A NOTE ON SPECTRAL AND PRESPECTRAL OPERATORS

by

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ABSTRACT. Results known for spectral and prespectral operators in weakly complete Banach spaces are extended to Banach spaces with the Bessaga-Pelczyński property.

1. Preliminaries. In this note, by a Banach space we mean a non-zero complex Banach space. Because of varying terminology in literature, we give the following definition.

DEFINITION 1.1. Let $\Sigma$ be a $\sigma$-algebra of subsets of a non-empty set $S$ and $X$ a Banach space. If $\Gamma$ is a total set of $X^*$, the dual of $X$, and $E(\cdot)$ is a Boolean homomorphism of $\Sigma$ on a boolean algebra of projections on $X$, we say that $E(\cdot)$ is a spectral measure of class $(\Sigma,\Gamma)$ when $E(\Sigma)$ is norm bounded in the Banach algebra $B(X)$ of all bounded operators in $X$ and

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when $X^*E(\cdot)x$ is countably additive on $\Sigma$, for each $x \in X$ and $X^* \in \Gamma$.

When $E(\cdot)$ is a spectral measure of class $(\Sigma, X^*)$, by
Orlicz-Pettis' theorem, $E(\cdot)$ is countably additive on $\Sigma$
in the strong operator topology, and in this case $E(\cdot)$ is
called a spectral measure on $\Sigma$.

An operator $T \in B(X)$ is called a prespectral operator of
class $\Gamma$ if there is a spectral measure $E(\cdot)$ of class $(\Sigma_p, \Gamma)$
where $\Gamma$ is total in $X^*$ and $\Sigma_p$ is the $\sigma$-algebra of all Borel
sets in $\mathcal{C}$, such that for each $\delta$ in $\Sigma_p$, $E(\delta)T = TE(\delta)$ and
the spectrum of $T$ restricted to $E(\delta)X$ is contained in the
closure of $\delta$. Then $E(\cdot)$ is called a resolution of the identity of class $\Gamma$ for $T$.

A prespectral operator $T$ of class $X^*$ is called a spectral operator. In this case, $T$ has a unique resolution of
the identity.

DEFINITION 1.2. A Banach space $X$ is said to have the
Bessaga-Pelczyński (B-P) property if every weakly unconditionally convergent series in $X$ is strongly unconditionally convergent.

LEMMA 1.3. Let $\Omega$ be a compact $T_2$ space and $X$ a Banach
space with the B-P property. Then every bounded linear
transformation $T: C(\Omega) \to X$ is weakly compact, where $C(\Omega)$
is the Banach algebra of all complex continuous functions
on $\Omega$.

Proof. By theorem 5 of Bessaga and Pelczyński [1], the
hypothesis on $X$ is equivalent to the fact that $X$ does not contain an isomorphic copy of $c_0$. Now the lemma follows from Theorem 15, p.160 of Diestel and Uhl [2].
2. Main results. In what follows \( X \) will denote a complex Banach space with the B-P property. Since a weakly complete Banach space has the B-P property (see [1]), all the results in this section hold for weakly complete Banach spaces also.

**Theorem 2.1.** Let \( A \) be an algebra of operators in \( X \) and let it be the image under a continuous homomorphism \( \phi \) of the Banach algebra \( C(\Omega) \) of all complex functions on a compact Hausdorff space \( \Omega \). Then there exists a uniquely determined spectral measure \( E(\cdot) \) in \( X \) on the Borel sets of \( \Omega \), for which

\[
\phi(f) = \int f(\lambda)E(d\lambda), \quad f \in C(\Omega).
\]

**Proof.** For a fixed \( x \in X \), consider the map \( T_x : C(\Omega) \to X \), defined by \( T_x f = \phi(f)x \). Since \( T_x \) is a continuous linear transformation and \( X \) has the B-P property, by Lemma 1.3 we have that \( T_x \) is weakly compact. Now following the same argument as in the proof of Theorem 2.5 of Chapter XVII of Dunford and Schwartz [4], the present theorem is established.

We use the terms complete and \( \sigma \)-complete boolean algebra of projections on \( X \) in the sense of Definition 3.1 of Chapter XVII of Dunford and Schwartz [4].

**Lemma 2.2.** A strongly closed bounded boolean algebra of projections in \( X \) is complete.

**Proof.** The proof of Lemma 3.5 of Chapter XVII of Dunford and Schwartz [4] holds here, except that we appeal to Theorem 2.1. above, instead of Theorem 2.5 of Chapter XVII of [4].

**Theorem 2.3.** A bounded boolean algebra of projections on
X is complete if and only if it is strongly closed.

Proof. The theorem follows from Lemma 2.2. above and from Corollary 3.7 of Chapter XVII of [4].

COROLLARY 2.4. Every bounded boolean algebra of projections \( B \) on \( X \) can be embedded in a \( \sigma \)-complete boolean algebra of projections on \( X \), contained in \( \bar{B}^S \) (the closure in the strong operator topology).

Proof. When \( B \) is norm bounded by a finite positive number \( M \), then it is easily verified that \( \bar{B}^S \) is a boolean algebra of projections and that \( \|E\| \leq M \), \( E \in \bar{B}^S \). Now the corollary follows from Theorem 2.3. ■

The above corollary leads to the following interesting result about prespectral operators.

THEOREM 2.5. Every prespectral operator \( T \) on \( X \) is a spectral operator.

Proof. Let \( T \) be a prespectral operator of class \( \Gamma \) on \( X \), with \( E(\cdot) \) a resolution of the identity of class \( \Gamma \). Then by Definition 1.1 and by Corollary 2.4 there exists a \( \sigma \)-complete boolean algebra \( B \) of projections on \( X \), which contains the range of \( E(\cdot) \) on \( \Sigma_p \). If \( \{\delta_n\} \) is a sequence of mutually disjoint sets in \( \Sigma_p \), then \( F_n = \sum_{k=1}^{n} E(\delta_k) \) is in \( B \) and \( \{F_n\} \) is a non-decreasing sequence of projections in \( B \). Therefore, by Lemma 3.4 of Chapter XVII of [4], for each \( x \in X \)

\[
\lim_{n \to \infty} F_n x = (V_{F_n}) x.
\]

In other words,

\[
(1) \quad (V_{F_n}) x = \lim_{n \to \infty} \sum_{k=1}^{n} E(\delta_k) x, \quad x \in X.
\]
But, \( x^* E(\tau)x \) is countably additive for each \( x \in X \) and \( x^* \in \Gamma \) and therefore from (1), we have

\[
(2) \quad x^* (VF_n)x = \sum_{k=1}^{\infty} x^* E(\delta_k)x = x^* (U\delta_k)x, \quad x \in X, \quad x^* \in \Gamma.
\]

Because \( \Gamma \) is total, (1) and (2) give that

\[
(3) \quad (VF_n)x = E(U\delta_k)x, \quad x \in X.
\]

From (1) and (3) it, follows that \( E(\tau) \) is countably additive in the strong operator topology. Hence \( T \) is spectral.

**THEOREM 2.6.** Let \( T_1 \) and \( T_2 \) be two commuting spectral operators on \( X \), with \( E_1(\tau) \) and \( E_2(\tau) \) as their respective resolutions of the identity and \( S_1 \) and \( S_2 \) as their respective scalar parts. If the boolean algebra of projections \( B \) determined by \( E_1(\tau) \) and \( E_2(\tau) \) is bounded, then \( T_1 + T_2 \) and \( T_1 T_2 \) are spectral with \( S_1 + S_2 \) and \( S_1 S_2 \) as their respective scalar parts and \( G_1(\tau) \) and \( G_2(\tau) \) as their respective resolutions of the identity, where

\[
G_1(\delta)x = \int E_2(\delta - \lambda)E_1(d\lambda)x, \\
G_2(\delta)x = \int E_2(\delta / \lambda)E_1(d\lambda)x
\]

for each \( x \in X \) and \( \delta \in \Sigma \).

**Proof.** Let \( T_1 = S_1 + N_1 \) and \( T_2 = S_2 + N_2 \) be the canonical decompositions of \( T_1 \) and \( T_2 \). Then by the generalized Fuglede-des theorem (Corollary 3.7, Chapter XV of [4]) \( S_1, S_2, N_1 \) and \( N_2 \) commute with each other and hence \( N_1 + N_2 \) and \( N_1 S_2 + N_2 S_1 + N_1 N_2 \) are quasi-nilpotent. Now the theorem follows from Theorem 4.5 of Chapter XV of [4], Corollary 2.4 above.
and Theorem 10 of Panchapagesan [5].

**REMARK 1.** In view of theorems 2.1 and 2.3 and corollary 2.4 of this paper, then theorems 6.10, 6.12, 6.14, 6.21, 6.22, and note 6.15 of Dowson [3], and corollaries 2.12, 2.13, 2.14, theorems 3.18, 3.19, 3.20, and corollary 3.28 of Chapter XVII of Dunford and Schwartz [4], which are known to be true for weakly complete Banach spaces, continue to be valid for Banach spaces with the B-P property.

It is also possible to extend the theorems of characterisations of spectral operators in weakly complete Banach spaces to operators in Banach spaces with the B-P property. To this end we need the following theorem.

**THEOREM 2.7.** Let \( E(\cdot): R \to B(X) \) be a Boolean homomorphism of the algebra of sets \( R \) on a boolean algebra of projections on \( X \) and let \( E(\cdot) \) be countably additive in the strong operator topology on \( R \), in the sense that whenever \( \{\delta_n\} \) is a pairwise disjoint sequence of members of \( R \) with \( \bigcup_{n \in \mathbb{N}} \delta_n \in R \), then

\[
E(\bigcup_{n \in \mathbb{N}} \delta_n)x = \sum_{n \in \mathbb{N}} E(\delta_n)x, \quad x \in X.
\]

Then a necessary and sufficient condition so that \( E(\cdot) \) can be extended uniquely to a spectral measure \( \overline{E}(\cdot) \) on \( S(R) \), the \( \sigma \)-algebra generated by \( R \), is that the range of \( E(\cdot) \) is norm bounded in \( B(X) \).

**Proof.** If such an extension \( \overline{E}(\cdot) \) on \( S(R) \) exists, then as in the second part of the proof of Corollary 3.10 of Chapter XVII of [4] we see that \( \overline{E}(S(R)) \) is a \( \sigma \)-complete boolean algebra of projections on \( X \) and hence is norm bounded by Lemma 3.3 of Chapter XVII of [4]. Conversely, if
E(R) is norm bounded in B(X), then E(R) is bounded boolean algebra of projections on X. Hence from Theorem 8 of Panchapagesan [5] and Corollary 2.4 above, the sufficiency of the condition follows.

**REMARK 2.** As a consequence of the above theorem, then theorems 3.14, 4.5, 5.2, and 5.15 of Chapter XVI of Dunford and Schwartz [4], which are known to be true for weakly complete Banach spaces, continue to be valid for Banach spaces with the B-P property.

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**REFERENCES**


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