infty$, $j/n \leqslant K$, $i/j \to 0$, then $P_n(i,j)$ behaves as described in Theorem 2.1.

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