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ω-CLOSED MAPPINGS*

by

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ABSTRACT. In this paper the concepts of ω-closed set, ω-closed mapping and P^*-spaces are defined and the following are the main results: (a) Let f be a continuous ω-closed mapping of a space X onto a space Y such that $f^{-1}(y)$ is Lindelöf for each Y' in Y. Then X is Lindelöf if Y is so. (b) Let f be a continuous ω-closed mapping of a regular space X onto a space Y. Then X is paracompact (strongly paracompact) if Y is paracompact (strongly paracompact) and for each y in Y, $f^{-1}(y)$ is paracompact relative to X (Lindelöf). (c) Let X be a Lindelöf space and Y be a P^*-space, then the projection $P:X\times Y \to Y$ is an ω-closed mapping. Hence, X×Y is Lindelöf (paracompact, strongly paracompact) if and only if Y is so.

RESUMEN. Se introducen las nociones de conjunto ω-cerrado, función ω-cerrada y espacio P^*, generalizando las de conjunto cerrado, función cerrada y espacio P (donde todo G_δ es abierto), respectivamente. Se demuestra que las imagenes inversas de funciones continuas ω-cerradas preservan (a) La propiedad de Lindelöf en caso de que cada fibra sea Lindelöf,

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(b) paracompacidad (paracompacidad fuerte) si el dominio es regular y cada fibra es relativamente paracompacta (Lindelöf). Si X es Lindelöf y Y es un espacio P^*, entonces la proyección $X\times Y \to Y$ es ω-cerrada y por tanto: $X\times Y$ es Lindelöf (paracompacto, fuertemente paracompacto) sí y sólamente si Y lo es.

1. Introductión. In this paper we shall introduce a new kind of mappings, namely ω-closed mappings, which are strictly weaker than closed mappings, then we show that the Lindelöf property is preserved by counter images of ω-closed mappings with Lindelöf counter images of points. Also we show that the paracompactness (strong paracompactness) property is preserved by taking counter images of ω-closed mappings with regular domains, if the inverse image of each point in the range is paracompact relative to the domain (Lindelöf, respectively).

Secondly, we define the concept of P^*-space as a generalization of P-space, then we show that if X in a Lindelöf space, the projection $P:X\times Y \to Y$ is ω-closed for any P^*-space Y. Also we use P^*-spaces to obtain some product theorems concerning Lindelöf (paracompact, strongly paracompact) spaces.

Finally we discuss some counter examples relevant to the given definitions and theorems.

2. Preliminaries. In general, we follow closely the notions, set theoretical terminology and topological conventions used by Engelking [2]. Cetain other conventions are explained in this section. For any set X, $|X|$ denotes the cardinal number of X.

2.1. DEFINITION. (Aull [1]). A subset F of a space (X,τ) is called *paracompact relative to* X, if every open cover of F by members of τ has a locally finite refinement in X by members of τ.

Recall that a point x of a space X is called a *condensation point* of the set $A \subset X$ if an arbitrary neigborhood (nbd) of the point x contains an uncountable subset of this set.

2.2. DEFINITION. A subset of a space X is called ω-*closed* if it contains all its condensation points. The complement of an ω-closed set is called ω-*open* set, also $Cl^{\omega}A$ will denote the intersection of all ω-closed sets which contains A.

Observe that A is ω-open if and only if for every $x \in A$ there is an open nbd U of x with $|U \setminus A| \leq \omega$.

2.3. DEFINITION. A mapping $f:X \to Y$ is called a ω-*closed* mapping if it maps closed sets onto ω-closed sets.

A mapping $f:X \to Y$ is called *Lindelöf* mapping if for each Lindelöf closed subset K of Y, $f^{-1}(K)$ is Lindelöf.

2.4. DEFINITION. (Gillman and Jerison [3]). A space X is called P-*space* if and only if the intersection of countably many open sets is an open set.

2.5. DEFINITION. A space X is called a P^{*}-*space* if the intersection of countably many open sets is an ω-open set.

3. ω - closed mappings.

3.1. THEOREM. (i) *An ω-closed subset of a Lindelöf space is Lindelöf.*
(ii) *If* $f:X \to Y$ *is a continuous mapping from* X *onto* Y, *then the following are equivalent:* (a) f *is ω-closed;* (b) *for each* $y \in Y$ *and any open set* U *such that* $f^{-1}(y) \in U$, *there exists an ω-open set* O_y *such that* $y \in O_y$ *and* $f^{-1}(O_y) \subset U$.
(iii) *Every Lindelöf, ω-open subset* A *of a space* X *is of the form* $G \setminus B$, *where* G *is open and* B *is a countable set, in particular* A *is a* G_δ*-set.*

Proof. The proof of (i) and (ii) is an easy consequence of the definition.

(iii) For each $x \in A$ there is a nbd U_x of x such that $|U_x \cap (X-A)| \leq \omega$. Now $\{U_x \mid x \in A\}$ is an open cover of A so it has a countable subcover $U_1, U_2, \ldots$; $A \subset \bigcup_{i=1}^{\infty} U_i$, where $|U_i \cap (X-A)| \leq \omega$, for each $i = 1,2,\ldots$, Now $U_i \cap (X-A) = \bigcup_{m=1}^{\infty}\{x_{i,m}\}$. Therefore,

$$A = \cup |U_i \setminus \bigcup_{m=1}^{\infty}\{x_{i,m}\}| = \bigcup_{i=1}^{\infty}(U_i \setminus B) \quad \text{where}$$

$$B \subset \bigcup_{m=1}^{\infty}\{x_{i,m}\}. \blacksquare$$

3.2. COROLLARY. *Let* X *be hereditary Lindelöf space. Then every ω-open subset of* X *is a* G_δ*-set. In particular every ω-open subset of the real line is a* G_δ*-set.*

The converse of part (iii) of theorem 3.1 is not true. For example take $A = [a,b) \subset R$, then A is a Lindelöf, G_δ-set but A is not ω-open. The following theorem is a generalization of the well known theorem that the Lindelöf property is preserved under taking counter images by closed continuous mappings with Lindelöf counter images of points.

3.3. **THEOREM.** *Let* f *be a continuous* ω-*closed mapping of a space* X *onto a space* Y *such that* $f^{-1}(y)$ *is Lindelöf for each* y *in* Y. *Then* X *is Lindelöf if* Y *is so.*

Proof. Let $\underset{\sim}{U} = \{U_\alpha \mid \alpha \in \Lambda\}$ be an open cover of X. Since $f^{-1}(y)$ is Lindelöf, $f^{-1}(y) \subset \bigcup_{i=1}^{\infty} U_{\alpha_i}$. Denote $O_y = Y - f(X - \bigcup_{i=1}^{\infty} U_{\alpha_i})$. Since f is ω-closed, O_y is ω-open for each $y \in Y$, so there exists an open nbd O_y' of y such that $[O_y' \cap (X - O_y)] \leqslant \omega$. Now $O_y' = [O_y \cap O_y'] \cup [O_y' \cap (X - O_y)]$. Therefore $f^{-1}(O_y')$ is contained in a union of countably many members of $\underset{\sim}{U}$. Since $\{O_y' \mid y \in Y\}$ is an open cover of Y and Y is Lindelöf, $\{O_y' \mid y \in Y\}$ has a countable subcover. Therefore, X is the union of countably many members of $\{f^{-1}(O_y') \mid y \in Y\}$, since each $f^{-1}(O_y')$ is contained in the union of countably many members of $\underset{\sim}{U}$. Consequently, X is the union of countably many members of $\underset{\sim}{U}$. Hence, X is Lindelöf. ■

3.4. **THEOREM.** *Let* f *be* ω-*closed continuous mapping of a regular space* X *onto a space* Y'

(i) *If* Y *is paracompact and* $f^{-1}(y)$ *is paracompact relative to* X *for each* y *in* Y, *then* X *is paracompact.*

(ii) *If* Y *is strongly paracompact and* $f^{-1}(y)$ *is Lindelöf for each* y *in* Y, *then* X *is strongly paracompact.*

Proof. (i) Let $\underset{\sim}{U}$ be an open cover of X. It sufficies to show that $\underset{\sim}{U}$ has a σ-locally finite refinement. Since $f^{-1}(y)$ is paracompact relative to X, for each y in Y, $\underset{\sim}{U}$ has an open locally finite refinement in X which cover $f^{-1}(y)$, say $\underset{\sim}{A}_y = \{A \mid \alpha \in \Lambda_y\}$. Denote $O_y = Y - f(X - \bigcup_{\alpha \in \Lambda_y} A_\alpha)$. Since f is ω-closed, O_y is ω-open for each y in Y. Hence there exists an open nbd O_y' of y such that $|O_y' \cap (X - O_y)| \leq \omega$. Put $O_y \cap O_y' = G_y$, $O_y' \cap (X - O_y) = H_y$. Then $f^{-1}(H_y)$ is contained in the union of a σ-locally finite refinement of $\underset{\sim}{U}$ whose members are open in X. Also $f^{-1}(G_y) \subset \bigcup_{\alpha \in \Lambda_y} A_\alpha$. Therefore $f^{-1}(O_y')$ is covered by a σ-locally finite refinement B_y of $\underset{\sim}{U}$ whose members are open in X. Since Y is paracompact, $\{O_y' \mid y \in Y\}$ han an open locally finite refinement V which covers Y. Let $S = \{f^{-1}(V) \cap B \mid f^{-1}(V) \subset f^{-1}(O_y'), B \in B_y, V \in \underset{\sim}{V}\}$. It is easy to see that $\underset{\sim}{S}$ is an open σ-locally finite refinement of $\underset{\sim}{U}$.

The proof of (ii) can be obtained by a similar method (one uses the characterization of strongly paracompactness in terms of star countable refinements).

3.5. COROLLARY. (i) (Ponomarev [8]). *Let* f *be a closed continuous mapping of a regular space* X *onto a strongly paracompact space* Y *such that* $f^{-1}(y)$ *is Lindelöf for each* y *in* Y. *Then* X *is strongly paracompact.*

(ii) (Hanai [4]). *Let* f *be a closed continuous mapping of a regular space* X *onto a paracompact space* Y *such that for each* y *in* Y, $f^{-1}(y)$ *is compact. Then* X *is paracompact.*

3.6. THEOREM. *Let* $f: X \to Y$ *be continuous mapping from* X *onto* Y, *where* Y *is locally Lindelöf Hausdörff* P^**-space. Then the following are equivalent:*

(i) f *is an ω-closed mapping and for each* $y \in Y$, $f^{-1}(y)$ *is Lindelöf;*

(ii) *f is a Lindelöf mapping.*

Proof. (i) ⇒ (ii) follows from theorem 3.3. (ii) ⇒ (i), let $f:X \to Y$ be a Lindelof continuous mapping, where Y is a locally Lindelöf, Hausdorff, P^*-space. It sufficies to show that f is ω-closed. Let F be a closed subset of X. Assume that f(F) is not ω-closed so there exists a point $y_0 \in Y \setminus f(F)$ such that for every nbd V of y_0, $|V \cap f(F)| > \omega$. Since Y is locally Lindelöf, there is an open nbd G of y_0 such that $C\ell G$ is Lindelof. Observe now $f(F) \cap C\ell G$ is not Lindelöf. Indeed, if it is, then it is easy to see that it is ω-closed, so there exists a nbd M of y_0 such that $|M \cap f(F)| \leq \omega$, which is impossible. Now $C\ell G$ is Lindelöf so $f^{-1}(C\ell G)$ is Lindelöf and $F \cap f^{-1}(C\ell G)$ is a Lindelöf subset of X. Therefore $f|F \cap f^{-1}(C\ell G)| = f(F) \cap C\ell G$ is Lindelöf which is a contradiction. Hence f(F) is ω-closed.

3.7. **THEOREM.** *Let* X *be a Lindelöf space and* Y *be a* P^**-space. Then the projection* $p:X\times Y \to Y$ *is an* ω*-closed mapping.*

Proof. Let $y \in Y$ and U be any open set in X×Y such that $p^{-1}(y) = X\times\{y\} \in U$. For each $(x,y) \in X\times\{y\}$, let O_x and $O_{y(x)}$ be open nbds of x and y such that $(x,y) \in O_x \times O_{y(x)} \subset U$. Now $\{O_x \mid x \in X\}$ is an open cover of X, therefore it has a countable subcover $\{O_{x_i}\}_{i=1}^{\infty}$. Hence $X\times\{y\} \subset \bigcup_{i=1}^{\infty} O_{x_i} \times O_{y(x_i)} \subset U$. Let $O_y = \bigcap_{i=1}^{\infty} O_{y(x_i)}$ then $X\times\{y\} \subset \bigcup_{i=1}^{\infty} O_{x_i} \times O_y \subset U$ and O_y is an ω-open set since Y is a P^*-space. Thus, for each y in Y, there is an ω-open set O_y such that $y \in O_y$ and $p^{-1}(O_y) \subset U$. Therefore by theorem 3.1, P is an ω-closed mapping.

3.8. **THEOREM.** *Let* Y *be a topological space such that* *there exists an F_σ-set which is not ω-closed and let X be a topological space. If the projection $p:X\times Y \to Y$ is ω-closed, then X is countably compact.*

<u>*Proof*</u>. Let $\bigcup_{i=1}^{\infty} A_i$ be an F -subset of Y which is not ω-closed. Let X be not countably compact. Then there exists a decreasing sequence $\{B_i\}_{i=1}^{\infty}$ of closed subsets of X such that $\bigcap_{i=1}^{\infty} B_i = \emptyset$. Let $F = \bigcup_{i=1}^{\infty}(A_i \times B_i)$, then we can see that F is a closed subset of X×Y. Now, for every point (x,y) in X×Y, p(x,y) = y. Then $P(F) = \bigcup_{i=1}^{\infty} A_i$ is not ω-closed. Therefore the projection is not ω-closed. Hence the result.

3.9. THEOREM. *A space Y is a P^*-space if and only if for any Lindelöf space X the projection $p:X\times Y \to Y$ is an ω-closed mapping.*

<u>*Proof.*</u> The necessity part follows from theorem 3.7. For the sufficiency of the condition, assume that Y is not a P^* and for any Lindelöf space X the projection $p:X\times Y \to Y$ is ω-closed. Let X = ℝ, the set of real numbers with the usual topology, then by theorem 3.8, X is countably compact which is a contradiction.

3.10. THEOREM. *Let $X = \prod_{\alpha\in n} X_\alpha$, where n is infinite, if for each α_0 in n the projection $\pi_\alpha: \prod_{\alpha\leq\alpha_0} X_\alpha \longrightarrow \prod_{\alpha<\alpha_0} X_\alpha$ is ω-closed, then for any α_0 the projection $\pi^{\alpha_0}: X \longrightarrow \prod_{\alpha<\alpha_0} X_\alpha$ is also ω-closed.*

<u>*Proof.*</u> Let $A \subset X$ be such that $\pi^{\alpha_0}(A)$ is not ω-closed we want to show that A is not closed. Choose a point x^0 in $Cl^\omega(\pi^{\alpha_0}(A)) \setminus \pi^{\alpha_0}(A)$ and for each $\alpha < \alpha_0$, let x_α be the α^{th} coordinate of x^0. Let $\beta \geqslant \alpha_0$ and suppose inductively that for each $\alpha < \beta$, x_α has been chosen so that

letting x^{β} to be the point (x_{α}) of $\prod_{\alpha<\beta} X_{\alpha}$, we have that x^{β} is in the $Cl^{\omega}(\pi^{\beta}(A))$. Now $\pi_{\beta}(Cl^{\omega}\pi^{\beta+1}(A))$ is a set which contains $\pi^{\beta}(A)$, hence there exists a point x_{β} in X such that (x_{β},x^{β}) is in $Cl^{\omega}(\pi^{\beta+1}(A))$. Thus we construct inductively a point $x\in X$ which is not in A (since $\pi^{\alpha_0}(x) = x^0 \notin A$) and such that for each $\beta\in n$, $\pi^{\beta}(x)\in Cl^{\omega}(\pi^{\beta}(A))$. Clearly x must be in $Cl^{\omega}(A)$ therefore, A is not ω-closed.

3.11. THEOREM. *Let* X *and* Y *be two spaces each with the property that every Lindelöf subset is ω-closed. If* $f:X \to Y$ *and* f *(considered as a subspace of* X×Y*) is Lindelöf, then* f *is weakly continuous (i.e., for every open set* $U \subset Y$, $f^{-1}(U)$ *is ω-open).*

Proof. Let $p_1:X\times Y \to X$ and $p_2:X\times Y \to Y$ be the projections, then X and range f are Lindelöf sets, as images of Lindelöf sets under P_1 and P_2. Let $P_1^* = P_1 \restriction f$. Observe that P_1^* is ω-closed. Indeed, if A f is closed, then A is Lindelöf, so $P_1^*(A)$ is Lindelöf, hence it is ω-closed. Since f is a function defined on X, P_1^* is a bijection onto X. This, toghether with the fact that P_1^* is ω-closed, implies that for every open set $V\subset f$, $P_1^*(V)$ is ω-open in X. Now $f = P_2 \circ P_1^{*-1}$, hence f has the required property.

4. Product Theorems. In this section we use the results from the previous sections to obtain some product theorems for Lindelof (paracompact, strongly paracompact) spaces.

4.1. THEOREM. *Let* X *be a Lindelof space and* Y *be a* P^**-space, Then the following holds:*

(i) $X\times Y$ *is Lindelof if and only if* Y *is so,*

(ii) $X\times Y$ *is paracompact if and only if* Y *is so,*

(iii) $X\times Y$ *is strongly paracompact if and only if* Y *is so.*

Proof. The proof follows from theorem 3.7, 3.3 and 3.4.

4.2. **THEOREM.** *Let* $\underset{\sim}{X} = \{X_n\}_{n=1}^{\infty}$ *be a family of Hausdorff Lindelöf* P^**-spaces and* $X = \prod_{n=1}^{\infty} X_n$. *If each finite subproduct of* X *is a* P^**-space, then* X *is a Lindelöf space.*

Proof. Let X_{n_o} be an element of $\underset{\sim}{X}$. Then $(\prod_{i=1}^{n-1} X_i)\times X_{n_o}$ is a Lindelöf P^*-space by theorem 4.1. Therefore the projection $(\prod_{i=1}^{n} X_i)\times X_{n_o} \to (\prod_{i=1}^{n-1} X_i)\times X_{n_o}$ is ω-closed. Hence by theorem 3.10 the projection $p: X\times X_{n_o} \to X_{n_o}$ is ω-closed. Therefore by theorems 3.7 and 3.6 X is a Lindelöf space.

4.3. **COROLLARY.** (Noble [7]). *The product of countably many Hausdorff-Lindelöf P-spaces is Lindelöf.*

5. Counterexamples.

In this section we discuss various counterexamples relevant to the definition and theorems in the previous sections. We shall start with examples concerning the ω-closed mappings.

5.1. **EXAMPLE.** Given a topological space (X,τ_1) we define a new topology τ on X as follows: $G \in \tau$ if and only if G is an ω-open set in τ_1, then τ will be an exapansion of τ_1 and therefore will be $T_{2\frac{1}{2}}$ and Urysohn if τ_1 is so. Now the identity mapping $f:(X,\tau) \to (X,\tau_1)$ will be an ω-closed mapping which is not closed.

In the special case when (X,τ_1) is hereditary Lindelöf, we have by theorem 3.1 that $G \in \tau$ if and only if $G = U \setminus B$ where U is open in X and B is a countable subset of X. If we define on X the cocountable topology τ_2 which is obtained by taking countable subsets of X as closed sets, then one can show in this case where X is hereditary Lindelöf that the topology τ is the smallest topology generated by $\tau_1 \cup \tau_2$. Also (X,τ) is Lindelöf, indeed if $\{U_\alpha - B_\alpha \mid \alpha \in \Lambda\}$ is an open cover of X ($U_\alpha \in \tau_1$, B is countable for each α), then $\{U_\alpha \mid \alpha \in \Lambda\}$ covers X and has a countable subcover $\{U_{\alpha_i}\}_{i=1}^{\infty}$. Therefore $\{U_{\alpha_i} \setminus B_{\alpha_i}\}_{i=1}^{\infty}$ covers all but countably many point of X, so all of X can be covered by some countable subcollection of $\{U_\alpha - B_\alpha \mid \alpha \in \Lambda\}$.

The above example shows that there exists a continuous ω-closed mapping from a Lindelöf space X onto a Lindelöf space Y which is not closed. Thus indeed theorem 3.3 is more general than the one in which the map would be assumed to be closed.

5.2. **EXAMPLE.** Let f be a mapping from a discrete countable space X onto the space of rationals Y. Then f is a continuous ω-closed mapping. However, f is not closed. Also for each y in Y, $f^{-1}(y)$ is Lindelöf and X,Y are Lindelöf spaces, consequently strong paracompact (hence paracompact) spaces. Thus indeed theorem 3.4 is more general than the in which the mapping would be assumed to be closed.

5.3. We shall now turn to P^*-spaces. Observe that any space without any condensation point is a P^*-space but not a P-space. According to the above any countable space is a P^*-space. As an example of an uncountable P^*-space we can mention the space of countable ordinals as well as

$N \cup \underset{\sim}{R}$ (see [6]). By $N \cup \underset{\sim}{R}$ we mean a space which is defined on the set theoretic union of a countable set N and an almost disjoint collection $\underset{\sim}{R}$ of subsets of N in the following way: each point of N is isolated and $\lambda \in R$ has a nbd basis $\{\{\lambda\} \cup (\lambda - F) \mid F \text{ is a finite subset of } N\}$. $N \cup \underset{\sim}{R}$ is first countable, locally compact and 0-dimensional. Also $N \cup \underset{\sim}{R}$ has no condensation point therefore it is a P^*-space but clearly not a P-space.

The following example is discussed in connection with theorem 4.1 and 4.2.

5.4. EXAMPLE. Consider the Soregenfrey plane S×S where S is the Sorgenfrey line. It is known that S is Lindelöf (hence paracompact and strongly paracompact) but S×S is not normal (thus not paracompact and certainly not strongly paracompact).

6. Remarks on Generalizations. In this section we state some rather straighforward generalizations of the previous results to higher cardinalities. For this purpose we recall the following definition: a space X is *[k,∞]-compact* if and only if every open cover has a subcover with cardinality $\leqslant k$. Now we generalize the definitions of ω -closed subsets, ω -closed mapping and P^*-spaces. A subset A of a space X is called *k-open* if and only if for every $x \in A$ there is an open nbd U of x with $|U \setminus A| \leqslant k$, and a set B is *k-closed* if it is the complement of an k-open set. A mapping $f: X \to Y$ is called k-closed if it maps closed sets onto k-closed sets. A space X is called P^*-space if

and only if the intersection of any family of open set with cardinality at most k is k-open.

Using the above definition and similar methods to those used before, we can generalize some of the previous results to higher cardinalities. For instance, theorem 3.3 will become: let $f:X \to Y$ be a continuous k-closed mapping of a space X onto a space Y such that $f^{-1}(y)$ is $[k,\infty]$-compact for each y in Y. Then X is $[k,\infty]$-compact if Y is so. Theorem 3.7 will become: Let X be an $[k,\infty]$-compact space and Y be a P^*-space, then the projection $p:X\times Y \to Y$ is k-closed mapping. Part (i) of theorem 4.1 will become: Let X be an $[k,\infty]$-compact space and Y be P^*-space. Then $X\times Y$ is $[k,\infty]$-compact if and only if Y is so.

We will conclude with the following observation: the definition of k-closed set a particular case of a schema considered in Kuratowski [5]: let P be a hereditary and additive property of sets, for a given set A of a space X; A^0, the set of P-*condensation points*, is the set of all points x such that for every nbd U of x, $U \cap A$ does not have P; A is called P-closed if $A^0 \subset A$. P-*closed* maps are defined in an obvious way. If P is the property *to be of cardinality* $\leqslant$ k, it is easy to verify that k-closed sets coincide with P-closed sets.

Observe now that theorem 3.3 states that the counter image of a Lindelöf under continuous P-closed mapping with Lindelöf fibers (= counter images of points) is Lindelöf provided that P is the property: *to be of the cardinality* $\leqslant \omega$.

Contrary to what could be expected the above fails if

P is the property to be Lindelöf. A counter example is obtained by taking the identity mapping of the reals with the discrete topology onto the reals with the standard topology.

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