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A NOTE ON SPECTRAL AND PRESPECTRAL OPERATORS

by

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ABSTRACT. Results known for spectral and prespectral operators in weakly complete Banach spaces are extended to Banach spaces with the Bessaga-Pelczyński property.

1. Preliminaries. In this note, by a Banach space we mean a non-zero complex Banach space. Because of varying terminology in literature, we give the following definition.

DEFINITION 1.1. Let Σ be a σ-algebra of subsets of a non-empty set S and X a Banach space. If Γ is a total set of X^*, the dual of X, and $E(\cdot)$ is a Boolean homomorphism of Σ on a boolean algebra of projections on X, we say that $E(\cdot)$ is a *spectral measure of class* (Σ,Γ) when $E(\Sigma)$ is norm bounded in the Banach algebra B(X) of all bounded operators in X and

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when $X^*E(\cdot)x$ is countably additive on Σ, for each $x \in X$ and $x^* \in \Gamma$.

When $E(\cdot)$ is a spectral measure of class (Σ, X^*), by Orlicz-Pettis' theorem, $E(\cdot)$ is countably additive on Σ in the strong operator topology, and in this case $E(\cdot)$ is called a *spectral measure on* Σ.

An operator $T \in B(X)$ is called a *prespectral operator of class* Γ if there is a spectral measure $E(\cdot)$ of class (Σ_p, Γ) where Γ is total in X^* and Σ_p is the σ-algebra of all Borel sets in $\mathbb{C}$, such that for each δ in Σ_p, $E(\delta)T = TE(\delta)$ and the spectrum of T restricted to $E(\delta)X$ is contained in the closure of δ. Then $E(\cdot)$ is called a *resolution of the identity of class* Γ *for* T.

A prespectral operator T of class X^* is called a *spectral operator*. In this case, T has a unique resolution of the identity.

DEFINITION 1.2. A Banach space X is said to have the *Bessaga-Pelczyński* (B-P) property if every weakly unconditionally convergent series in X is strongly unconditionally convergent.

LEMMA 1.3. *Let* Ω *be a compact* T_2 *space and* X *a Banach space with the* B-P *property. Then every bounded linear transformation* $T: C(\Omega) \to X$ *is weakly compact, where* $C(\Omega)$ *is the Banach algebra of all complex continuous functions on* Ω .

<u>*Proof*</u>. By theorem 5 of Bessaga and Pelczyński [1], the hypothesis on X is equivalent to the fact that X does not contain an isomorphic copy of c_0. Now the lemma follows from Theorem 15, p.160 of Diestel and Uhl [2].

2. <u>Main results</u>. In what follows X will denote a complex Banach space with the B-P property. Since a weakly complete Banach space has the B-P property (see [1]), all the results in this section hold for weakly complete Banach spaces also.

THEOREM 2.1. *Let* A *be an algebra of operators in* X *and let it be the image under a continuous homomorphism* ϕ *of the Banach algebra* $C(\Omega)$ *of all complex functions on a compact Hausdorff space* Ω. *Then there exists a uniquely determined spectral measure* $E(\cdot)$ *in* X *on the Borel sets of* Ω, *for which.*

$$\phi(f) = \int_{\Omega} f(\lambda)E(d\lambda), \qquad f \in C(\Omega).$$

Proof. For a fixed $x \in X$, consider the map $T_x : C(\Omega) \to X$, defined by $T_x f = \phi(f)x$. Since T_x is a continuous linear transformation and X has the B-P property, by Lemma 1.3 we have that T_x is weakly compact. Now following the same argument as in the proof of Theorem 2.5 of Chapter XVII of Dunford and Schwartz [4], the present theorem is established. ∎

We use the terms *complete* and *σ-complete* boolean algebra of projections on X in the sense of Definition 3.1 of Chapter XVII of Dunford and Schwartz [4].

LEMMA 2.2. *A strongly closed bounded boolean algebra of projections in* X *is complete.*

Proof. The proof of Lemma 3.5 of Chapter XVII of Dunford and Schwartz [4] holds here, excepto that we appeal to Theorem 2.1. above, instead of Theorem 2.5 of Chapter XVII of [4].

THEOREM 2.3. *A bounded boolean algebra of projections on*

X is complete if and only if it is strongly closed.

Proof. The theorem follows from Lemma 2.2. above and from Corollary 3.7 of Chapter XVII of [4].

COROLLARY 2.4. *Every bounded boolean algebra of projections B on X can be embedded in a σ-complete boolean algebra of projections on X, contained in* $\bar{B}^s$ *(the closure in the strong operator topology).*

Proof. When B is norm bounded by a finite positive number M, then it is easily verified that $\bar{B}^s$ is a boolean algbera of projections and that $\|E\| \leq M$, $E \in \bar{B}^s$. Now the corollary follows from Theorem 2.3. ∎

The above corollary leads to the following interesting result about prespectral operators.

THEOREM 2.5. *Every prespectral operator T on X is a spectral operator.*

Proof. Let T be a prespectral operator of class Γ on X, with E(•) a resolution of the identity of class Γ. Then by Definition 1.1 and by Corollary 2.4 there exists a σ-complete boolean algebra B of projections on X, which contains the range of E(•) on Σ_p. If $\{\delta_n\}$ is a sequence of mutually disjoint sets in Σ_p, then $F_n = \sum_{k=1}^{n} E(\delta_k)$ is in B and $\{F_n\}$ is a non-decreasing sequence of projections in B. Therefore, by Lemma 3.4 of Chapter XVII of [4], for each $x \in X$

$$\lim_n F_n x = (\bigvee_1^\infty F_n)x.$$

In other words,

$$(1) \qquad (\bigvee_1^\infty F_n)x = \lim_{n\to\infty} \sum_1^n E(\delta_k)x, \qquad x \in X.$$

But, $x^*E(\cdot)x$ is countably additive for each $x \in X$ and $x^* \in \Gamma$ and therefore from (1), we have

$$(2) \qquad x^*(\bigvee_1^\infty F_n)x = \sum_{k=1}^\infty x^*E(\delta_k)x = x^*E(\bigcup_1^\infty \delta_k)x, \quad x \in X, \quad x^* \in \Gamma.$$

Because Γ is total, (1) and (2) give that

$$(3) \qquad (\bigvee_1^\infty F_n)x = E(\bigcup_1^\infty \delta_k)x, \qquad x \in X.$$

From (1) and (3) it, follows that $E(\cdot)$ is countably additive in the strong operator topology. Hence T is spectral.

THEOREM 2.6. *Let* T_1 *and* T_2 *be two commuting spectral operators on* X, *with* $E_1(\cdot)$ *and* $E_2(\cdot)$ *as their respective resolutions of the identity and* S_1 *and* S_2 *as their respective scalar parts. If the boolean algebra of projections* B *determined by* $E_1(\cdot)$ *and* $E_2(\cdot)$ *is bounded, then* T_1+T_2 *and* T_1T_2 *are spectral with* S_1+S_2 *and* S_1S_2 *as their respective scalar parts and* $G_1(\cdot)$ *and* $G_2(\cdot)$ *as their respective resolutions of the identity, where*

$$G_1(\delta)x = \int E_2(\delta-\lambda)E_1(d\lambda)x ,$$

$$G_2(\delta)x = \int E_2(\delta/\lambda)E_1(d\lambda)x$$

for each $x \in X$ *and* $\delta \in \Sigma_p$.

Proof. Let $T_1 = S_1+N_1$ and $T_2 = S_2+N_2$ be the canonical decompositions of T_1 and T_2. Then by the generalized Fugledes theorem theorem (Corollary 3.7, Chapter XV of [4]) S_1, S_2, N_1 and N_2 commute with each other and hence N_1+N_2 and $N_1S_2+N_2S_1+N_1N_2$ are quasi-nilpotent. Now the theorem follows from Theorem 4.5 of Chapter XV of [4] , Corollary 2.4 above and Theorem 10 of Panchapagesan [5]. ∎

REMARK 1. In view of theorems 2.1 and 2.3 and corollary 2.4 of this paper, then theorems 6.10, 6.12, 6.14, 6.21, 6.22, and note 6.15 of Dowson [3], and corollaries 2.12, 2.13, 2.14, theorems 3.18, 3.19, 3.20, and corollary 3.28 of Chapter XVII of Dunford and Schwartz [4], which are known to be true for weakly complete Banach spaces, continue to be valid for Banach spaces with the B-P property.

It is also possible to extend the theorems of characterisations of spectral operators in weakly complete Banach spaces to operators in Banach spaces with the B-P property. To this end we need the following theorem.

THEOREM 2.7. *Let* $E(\cdot):R \to B(X)$ *be a Boolean homomorphism of the algebra of sets* R *on a boolean algebra of projections on* X *and let* $E(\cdot)$ *be countably additive in the strong operator topology on* R*, in the sense that whenever* $\{\delta_n\}$ *is a pairwise disjoint sequence of members of* R *with* $\bigcup_1^\infty \delta_n \in R$*, then*

$$E\left(\bigcup_1^\infty \delta_n\right)x = \sum_1^\infty E(\delta_n)x, \qquad x \in X.$$

Then a necessary and sufficient condition so that $E(\cdot)$ *can be extended uniquely to a spectral measure* $\overline{E}(\cdot)$ *on* $S(R)$*, the σ-algebra generated by* R*, is that the range of* $E(\cdot)$ *is norm bounded in* $B(X)$.

Proof. If such an extension $\overline{E}(\cdot)$ on $S(R)$ exists, then as in the second part of the proof of Corollary 3.10 of Chapter XVII of [4] we see that $\overline{E}(S(R))$ is a σ-complete boolean algebra of projections on X and hence is norm bounded by Lemma 3.3 of Chapter XVII of [4]. Conversely, if

E(R) is norm bounded in B(X), then E(R) is bounded boolean algebra of projections on X. Hence from Theorem 8 of Panchapagesan [5] and Corollary 2.4 above, the sufficiency of the condition follows.

REMARK 2. As a consecuence of the above theorem, then theorems 3.14, 4.5, 5.2, and 5.15 of Chapter XVI of Dunford and Schwartz [4], which are known to be true for weakly complete Banach spaces, continue to be valid for Banach spaces with the B-P property.

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