RANDOM FIXED POINT THEOREMS FOR
ULTIMATELY COMPACT OPERATORS

by

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RESUMEN. La clase de los operadores finalmente compactos en el sentido de Sadovski contiene las clases de operadores condensantes, compactos y contractivos. Se deducen teoremas de punto fijo para operadores estocásticos finalmente compactos superiormente semicontinuos, usando el grado de Leray-Schauder y sus generalizaciones a opera-
dores determinísticos.

ABSTRACT. Ultimately compact operators in the sense of Sadovski contain the classes of condensing, of com-
pact and contractive operators. Fixed-point theorems are derived for upper semicontinuous ultimately compact stochastic operators using the Leray-Schauder degree and its generalizations for deterministic operators.

Introduction. An appropriate starting point for stoch-

* This work is partially supported by the Deutscher Akademis-
cher Austauschdienst, D-5300 Bonn.
tence problems for differential equations under Carathéodory-
conditions (see Coddington, Levinson [4] and Engl [7]). The
 corresponding problems for multivalued differential equations
lead to the consideration of random fixed-points for stochastic
multifunctions $T(\cdot, \cdot) : W \times X \to 2^X$, where $X$ is a separable Banach
space.

If $(w, \cdot)$ is a continuous operator with respect to the Haus-
dorff distance in $2^X$ for each $w \in W$, the problem has been solved
by Kannan and Salehi [11] and by Engl [7, Theorem 6]. Their the-
orem says that $T$ always has a random fixed-point if the corre-
sponding deterministic operator $T(w, \cdot)$ has a fixed-point for
each $w \in W$.

However, most fixed-point theorems and the Leray-Schauder
degree for multifunctions refer to the larger class of upper
semicontinuous (u.s.c.) multifunctions. The main difficulty
that arises here is that generally the operator $T(\cdot, \cdot)$ is not
jointly measurable on $W \times X$. For compact u.s.c. stochastic opera-
tors, Engl derived in [7, Theorem 16] a random version of the
Schauder-Kakutani fixed-point theorem.

In our present article we do not need the compactness of
$T(w, \cdot)$ and can so derive fixed-point theorems for ultimately com-
pact u.s.c. random operators. This gives us for example the
stochastic version of the theorem of Krasnoselski for the sum of
a compact and a contractive multifunction.

A survey about the development of problems and theorems in
this area until 1976 may be found in the publication of Bharucha-
Reid [2]. We do not treat here measurability of solutions of
equations of the type $Lu + Nu = 0$ where $L$ is a random linear opera-
tors and $N$ a random nonlinear operator (see Kannan and Salehi
[12]).
§1. Basic definitions and properties.

DEFINITION 1. (a) Let $X$ always be a real separable Banach space. We denote by

\[ P(X) := \{ M \subset X ; M \neq \emptyset \} \]

\[ B(X) := \{ M \subset X ; M \neq \emptyset \text{ bounded} \} \]

\[ A(X) := \{ M \subset X ; M \text{ closed}, M \neq \emptyset \} \]

\[ C(X) := \{ M \subset X ; M \text{ convex}, M \neq \emptyset \} \]

\[ K(X) := \{ M \subset X ; M \text{ compact}, M \neq \emptyset \} \]

\[ O(X) := \{ M \subset X ; M \text{ open}, M \neq \emptyset \} \]

\[ KC(X) := K(X) \cap C(X), \text{ and analogously other combinations.} \]

(b) Let $(W,A)$ always be measurable space, where $A$ is a $\sigma$-algebra of subsets of $W$. $(W,A,\mu)$ means a $\sigma$-finite measure space, where $\mu : A \to [0,\infty]$ is a $\sigma$-additive function with $\mu(\emptyset) = 0$. By $B$ we denote the $\sigma$-algebra of Borel subsets of $X$.

DEFINITION 2. Let $C : W \to P(X)$ be a multifunction.

(a) $\tilde{C} : W \to A(X)$ is defined by $\tilde{C}(w) = \overline{C(w)}$, $w \in W$.

(b) $C$ is measurable iff for each open $D \subset X$ we have

\[ \{ w \in W : C(w) \cap D \neq \emptyset \} \in A. \]

(c) $C$ is separable iff $C$ is measurable and there exists a countable subset $Z \subset X$ with $C(w) = \overline{Z \cap C(w)}$ for all $w \in W$.

(d) $Gr(C) := \{(w,x) \in W \times X ; x \in C(w)\}$, the graph of $C$.

LEMMA 3. Let $C : W \to P(X)$ be a multifunction.

(a) If $C(w)$ is independent of $w$, then $\tilde{C}$ is measurable.

(b) If $W$ is countable and $C$ measurable, then $C$ is separable.

(c) If $C$ is measurable, $\text{int}C(w) \neq \emptyset$ for all $w \in W$, $C(w) = \overline{\text{int}C(w)}$ for all $w \in W$, then $C$ is separable.

(d) If $C : W \to O(X)$ is measurable, then $\tilde{C}$ is separable.

Proof. (a) and (b) are obvious, (c) follows from the demon-
stratification of proposition 4 in [7], (d) is an immediate consequence of (c) and proposition 2.6 in [9] (Our measurable multifunctions are called meakly measurable in [9]).

**DEFINITION 4.** Let \( S \subseteq X \) and \( f : S \to P(X) \) be a multifunction, 
(a) \( f \) is u.s.c. on \( S \) if and only if for each \( x \in S \) and each open \( V \supset f(x) \) there exists an open neighborhood \( U \) of \( x \) with \( f(U \cap S) \subseteq V \).
(b) \( f \) is closed on \( S \) if and only if for each sequence \( x_n \in S \), \( x_n \to x \in S \) and \( y_n \in f(x_n) \), \( y_n \to y \in X \), we have \( y \in f(x) \).

**LEMMA 5.** Let \( S \subseteq X \) and \( f : S \to P(X) \) a multifunction.
(a) If \( S \) is a closed subset of \( X \), we have the following equivalence: \( f \) is closed iff \( Gr(f) \) is a closed subset of \( X \times X \).
(b) If \( f \) is u.s.c. and \( f(x) \) closed for each \( x \in S \), then \( f \) is closed.
(c) If \( f \) is closed and \( f(\{x, x_1, x_2, x_3, \ldots \}) \) is relatively compact for each convergent sequence \( x_n \to x \) \((n \to \infty)\) with \( x_n \in S \), \( x \in S \), then \( f \) is u.s.c.
(d) \( f \) is u.s.c. iff \( \{x \in S \mid f(x) \cap A \neq \emptyset \} \) is a closed subset of \( S \) for each closed \( A \) of \( X \).
(e) If \( f \) is closed on \( S \), then \( f(L) \) is a closed subset of \( X \) for each compact subset \( L \) of \( S \).
(f) Assume that \( f \) is u.s.c. on \( S \), that \( L \) is a compact subset of \( S \), and that \( f(x) \) is relatively compact for each \( x \in S \). Then \( f(L) \) is a relatively compact set.
(g) Let \( f : S \to K(X) \) be u.s.c. Then \( f(L) \) is compact for each compact \( L \subseteq S \).

**Proof.** (a) obvious.
(b) We assume \( x_n \to x \), \( x_n \in S \), \( x \in S \), \( y_n \in f(x_n) \), \( y_n \to y \in X \). If \( V \) is an open neighborhood of \( f(x) \), then there exists another open neighborhood \( U \) of \( x \) such that \( f(U \cap S) \subseteq V \). This implies \( f(x_n) \subseteq \bar{V} \) for all \( n > n_0 \), or \( y_n \in \bar{V} \) for all \( n > n_0 \). Then it follows that \( y \in \bar{V} \), and finally \( y \in f(x) \), since \( f(x) = \bigcap \{\bar{V} \mid V \text{ open}, V \supset f(x) \} \).
(c) If \( f \) were not u.s.c. then there would exist a \( x \in S \) and an open \( V \supset f(x) \) such that for each open neighborhood \( U \) of \( x \) we should have \((X \setminus V) \cap (U \cap S) \neq \emptyset\). Then there exists \( x_n \in S \) with \( \|x_n - x\| < \frac{1}{n} \) and an element \( y_n \in f(x_n) \) with \( y_n \notin V \). We may assume \( y_n \to y \) for some \( y \in X \). Then we should \( y \notin V \) and therefore \( y \notin f(x) \). So our assumption leads to a contradiction.

(d) See ([1], p.115).

(e) Let \( y_n \in f(L) \), \( y_n \to y \). Therefore \( y_n \in f(x_n) \) for some \( x_n \in L \). We may assume \( x_n \to x \in L \). From the closedness of \( f \) it follows that \( y \in f(x) \subseteq f(L) \).

(f) Considering a sequence \( y_n \in f(L) \), we have \( y_n \in f(x_n) \) for some \( x_n \in L \), and without loss of generality we again assume \( x_n \to x \in L \). \( f \) is u.s.c., therefore \( d(y_n, f(x)) \to 0 \) \((n \to \infty)\) and so there exists a sequence \( u_n \in f(x) \) such that \( \|y_n - u_n\| \to 0 \). We again take \( u_n \to u \) for some \( u \in f(x) \). This means \( y_n \to u \).

(g) By (a) and (e) the set \( f(L) \) is closed, and by (f) we know that \( f(L) \) is relatively compact. Therefore \( f(L) \) is compact, (see also [1]).

**REMARK.** In the preceding lemma we need not the separability of \( X \).

**DEFINITION 6.** Let \( C:W \to P(X) \) be measurable and \( T(\cdot, \cdot):\text{Gr}(C) \to P(X) \) a multifunction.

(a) \( T \) is called a **stochastic** (or random) **operator** if and only if
\[
\{w \in W ; x \in C(x), T(w, x) \cap D \neq \emptyset\} \in \mathcal{A}
\]
for each \( x \in X \) and for each open \( D \subseteq X \).

(b) A function \( x(\cdot):W \to X \) is called a **stochastic** (or random) **fixed-point** of \( T \) if and only if

1. \( x(\cdot) \) is a \((\mathcal{A}, \mathcal{B})\)-measurable function, \( x(w) \in C(w) \) for all \( w \in W \);
2. \( x(w) \in T(x, x(w)) \) for \( \mu \)-almost all \( w \in W \).
(c) T is called \textit{u.s.c. stochastic operator} if and only if

(1) T is stochastic operator,
(2) \( T(w,\cdot):C(w) \to \mathbb{P}(X) \) is u.s.c. on \( C(w) \) for each \( w \in W \).

§2. Construction of a jointly measurable multifunction \( H \). If for a stochastic operator \( T(\cdot,\cdot) \) there exists an element \( x(w) \in C(w) \) with \( x(w) \in T(w,x(w)) \) for each \( w \in W \), then it does not necessarily exists a stochastic fixed-point of \( T \). For a counterexample see \([8]\) or \([7]\). More regularity properties of \( T \) are required.

Unfortunately, an u.s.c. stochastic operator \( T \) is not jointly measurable with respect to both variables \( (w,x) \). A counterexample may be found in \([7]\). But we need such a property in our demonstrations. So we pass to another u.s.c. stochastic operator \( H(\cdot,\cdot) \) which additionally is jointly measurable. The idea for the construction of this new operator \( H \) stems from the proof of the well-known fact that a function \( g(\cdot,\cdot) \) is jointly measurable if it satisfies a Carathéodory-condition. That means \( g \) has to be measurable with respect to \( w \) and continuous with respect to \( x \) (see Scorza-Dragoni \([17]\) and Neubrunn \([13]\)). This idea was successfully modified and applied in \([6]\). Despite of \( H(w,x) \subseteq T(w,x) \) we can show that we do not loose too many fixed-points replacing \( T \) by \( H \).

\textbf{DEFINITION 7.} Let \( A \) and \( B \) be two nonempty subsets of \( X \)

(a) For \( x \in X \) we denote by \( d(x,B) := \inf\{\|x-b\|; b \in B\} \), the \textit{distance}
of \( x \) to \( B \).
(b) \( e(A,B) := \sup\{d(x,B); x \in A\} \) is called the \textit{excess} of \( A \) over \( B \) where the supremum is taken in \([0,\infty]\).
(c) The Hausdorff distance of A and B is defined by \( D(A,B) := \max\{d(A,B), d(B,A)\} \).

We refer to [3, chapter II, §1] for elementary properties.

**Proposition 8.** Let \((W,A)\) be a measurable space, \(X\) a separable real Banach space, \(C:W \rightarrow A(X)\) separable, \(Z\) like in definition 2(c), \(T:Gr(C) \rightarrow KC(X)\) a u.s.c. stochastic operator, and for \(x \in C(w)\),

\[
H(w,x) := \bigcap_{n \in \mathbb{N}} \text{conv} \left\{ \bigcup_{z \in Z \cap C(w)} T(w,z) \, : \, \|z-x\| < \frac{1}{n} \right\}.
\]

Then this so defined multifunction has the following properties:

(a) \(H(w,x) \subseteq T(w,x)\) for all \((w,x) \in Gr(C)\)

(b) \(H(w,x) = T(w,x)\) for all \((w,x) \in Gr(C), x \in Z \cap C(w)\)

(c) \(H(w,x) \neq \emptyset\) for all \((w,x) \in Gr(C)\)

(d) \(H(w,\cdot):C(w) \rightarrow KC(X)\) is u.s.c. for each \(w \in W\)

(e) \(H(\cdot,\cdot)\) is \((A \times B, B)\)-measurable.

**Proof.** For fixed \(n \in \mathbb{N}\) and \((w,x) \in Gr(C)\) we set \(T_N(w,x) := \bigcup_{z \in Z \cap C(w)} T(w,z) \, : \, \|z-x\| < \frac{1}{N} \). By \(H_N\) we denote the closure of the convex hull of \(T_N\), \(H_N := \text{conv} T_N\). So we have

\[
H_N(w,x) = \bigcap_{n \in \mathbb{N}} H_N(w,x).
\]

Clearly, \(T_N(w,x) \neq \emptyset\) for all \((w,x) \in Gr(C)\).

In the demonstration we will omit the variables \((w,x)\) if confusion is not possible.

(a) Let \(\varepsilon > 0\) be given and \(U_\varepsilon(Tx) := \{y \in X \, : \, d(y,Tx) < \varepsilon\} \). There exists a \(N \in \mathbb{N}\) such that \(Tz \subseteq U_\varepsilon(Tx)\) for all \(z \in C(w)\) with \(\|z-x\| < \frac{1}{N}\). This implies \(T_N(w,x) \subseteq U_\varepsilon(Tx)\). By the convexity of \(U_\varepsilon(Tx)\) we can conclude \(H_N(w,x) = \text{conv} T_N \subseteq U_\varepsilon(Tx) \subseteq U_{2\varepsilon}(Tx)\) and so \(H(w,x) \subseteq U_{2\varepsilon}(Tx)\). This means \(H(w,x) \subseteq \bigcap_{\varepsilon > 0} U_{2\varepsilon}(Tx) = T(w,x)\).

The last equality is a consequence of the closedness of \(T(w,x)\).

(b) For \(x \in Z \cap C(w)\) we have \(T(w,x) \subseteq T_N(w,x) \subseteq H_N(w,x)\) for each \(N \in \mathbb{N}\), therefore \(T(w,x) \subseteq H(w,x)\), and by (a) equality holds.

(c) By construction, the set \(H(w,x)\) is convex and closed.
and, therefore by (a), compact. For all $n \in \mathbb{N}$ we already know that $T_n(w,x) \neq \emptyset$, $(w,x) \in \text{Gr}(C)$. We choose $y_n \in T_n(w,x)$ and can find a $z_n \in Z \cap C(w)$ with $\|z_n-x\| < \frac{1}{n}$ and $y_n \in T(w,x_n)$. Therefore $\lim_{n \to \infty} z_n = x$ and $\bigcup_{n} \{y_n\} \subset \bigcup_{n} T(z_n)$, where the last set is relatively compact by Lemma 5(g). Again we take without loss of generality $\lim_{n \to \infty} y_n = y$ for some $y \in X$. For the moment we fix $N \in \mathbb{N}$ and get for all $n > N$: $y_n \in T_n \subset \text{conv}T_n = H_N$. But $H_N$ is closed, si $y \in H_N$. Making this conclusion for each $N \in \mathbb{N}$ gives us finally $y \in H(w,x)$.

(d) $H(w,x) \in \text{KC}(X)$ is evident by the preceding observations. Applying lemma 5(c) we show that $H(w,\cdot)$ is u.s.c. on $C(w)$. For a compact subset $L$ of $C(w)$ we deduce from (a) that $H(L) = \bigcup_{x \in L} H(x)$ $\subset \bigcup_{x \in L} T(x) = T(L)$. By lemma 5(g) the set $T(L)$ is compact and therefore $H(L)$ relatively compact. The only thing still to show is the closedness of the map $H(w,\cdot)$. Let be $x_n \in C(w)$, $x \in C(w)$, $\lim_{n \to \infty} x_n = x$, $y_n \in H(x_n)$, $y \in X$, $\lim_{n \to \infty} y_n = y$. For fixed $N \in \mathbb{N}$ it exists a $n_0$ such that $\|x_n-x\| < \frac{1}{2N}$ for all $n > n_0$. For $n > n_0$ we have $y_n \in H(x_n) \subset H_2N(x_n) \subset H_N(x)$ because $\|z-x\| \leq \|z-x_n\| + \|x_n-x\|$ for all $z \in Z \cap C(w)$. The set $H_N(x)$ is closed, therefore $\lim_{n \to \infty} y_n = y \in H_N(x)$. Thus $y \in \bigcap_{N \in \mathbb{N}} H_N(x) = H(x)$.

(e) The multifunction $T_n(\cdot,\cdot):\text{Gr}(C) \to P(X)$ is $(A \times B, B)$-measurable. For a demonstration see the first part of the proof of proposition 5(3) in [6]. We conclude from the proposition 2.6 and theorem 9.1 in [9] that $T_n$ and also $H_n = \text{conv}T_n = \text{conv}T_n$ are $(A \times B, B)$-measurable multifunctions on $\text{Gr}(C)$. Taking in the moment for granted that $\lim_{n \to \infty} d(x,H_n(w,y)) = d(x,H(w,y))$ for $x \in X$, $(w,y) \in \text{Gr}(C)$, we can bring to an end the proof of (e) as follows: since $H_n(\cdot,\cdot)$ is measurable we have that $d(H_n(\cdot,\cdot))$ is measurable for each $x \in X$. Applying once more theorem III,9 in [3] gives the measurability of $H(\cdot,\cdot)$. For the rest of the proof we firstly show

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lim \( e(T_n(w,x), H(w,x)) = 0 \), \( (w,x) \in \text{Gr}(C) \). \hspace{1cm} (1)

If (1) is not valid then there exists a \( \varepsilon_0 > 0 \) such that without loss of generality \( e(T_n, H) > 2\varepsilon_0 \) for all \( n \in \mathbb{N} \). Then there exist \( y_n \in T_n \), \( d(y_n, H) > 2\varepsilon_0 \) and \( y_n \in T(z_n) \) for some \( z_n \in Z \cap C(w) \) with \( \|z_n - x\| < \frac{1}{n} \). So \( \bigcup_{n \in \mathbb{N}} \{y_n\} \subseteq \bigcup_{n \in \mathbb{N}} T(z_n) \), and \( \bigcup_{n \in \mathbb{N}} \{y_n\} \) is relatively compact by lemma 5(g). We may assume \( \lim_{n \to \infty} y_n = y \) for some \( y \in X \).

By (d) and lemma 5(b) it follows \( y \in H(w,x) \). This is a contradiction to \( d(y_n, H) > 2\varepsilon_0 \) for all \( n \in \mathbb{N} \). Secondly, let us show

\[
\lim_{n \to \infty} e(H_n, H) = 0 , \quad (w,x) \in \text{Gr}(C). \hspace{1cm} (2)
\]

For any given \( \varepsilon > 0 \) there exists in view of (1) a \( n_0 \) with \( e(T_n, H) < \frac{\varepsilon}{2} \) for all \( n > n_0 \), and, as a consequence, we have \( T_n \subseteq U_{\varepsilon/2}(H) \) for all \( n > n_0 \) and also \( H_n = \text{conv} T_n \subseteq U_{\varepsilon}(H) \) for all \( n > n_0 \), since \( U_{\varepsilon/2}(H) \) as \( \varepsilon/2 \)-neighborhood of the convex \( H \) is convex, too. Observing \( e(U_{\varepsilon}(H), H) \leq \varepsilon \) and [3, page 38] one obtains \( e(H_n, H) \leq e(H_n, U_{\varepsilon}(H)) + e(U_{\varepsilon}(H), H) \leq \varepsilon \) for all \( n > n_0 \). Thirdly we get

\[
\lim_{n \to \infty} D(H_n, H) = 0 , \quad (w,x) \in \text{Gr}(C), \hspace{1cm} (3)
\]

using result (2) and \( H \subseteq H_n \) which implies \( e(H, H_n) = 0 \). Now the desired result follows at once of (3) and the inequality

\[ |d(x, H_n) - d(x, H)| \leq D(H_n, H). \]

**Lemma 9.** Let be \( (\Omega, F) \) a measurable space, \( R: \Omega \to A(X) \) a measurable multifunction, \( r: \Omega \to X \) a measurable function. Then \( d(r(\cdot), R(\cdot)): \Omega \to \mathbb{R} \) is measurable.

**Proof.** Lemma 6 in [6].

**Theorem 10.** Let \( (W, A, \mu) \) be a \( \sigma \)-finite measure space, \( X \) a real separable Banach space, \( C: W \to A(X) \) separable, \( T: \text{Gr}(C) \to \text{KC}(X) \) a u.s.c. stochastic operator, \( H \) like in propo-
sition 8, and $H(w) := \{x \in C(w) ; x \in H(w,x)\} \neq \emptyset$ for all $w \in W$. Then,

(a) there exists a stochastic fixed-point $x(\cdot):W \to X$ of $H$ and $T$;
(b) if in addition $(w,A,\mu)$ is a complete measure space then

$H:W \to A(X)$ is measurable, and there exists a stochastic fixed-point $x(\cdot):W \to X$ which fulfills $x(w) \in H(w,x(w)) \subset T(w,x(w))$ for all $w \in W$.

**Proof.** For $(w,x) \in Gr(C)$ define $\tilde{x}(w,x) := x$. It is easily verified that this function $\tilde{x}(\cdot,\cdot):Gr(C) \to X$ is $(A \times B,B)$-measurable. By proposition 8(e) and lemma 9 the function $N(w,x) := d(\tilde{x}(w,x),H(w,x)) = d(x,H(w,x))$ is $(A \times B,B(R))$-measurable. Furthermore

$$Gr(H) = \{(w,x) ; x \in C(w), x \in H(w,x)\}
= \{(w,x) \in Gr(C) ; d(x,H(w,x)) = 0\}
= N^{-1}(0) \in A \times B.$$

(a) Apply the theorem of Aumann (Theorem 5.2 in [9]) to the multifunction $H$.

(b) By proposition 8 (d) the multifunction $H(w,\cdot)$ is u. s.c. This implies $H(w) = \overline{H(w)}$, and so the measurability of $H$ by theorem 3.5 in [9]. Now we apply the theorem of Kuratowski, Ryll-Nardzewski (Theorem 5.1 in [9]).

Now we will show the existence of a random fixed-point when $T$ is a continuous stochastic operator. The second part of the following theorem has already been proven in Theorem 6 of [7].

**THEOREM 11.** Let $(W,A,\mu)$ be a $\sigma$-finite measure space, $X$ a real separable Banach space, $C:W \to A(X)$ separable, $T:Gr(C) \to AB(X)$ a continuous stochastic operator, that is

$$\lim_{n \to \infty} D(T(w,x_n),T(w,x)) = 0$$

for each $w \in W$ and for each sequence $x_n \in C(w)$ with $\lim_{n \to \infty} x_n = x$. Then it follows that

(a) $T$ is $(A \times B,B)$-measurable.
§3. Random fixed-point theorems for ultimately compact stochastic operators.

DEFINITION 12. Let $D \neq \emptyset$ be a closed subset of the Banach space $X$, $f:D \to P(X)$ a multifuction. We denote by $\alpha, \beta, \delta$ ordinal numbers. By transfinite induction we define the sets

$$f_0 := \text{conv } f(D)$$

$$f_\alpha := \text{conv } f(D \cap f\alpha^{-1})$$

if $\alpha^{-1}$ exists

$$f_\alpha := \bigcap_{\beta < \alpha} f_\beta$$

if $\alpha^{-1}$ does not exist,

which have the following well-known properties:

(a) each $f_\alpha$ is closed and convex
(b) $f_\alpha \subseteq f_\beta$ if $\alpha > \beta$, and hence $f(D \cap f_\beta) \subseteq \overline{\text{conv}}(D \cap f_\beta) = f_{\beta+1} \subseteq f_\beta$.

(c) There exists an ordinal number $\delta$ such that $f_\beta = f_\delta$ for all $\beta > \delta$. We denote this limit set $f_\delta$ by $f_\infty$. Thus $f(D \cap f_\infty) \subseteq \overline{\text{conv}}(D \cap f_\infty) = f_{\delta+1} = f_\delta = f_\infty$, and we have $f_\infty = \emptyset$ if and only if $D \cap f_\infty = \emptyset$.

(d) If $x \in f(x)$ then $x \in f_\infty$.

If in addition the mapping $f$ is u.s.c. on $D$ and if $f(D \cap f_\infty)$ is relatively compact, then $f$ is called ultimately compact. This means that the limit set $f_\infty$ is compact.

**Lemma 13.** Let $D \neq \emptyset$ be a closed subset of the Banach space $X$, $g,f:D \to A(X)$ u.s.c. multifunctions with $g(x) \subseteq f(x)$ for all $x \in D$ and $f$ ultimately compact. Then, (a) $g$ is ultimately compact and $g_\infty \subseteq f_\infty$, (b) $\emptyset \neq g_\infty \subseteq D$ if $g(D) \subseteq D$ and $f_\infty \neq \emptyset$.

**Proof.** (a) Obvious; for details of the demonstration see for example the proof of theorem 14(b).

(b) We have $\emptyset \neq D \cap f_\infty$ compact. Because $g(D \cap f_\infty) \subseteq f(D \cap f_\infty)$ and lemma 5(b)(e), the set $g(D \cap g_\infty)$ is compact, moreover $\emptyset \neq g(D \cap f_\infty) \subseteq D$. We defined $Q_0 := g(D \cap f_\infty) \subseteq f_\infty \cap D$, $Q_{n+1} := g(Q_n)$ for all $n \in \mathbb{N}$. This is a decreasing sequence of sets where each $Q_n$ is nonempty and compact by lemma 5(b), (e). Hence $Q := \bigcap_{n=1}^{\infty} Q_n$ is compact and nonempty. We will show now $Q \subseteq g(Q)$; let $x \in Q = \bigcap_{n=1}^{\infty} g(Q_{n-1})$, hence $x = g(q_n)$ for some $q_n \in Q_n$ for all $n \in \mathbb{N}$. We may assume $\lim_{n \to \infty} q_n = q$ for some $q \in Q_0$ by the compactness of $Q_0$. Obviously $q \in Q$. From lemma 5(b) it follows $x \in g(q) \subseteq g(Q)$. A usual conclusion with transfinite induction gives us $Q \subseteq g_\alpha$ for each ordinal number $\alpha$, and so $\emptyset \neq Q \subseteq g_\infty$.

**Theorem 14.** Assume that $D \neq \emptyset$ is an open subset of the Banach space $X$, $A := D$, $g,f:A \to KC(X)$ u.s.c. multifunct-
tions, \( g(x) \subseteq f(x) \) for all \( x \in A \), \( f \) ultimately compact, and
\[
h(t,x) := t g(x) + (1-t) f(x)
\]
for \((t,x) \in [0,1] \times A\). Then
(a) \( h: [0,1] \times A \rightarrow KC(X) \) is a u.s.c. multifunction,
(b) \( h \) is ultimately compact, \( h_\infty \subseteq f_\infty \),
(c) \( \deg(J-g,D,0) = \deg(J-f,D,0) \) if \( 0 \not\in (J-f)(\emptyset) \), where
\( \deg(\cdot,\cdot,\cdot) \) is the generalization of the Leray-Schauder degree
introduced by Petryshyn and Fitzpatrick in [15].

**Proof.** (a) \( h(t,x) \) is a convex, compact set for fixed
\((t,x) \in [0,1] \times A\). For an arbitrary compact set \( L \subseteq [0,1] \times A \) we ver-
ify the compactness of \( h(L) \). Let \( w_n \in h(L) \) with \( w_n \in h(t_n,x_n) =
\)
t\( g(x_n) + (1-t_n) f(x_n) \) for some \((t_n,x_n) \in L\) which implies \( w_n =
\)
t\( u_n + (1-t_n) v_n \) for some \( u_n \in g(x_n), v_n \in f(x_n) \). We may take
\((t_n,x_n) \rightarrow (t,x) \in L \) and by lemma 5(g) also \( u_n \rightarrow u, v_n \rightarrow v \). Applying
lemma 5(b) we get \( u \in g(x), v \in f(x) \). Hence \( w_n = t_n u_n + (1-t_n) v_n +
\)
t\( w \in g(x) + (1-t) f(x) = h(t,x) \). We now show that \( h \) is
closed. Therefore let \((t_n,x_n) \in [0,1] \times A, (t_n,x_n) \rightarrow (t,x), w_n \in
\)
h\((t_n,x_n), w_n \rightarrow w \). Since \( L = \{(t_n,x_n),(t,x) ; n \in \mathbb{N}\} \) is a com-

pact subset of \([0,1] \times A\), one proves as before that \( w \in h(t,x) \).

Finally, from lemma 5(c) the desired result follows.

(b) It suffices to show \( h_\alpha \subseteq f_\alpha \) for each ordinal number.
At first \( h(t,x) = t g(x) + (1-t) f(x) \subseteq f(x) \) for all \((t,x) \in [0,1] \times A\).
Therefore \( h_0 = \text{conv}(h([0,1] \times A)) \subseteq \text{conv}(f(A)) = f_0 \), and taking \( h_\beta \subseteq f_\beta \)
for all ordinals \( \beta < \alpha \) we deduce \( h_\alpha = \text{conv}(h([0,1] \times (A \cap h_{\alpha-1}))) \subseteq
\)
\( \text{conv}(h([0,1] \times (A \cap f_{\alpha-1}))) \subseteq \text{conv}(f(A \cap f_{\alpha-1})) = f_\alpha \) if \( \alpha-1 \) exists, and
\( h_\alpha = \bigcap_{\beta < \alpha} h_\beta \subseteq f_\beta = f_\alpha \) if \( \alpha-1 \) does not exist.

(c) \( 0 \not\in (J-f)(\emptyset) \) is equivalent to \( x \not\in f(x) \) for all \( x \in \emptyset \). So
\( x \not\in h(t,x) \) for all \((t,x) \in [0,1] \times \emptyset \). So \( \deg(J-h(t,\cdot),D,0) \) is well
defined for each \( t \in [0,1] \) and independent of \( t \) by (a), (b) and

theorem 2.2 of [15].

**Remark 15.** (a) If in the above theorem we have
deg(J-f,D,0) \neq 0, \text{ then } f_\infty \neq \emptyset \text{ and also } g_\infty \neq \emptyset.

(b) In the article [16], Sadovski gives on pages 137, 138 the example of two ultimately compact functions f,g which satisfy for h(t,x) = tg(x)+(1-t)f(x) the condition x \in h(t,x) \text{ for } x \in \Theta D, t \in [0,1], \text{ but for which nevertheless } deg(J-f,D,0) \neq deg(J-g,D,0). \text{ Therefore, in contrast to the case of compact or condensing operators, our theorem 14 is not evident.}

(c) For definition 12, lemma 13, and theorem 14 we obviously do not need separability of X.

DEFINITION 16. Let C:W \rightarrow P(X) be measurable and T(\cdot,\cdot):Gr(\tilde{C}) \rightarrow P(X) a multifunction. T is called an **ultimately compact stochastic operator** if and only if:

(i) T is a stochastic operator,

(ii) T(w,\cdot):\tilde{C}(w) \rightarrow P(X) is ultimately compact for each w \in W.

THEOREM 17. Let be C:W \rightarrow \Omega(X) measurable, T:Gr(\tilde{C}) \rightarrow \mathcal{K}(X) an ultimately compact stochastic operator, x \notin T(w,x) \text{ for all } x \in \Theta C(w), w \in W, \text{ and finally } deg(J-T(w,\cdot),C(w),0) \neq 0 \text{ for all } w \in W. \text{ Then there exists a random fixed-point of } T.

**Proof.** By lemma 3(d) the multifunction \tilde{C} is separable. We apply proposition 8 and obtain for each w \in W the u.s.c. multifunction H(w,\cdot):\tilde{C}(w) \rightarrow \mathcal{K}(X), H(w,x) \subseteq T(w,x). \text{ From theorem 14, it follows } deg(J-H(w,\cdot),C(w),0) \neq 0 \text{ for all } w \in W. \text{ Thus there exists for all } w \in W \text{ an element } x(w) \in C(w), x(w) \in H(w,x(w)). \text{ Using the notation of theorem 10, we have } H(w) \neq \emptyset \text{ for all } w \in W, \text{ and so there exists a random fixed-point of } T \text{ by the same theorem 10.}

COROLLARY 18. Let C:W \rightarrow \Omega(X) be measurable, each C(w) a symmetric neighborhood of the origin, T:Gr(\tilde{C}) \rightarrow \mathcal{K}(X) an odd ultimately compact stochastic operator, and x \notin T(w,x) \text{ for all }
\( x \in \mathcal{C}(w), w \in W. \) Then there exists a stochastic fixed point of \( T. \)

**Proof.** For each \( x \in \mathcal{C}(w) \) the operator \( T \) satisfies the condition \( T(w,x) = -T(w,-x). \) Hence, by theorem 2.4 of \([15]\), \( \deg(J-T(w,\cdot),C(w),0) \) is an odd integer for all \( w \in W. \) Now we use the above theorem 17.

**THEOREM 19.** Let \( C: W \to \mathcal{AC}(X) \) be separable, and \( T: \text{Gr}(C) \to KC(X) \) an ultimately compact stochastic operator with \( T(w,C(w)) \in \mathcal{C}(w) \) and \( T_{\infty}(w,\cdot) \neq \emptyset \) for all \( w \in W. \) Then \( T \) has a stochastic fixed-point.

**Proof.** Passing to the operator \( H \) in proposition 8, we get \( H: \text{Gr}(C) \to KC(X) \) as ultimately compact stochastic operator with \( H(w,C(w)) \subset C(w) \) and \( H_{\infty}(w,\cdot) \neq \emptyset \) for all \( w \in W, \) by lemma 13. Theorem 3.6 of \([15]\) guarantees \( H(w) \neq \emptyset, \) and so by theorem 10 there exists a random fixed-point of \( T. \)

§4. Special cases.

**DEFINITION 20.** (a) For a bounded subset \( B \) of the Banach space \( X \) we define the Kuratowski-measure of noncompactness:

\[
\chi(B) := \inf\{\varepsilon > 0 : B \text{ admits a finite covering by sets of diameter } \leq \varepsilon\},
\]

and the Hausdorff-measure of noncompactness:

\[
\gamma(B) := \inf\{\varepsilon > 0 : B \text{ admits a finite } \varepsilon \text{-ball covering}\}.
\]

(Fundamental properties of \( \chi, \gamma \) may be found in \([5, \text{p.19}])\).

(b) A multifunction \( f:D \to K(X), \emptyset \neq D \) closed subset of \( X, \) is said to be \( \chi \)-condensing if and only if \( f \) is u.s.c.
on \( D, \) maps bounded sets to bounded sets, and satisfies \( \chi(f(B)) < \chi(B) \) for each bounded \( B \subset D \) which is not relatively compact. A
corresponding definition holds for \( \gamma \)-condensing multifunctions.

(c) A multifunction \( f:D \to K(X), \emptyset \neq D \) closed subset of \( X \), is said to be compact if and only if \( f \) is u.s.c. on \( D \) and maps bounded subsets of \( D \) to relatively compact sets.

(d) A multifunction \( f:D \to K(X), \emptyset \neq D \) closed subset of \( X \), is said to be a contraction (with constant \( k \)) if and only if there exists a \( k \in (0,1) \) such that \( D(f(x), f(y)) \leq k \| x-y \| \) for all \( x,y \in D \).

We now list up some well-known relations between these properties.

**Lemma 21.** Let \( D \) be a nonvoid closed subset of the Banach space \( X \) and \( f,g:D \to KC(X) \) multifunctions.

(a) If \( f \) is compact, then \( f \) is \( X \)-condensing and \( \gamma \)-condensing.

(b) If \( f \) is \( \chi \)-condensing or \( \gamma \)-condensing, then \( f \) is ultimately compact.

(c) Let \( D \) be bounded, \( f:D \to KC(X) \) compact, \( g:X \to KC(X) \) a contraction. Then the sum \( f+g:D \to KC(X) \) is \( \gamma \)-condensing.

(d) Let \( D \) be bounded, \( f:D \to KC(X) \) compact, \( g:D \to X \) a single-valued contraction. Then \( f+g:D \to KC(X) \) is \( \chi \)-condensing.

(e) Let \( D \) be bounded, \( f:D \to KC(X) \) compact, \( g:D \to KC(X) \) a contraction with \( k < \frac{1}{2} \). Then \( f+g:D \to KC(X) \) is \( \chi \)-condensing.

**Proof.** (a) Obvious, (b) see lemma 3.2 in [15], (c), (d), (e) see remark 3.9 in [15].

**Definition 22.** Let \( C:W \to P(X) \) be measurable and \( T(\cdot,\cdot):Gr(C) \to K(X) \) a multifunction. Exactly the same as before we may define now a \( \chi \)-condensing, \( \gamma \)-condensing, compact or contrative stochastic operator \( T \).

**Theorem 23.** (Type Kakatani, Schauder, Rothe). Let
C:W → AC(X) be separable, T:Gr(C) → KC(X) a χ or γ-condensing random operator with T(w, ∂C(w)) ⊂ C(w) for all w ∈ W. Then T has a random fixed-point.

Proof. The operator H, constructed in proposition 8, satisfies H:Gr(C) → KC(X), H(w, x) ⊂ T(w, x), H(w, ∂C(w)) ⊂ C(w) for all w ∈ W, and is a χ or γ-condensing stochastic operator, too. Suppose that intC(w₀) = ∅ for w₀ ∈ W. This implies ∂C(w₀) = C(w₀) and so H(w₀, C(w₀)) ⊂ C(w₀). By corollary 3.5 of [15] there exists a x₀ ∈ C(w₀) with x₀ ∈ H(w₀, x₀), which signifies H(w₀) ≠ ∅. Otherwise, D := intC(w₁) ≠ ∅ for w₁ ∈ W, then C(w₁) = D, ∂D = ∂C(w₁), D convex, H(w₁, ∂D) ⊂ ∂D. We apply now corollary 3.4 of [15], thus obtaining an element x₁ ∈ C(w₁) with x₁ ∈ H(w₁, x₁), that is H(w₁) ≠ ∅. Now use theorem 10. Finally, we add in passing that one can easily generalize the corollary 3.4 in [15] to arbitrary open convex sets, so that it might indeed be applied to our slightly more general situation here.

THEOREM 24. Let C:W → O(X) be measurable, 0 ∈ C(w) for all w ∈ W; T:Gr(C) → KC(X) a χ or γ-condensing stochastic operator with λx ∈ T(w, x) for all x ∈ ∂C(w), λ ≥ 1, w ∈ W. Then T has a stochastic fixed-point.

Proof. By theorem 3.2 in [15] we have deg(J-T(w,·),C(w),0) = 1 for all w ∈ W. Because of lemma 21(b) and theorem 17 there exists a random fixed-point of T.

COROLLARY 25. (type Krasnoselski). Let C:W → AC(X) be separable and each C(w) bounded. Let G(·,·):Gr(C) → KC(X) be a compact random operator, let S(·,·):Gr(C) → X be single-valued contractive (or S(·,·):Gr(C) → KC(X) be contractive with k < 1/2, or S(·,·):W×X → KC(X) be a contractive random operator) and let T := S+G fulfill the condition T(w, ∂C(w)) ⊂ C(w) for all w ∈ W.
Then $T$ has a stochastic fixed-point.

**Proof.** By parts (c), (d), (e) of lemma 21 the stochastic operator $T$ is $\chi$ or $\gamma$-condensing. Applying theorem 23 gives the desired result.

**FINAL REMARKS.** (a) In [7], theorem 23 was proven for compact random operators and corollary 25 for a single-valued stochastic operator $T = S + G$ where $S: \text{Gr}(C) \to X$ is contractive and $G: \text{Gr}(C) \to X$ compact.

(b) One may deduce another corollary of Krasnoselski type based on theorem 24, replacing the condition $T(w, \partial C(w)) \subseteq C(w)$ of corollary 25 by $\lambda x \in T(w, x)$ for all $x \in \partial C(w)$, $\lambda \geq 1$, $w \in W$.

(c) If the Banach space $X$ satisfies the condition of Opial (see [14]), if each $C(w)$ is weakly compact, and if we require that $S$ is nonexpansive, $G$ completely continuous, then we can derive a further result similar to that of corollary 25. This generalizes corollary 18 in [6]. We will not present the details, a proof is obvious after observing corollary 3.9 in [15].

(d) In the preceding sections we have paid attention only to those fixed-point theorems which are consequence of the Leray-Schauder degree. Clearly there are also other fixed-point theorems which have more in common with modifications of the Banach fixed-point principle, and which have stochastic versions, too. For a survey see the article of Ivanov [10].

**ADDENDUM.**

(1) A special case of this article above was also treated in the publication of S. Itoh: Measurable and condensing multi-valued mappings and random fixed-point theorems, Kodai Math. J. 2(1979) 3, 293-299.

(2) The notion of a separable multifunction (see definition 2c) is closely related to the notion of a almost uniformly...

*LITERATURE*


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(Recibido en noviembre de 1981)