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RANDOM FIXED POINT THEOREMS FOR ULTIMATELY COMPACT OPERATORS

by

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RESUMEN. La clase de los operadores finalmente compactos en el sentido de Sadovski contiene las clases de operadores condensantes, compactos y contractivos. Se deducen teoremas de punto fijo para operadores estocásticos finalmente compactos superiormente semicontinuos, usando el grado de Leray-Schauder y sus generalizaciones a operadores determinísticos.

ABSTRACT. Ultimately compact operators in the sense of Sadovski contain the classes of condensing, of compact and contractive operators. Fixed-point theorems are derived for upper semicontinuous ultimately compact stochastic operators using the Leray-Schauder degree and its generalizations for deterministic operators.

Introduction. An appropriate starting point for stochastic operators is the abstract fixed-point formulation of exis-

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tence problems for differential equations under Carathéodory-conditions (see Coddington, Levinson [4] and Engl [7]). The corresponding problems for multivalued differential equations lead to the consideration of random fixed-points for stochastic multifunctions $T(\cdot,\cdot): W\times X \to 2^X$, where X is a separable Banach space.

If $(w,\cdot)$ is a continuous operator with respect to the Hausdorff distance in 2^X for each $w \in W$, the problem has been solved by Kannan and Salehi [11] and by Engl [7, Theorem 6]. Their theorem says that T always has a random fixed-point if the corresponding deterministic operator $T(w,\cdot)$ has a fixed-point for each $w \in W$.

However, most fixed-point theorems and the Leray-Schauder degree for multifunctions refer to the larger class of upper semicontinuous (u.s.c.) multifunctions. The main difficulty that arises here is that generally the operator $T(\cdot,\cdot)$ is not jointly measurable on $W\times X$. For compact u.s.c. stochastic operators, Engl derived in [7, Theorem 16] a random version of the Schauder-Kakutani fixed-point theorem.

In our present article we do not need the compactness of $T(w,\cdot)$ and can so derive fixed-point theorems for ultimately compact u.s.c. random operators. This gives us for example the stochastic version of the theorem of Krasnoselski for the sum of a compact and a contractive multifunction.

A survey about the develepment of problems and theorems in this area until 1976 may be found in the publication of Bharucha-Reid [2]. We do not treat here measurability of solutions of equations of the type $Lu+Nu = 0$ where L is a random linear operators and N a random nonlinear operator (see Kannan and Salehi [12]).

§1. Basic definitions and properties.

DEFINITION 1. (a) Let X always be a real separable Banach space. We denote by

$P(X) := \{M \subset X ; M \neq \emptyset\}$

$B(X) := \{M \subset X ; M \neq \emptyset \text{ bounded}\}$

$A(X) := \{M \subset X ; M \text{ closed}, M \neq \emptyset\}$

$C(X) := \{M \subset X ; M \text{ convex}, M \neq \emptyset\}$

$K(X) := \{M \subset X ; M \text{ compact}, M \neq \emptyset\}$

$O(X) := \{M \subset X ; M \text{ open}, M \neq \emptyset\}$

$KC(X) := K(X) \cap C(X)$, and analogously other combinations.

(b) Let $(W, \mathcal{A})$ always be measurable space, where $\mathcal{A}$ is a σ-algebra of subsets of W. $(W, \mathcal{A}, \mu)$ means a σ-finite measure space, where $\mu : \mathcal{A} \to [0, +\infty]$ is a σ-additive function with $\mu(\emptyset) = 0$. By $\mathcal{B}$ we denote the σ-algebra of Borel subsets of X.

DEFINITION 2. Let $C : W \to P(X)$ be a multifunction.

(a) $\bar{C} : W \to A(X)$ is defined by $\bar{C}(w) = \overline{C(w)}$, $w \in W$.

(b) C is *measurable* iff for each open $D \subset X$ we have $\{w \in W : C(w) \cap D \neq \emptyset\} \in \mathcal{A}$.

(c) C is *separable* iff C is measurable and there exists a countable subset $Z \subset X$ with $C(w) = \overline{Z \cap C(w)}$ for all $w \in W$.

(d) $Gr(C) := \{(w,x) \in W \times X ; x \in C(w)\}$, the *graph* of C.

LEMMA 3. *Let* $C : W \to P(X)$ *be a multifunction.*

(a) *If* $C(w)$ *is independent of* w, *then* $\bar{C}$ *is measurable.*

(b) *If* W *is countable and* C *measurable, then* C *is separable.*

(c) *If* C *is measurable,* $\operatorname{int} C(w) \neq \emptyset$ *for all* $w \in W$, $C(w) = \overline{\operatorname{int} C(w)}$ *for all* $w \in W$, *then* C *is separable.*

(d) *If* $C : W \to O(X)$ *is measurable, then* $\bar{C}$ *is separable.*

Proof. (a) and (b) are obvious, (c) follows from the demonstration of proposition 4 in [7], (d) is an inmediate consequence of (c) and proposition 2.6 in [9] (Our measurable multifunctions are called meakly measurable in [9]).

DEFINITION 4. Let $S \subset X$ and $f:S \to P(X)$ be a multifunction,

(a) f is *u.s.c.* on S if and only if for each $x \in S$ and each open $V \supset f(x)$ there exists an open neighborhood U of x with $f(U \cap S) \subset V$.

(b) f is *closed* on S if and only if for each sequence $x_n \in S$, $x_n \to x \in S$ and $y_n \in f(x_n)$, $y_n \to y \in X$, we have $y \in f(x)$.

LEMMA 5. *Let be* $S \subset X$ *and* $f:S \to P(X)$ *a multifunction.*

(a) *If* S *is a closed subset of* X*, we have the following equivalence:* f *is closed iff* $Gr(f)$ *is a closed subset of* $X \times X$.

(b) *If* f *u.s.c. and* $f(x)$ *closed for each* $x \in S$*, then* f *is closed.*

(c) *If* f *is closed and* $f(\{x, x_1, x_2, x_3, \ldots\})$ *is relatively compact for each convergent sequence* $x_n \to x$ $(n \to \infty)$ *with* $x_n \in S$, $x \in S$*, then* f *is u.s.c.*

(d) f *is u.s.c. iff* $\{x \in S \; ; \; f(x) \cap A \neq \emptyset\}$ *is a closed subset of* S *for each closed* A *of* X.

(e) *If* f *is closed on* S*, then* $f(L)$ *is a closed subset of* X *for each compact subset* L *of* S.

(f) *Assume that* f *is u.s.c. on* S*, that* L *is a compact subset of* S*, and that* $f(x)$ *is relatively compact for each* $x \in S$*. Then* $f(L)$ *is a relatively compact set.*

(g) *Let* $f:S \to K(X)$ *be u.s.c. Then* $f(L)$ *is compact for each compact* $L \subset S$.

Proof. (a) obvious.

(b) We assume $x_n \to x$, $x_n \in S$, $x \in S$, $y_n \in f(x_n)$, $y_n \to y \in X$. If V is an open neighborhood of $f(x)$, then there exists another open neighborhood U of x such that $f(U \cap S) \subset V$. This implies $f(x_n) \subset \bar{V}$ for all $n > n_0$, or $y_n \in \bar{V}$ for all $n > n_0$. Then it follows that $y \in \bar{V}$, and finally $y \in f(x)$, since $f(x) = \bigcap \{\bar{V} \; ; \; V \text{ open}, V \supset f(x)\}$.

(c) If f were not u.s.c. then there would exist a $x\in S$ and an open $V\supset f(x)$ such that for each open neighborhood U of x we should have $(X\setminus V)\cap f(U\cap S)\neq\emptyset$. Then there exists $x_n\in S$ with $\|x_n-x\|<\frac{1}{n}$ and an element $y_n\in f(x_n)$ with $y_n\notin V$. We may assume $y_n\to y$ for some $y\in X$. Then we should $y\notin V$ and therefore $y\notin f(x)$. So our assumption leads to a contradiction.

(d) See ([1], p.115).

(e) Let $y_n\in f(L)$, $y_n\to y$. Therefore $y_n\in f(x_n)$ for some $x_n\in L$. We may assume $x_n\to x\in L$. From the closedness of f it follows that $y\in f(x)\subset f(L)$.

(f) Considering a sequence $y_n\in f(L)$, we have $y_n\in f(x_n)$ for some $x_n\in L$, and without loss of generality we again assume $x_n\to x\in L$. f is u.s.c., therefore $d(y_n,f(x))\to 0$ $(n\to\infty)$ and so there exists a sequence $u_n\in f(x)$ such that $\|y_n-u_n\|\to 0$. We again take $u_n\to u$ for some $u\in f(x)$. This means $y_n\to u$.

(g) By (a) and (e) the set f(L) is closed, and by (f) we know that f(L) is relatively compact. Therefore f(L) is compact, (see also [1]).

REMARK. In the preceeding lemma we need not the separability of X.

DEFINITION 6. Let $C:W\to P(X)$ be measurable and $T(\cdot,\cdot)$: $Gr(C)\to P(X)$ a multifunction.

(a) T is called a *stochastic* (or random) *operator* if and only if $\{w\in W\ ;\ x\in C(x),\ T(w,x)\cap D\neq\emptyset\}\in A$ for each $x\in X$ and for each open $D\subset X$.

(b) A function $x(\cdot):W\to X$ is called a *stochastic* (or random) *fixed-point* of T if and only if

(1) $x(\cdot)$ is a $(A,\mathcal{B})$-measurable function, $x(w)\in C(w)$ for all $w\in W$;

(2) $x(w)\in T(x,x(w))$ for μ-almost all $w\in W$.

(c) T is called *u.s.c. stochastic operator* if and only if

   (1) T is stochastic operator,

   (2) $T(w,\cdot):C(w) \to P(X)$ is u.s.c. on $C(w)$ for each $w \in W$.

§2. <u>Construction of a jointly measurable multi-function</u> H. If for a stochastic operator $T(\cdot,\cdot)$ there exists an element $x(w) \in C(w)$ with $x(w) \in T(w,x(w))$ for each $w \in W$, then it does not necessariyl exists a stochastic fixed-point of T. For a counterexample see [8] or [7]. More regularity properties of T are required.

Unfortunately, an u.s.c. stochastic operator T is not jointly measurable with respect to both variables (w,x). A counterexample may be found in [7]. But we need such a property in our demonstrations. So we pass to another u.s.c. stochastic operator $H(\cdot,\cdot)$ which additionally is jointly measurable. The idea for the construction of this new operator H stems from the proof of the well-known fact that a function $g(\cdot,\cdot)$ is jointly measurable if it satisfies a Carathéodory-condition. That means g has to be measurable with respect to w and continuous with respect to x (see Scorza-Dragoni [17] and Neubrunn [13]). This idea was successfully modified and applied in [6]. Despite of $H(w,x) \subset T(w,x)$ we can show that we do not loose too many fixed-points replacing T by H.

DEFINITION 7. Let A and B be two nonempty subsets of X

(a) For $x \in X$ we denote by $d(x,B) := \inf\{\|x-b\|;\ b \in B\}$, the *distance* of x to B.

(b) $e(A,B) := \sup\{d(x,B)\ ;\ x \in A\}$ is called the *excess* of A over B where the supremum is taken in $[0,+\infty]$.

(c) The *Hausdorff distance* of A and B is defined by $D(A,B) := \max\{e(A,B), e(B,A)\}$.

We refer to [3, chapter II, §1] for elementary properties.

PROPOSITION 8. *Let* $(W,\mathcal{A})$ *be a measurable space,* X *a separable real Banach space,* $C:W \to A(X)$ *separable,* Z *like in definition* 2(c), $T:Gr(C) \to KC(X)$ *a u.s.c. stochastic operator, and for* $x \in C(w)$,

$$H(w,x) := \bigcap_{n\in\mathbb{N}} \overline{\text{conv}} \{\bigcup_z T(w,z) \; ; \; z \in Z \cap C(w) \; , \; \|z-x\| < \tfrac{1}{n}\} \; .$$

Then this so defined multifunction has the following properties:

(a) $H(w,x) \subset T(w,x)$ *for all* $(w,x) \in Gr(C)$

(b) $H(w,x) = T(w,x)$ *for all* $(w,x) \in Gr(C)$, $x \in Z \cap C(w)$

(c) $H(w,x) \neq \emptyset$ *for all* $(w,x) \in Gr(C)$

(d) $H(w,\cdot):C(w) \to KC(X)$ *is u.s.c. for each* $w \in W$

(e) $H(\cdot,\cdot)$ *is* $(\mathcal{A}\times\mathcal{B},\mathcal{B})$*-measurable.*

<u>*Proof*</u>. For fixed $N \in \mathbb{N}$ and $(w,x) \in Gr(C)$ we set $T_N(w,x) := \bigcup_z \{T(w,z) \; ; \; z \in Z \cap C(w) \; , \; \|z-x\| < \frac{1}{N}\}$. By H_N we denote the closure of the convex hull of T_N, $H_N := \overline{\text{conv}T_N}$. So we have $H(w,x) = \bigcap_{N\in\mathbb{N}} H_N(w,x)$. Clearly, $T_N(w,x) \neq \emptyset$ for all $(w,x) \in Gr(C)$. In the demonstration we will omit the variables (w,x) if confusion is not possible.

(a) Let $\varepsilon > 0$ be given and $U_\varepsilon(Tx) := \{y \in X \; ; \; d(y,Tx) < \varepsilon\}$. There exists a $N \in \mathbb{N}$ such that $Tz \subset U_\varepsilon(Tx)$ for all $z \in C(w)$ with $\|z-x\| < \frac{1}{N}$. This implies $T_N(w,x) \subset U_\varepsilon(Tx)$. By the convexity of $U_\varepsilon(Tx)$ we can conclude $H_N(w,x) = \overline{\text{conv}T_N} \subset \overline{U_\varepsilon(Tx)} \subset U_{2\varepsilon}(Tx)$ and so $H(w,x) \subset U_{2\varepsilon}(Tx)$. This means $H(w,x) \subset \bigcap_{\varepsilon>0} U_{2\varepsilon}(Tx) = T(w,x)$. The last equality is a consequence of the closedness of T(w,x).

(b) For $x \in Z \cap C(w)$ we have $T(w,x) \subset T_N(w,x) \subset H_N(w,x)$ for each $N \in \mathbb{N}$, therefore $T(w,x) \subset H(w,x)$, and by (a) equality holds.

(c) By construction, the set H(w,x) is convex and closed, and, therefore by (a), compact. For all $n\in\mathbb{N}$ we already know that $T_n(w,x) \neq \emptyset$, $(w,x)\in Gr(C)$. We choose $y_n\in T_n(w,x)$ and can find a $z_n\in Z\cap C(w)$ with $\|z_n-x\| < \frac{1}{n}$ and $y_n\in T(w,x_n)$. Therefore $\lim_{n\to\infty} z_n = x$ and $\bigcup_n\{y_n\}\subset\bigcup_n T(z_n)$, where the last set is relatively compact by Lemma 5(g). Again we take without loss of generality $\lim_{n\to\infty} y_n = y$ for some $y\in X$. For the moment we fix $N\in\mathbb{N}$ and get for all $n > N$: $y_n\in T_N\subset \overline{\text{conv}T_N} = H_N$. But H_N is closed, si $y\in H_N$. Making this conclusion for each $N\in\mathbb{N}$ gives us finally $y\in H(w,x)$.

(d) $H(w,x)\in KC(X)$ is evident by the preceeding observations. Applying lemma 5(c) we show that $H(w,\cdot)$ is u.s.c. on $C(w)$. For a compact subset L of $C(w)$ we deduce from (a) that $H(L) = \bigcup_{x\in L} H(x) \subset\bigcup_{x\in L} T(x) = T(L)$. By lemma 5(g) the set $T(L)$ is compact and therefore $H(L)$ relatively compact. The only thing still to show is the closedness of the map $H(w,\cdot)$. Let be $x_n\in C(w)$, $x\in C(w)$, $\lim_{n\to\infty} x_n = x$, $y_n\in H(x_n)$, $y\in X$, $\lim_{n\to\infty} y_n = y$. For fixed $N\in\mathbb{N}$ it exists a n_0 such that $\|x_n-x\| < \frac{1}{2N}$ for all $n > n_0$. For $n > n_0$ we have $y_n\in H(x_n)\subset H_{2N}(x_n)\subset H_N(x)$ because $\|z-x\| \leq \|z-x_n\|+\|x_n-x\|$ for all $z\in Z\cap C(w)$. The set $H_N(x)$ is closed, therefore $\lim_{n\to\infty} y_n = y\in H_N(x)$. Thus $y\in\bigcap_{N\in\mathbb{N}} H_N(x) = H(x)$.

(e) The multifunction $T_n(\cdot,\cdot):Gr(C) \to P(X)$ is $(A\times\mathcal{B},\mathcal{B})$-measurable. For a demonstration see the first part of the proof of proposition 5(3) in [6]. We conclude from the proposition 2.6 and theorem 9.1 in [9] that $\bar{T}_n$ and also $H_n = \overline{\text{conv}T_n} = \overline{\text{conv}\bar{T}_n}$ are $(A\times\mathcal{B},\mathcal{B})$-measurable multifunctions on $Gr(C)$. Taking in the moment for granted that $\lim_{n\to\infty} d(x,H_n(w,y)) = d(x,H(w,y))$ for $x\in X$, $(w,y)\in Gr(C)$, we can bring to an end the proof of (e) as follows: since $H_n(\cdot,\cdot)$ is measurable we have that $d(H_n(\cdot,\cdot))$ is measurable for each $x\in X$. Applying once more theorem III,9 in [3] gives the measurability of $H(\cdot,\cdot)$. For the rest of the proof we firstly show

$$\lim_{n\to\infty} e(T_n(w,x), H(w,x)) = 0 , \quad (w,x) \in Gr(C). \qquad (1)$$

If (1) is not valid then there exists a $\varepsilon_0 > 0$ such that without loss of generality $e(T_n,H) > 2\varepsilon_0$ for all $n \in \mathbb{N}$. Then there exist $y_n \in T_n$, $d(y_n,H) > 2\varepsilon_0$ and $y_n \in T(z_n)$ for some $z_n \in Z \cap C(w)$ with $\|z_n - x\| < \frac{1}{n}$. So $\bigcup_{n\in\mathbb{N}} \{y_n\} \subset \bigcup_{n\in\mathbb{N}} T(z_n)$, and $\bigcup_{n\in\mathbb{N}} \{y_n\}$ is relatively compact by lemma 5(g). We may assume $\lim_{n\to\infty} y_n = y$ for some $y \in X$. By (d) and lemma 5(b) it follows $y \in H(w,x)$. This is a contradiction to $d(y_n,H) > 2\varepsilon_0$ for all $n \in \mathbb{N}$. Secondly, let us show

$$\lim_{n\to\infty} e(H_n,H) = 0 , \quad (w,x) \in Gr(C). \qquad (2)$$

For any given $\varepsilon > 0$ there exists in view of (1) a n_0 with $e(T_n,H) < \frac{\varepsilon}{2}$ for all $n > n_0$, and, as a consequence, we have $T_n \subset U_{\varepsilon/2}(H)$ for all $n > n_0$ and also $H_n = \overline{\text{conv} T_n} \subset U_\varepsilon(H)$ for all $n > n_0$, since $U_{\varepsilon/2}(H)$ as $\varepsilon/2$-neighborhood of the convex H is convex, too. Observing $e(U_\varepsilon(H),H) \leqslant \varepsilon$ and [3, page 38] one obtains $e(H_n,H) \leqslant e(H_n,U_\varepsilon(H)) + e(U_\varepsilon(H),H) \leqslant \varepsilon$ for all $n > n_0$. Thirdly we get

$$\lim_{n\to\infty} D(H_n,H) = 0 , \quad (w,x) \in Gr(C), \qquad (3)$$

using result (2) and $H \subset H_n$ which implies $e(H,H_n) = 0$. Now the desired result follows at once of (3) and the inequality $|d(x,H_n) - d(x,H)| \leqslant D(H_n,H)$.

LEMMA 9. *Let be (Ω,F) a measurable space, $R:\Omega \to A(X)$ a measurable multifunction, $r:\Omega \to X$ a measurable function. Then $d(r(\cdot),R(\cdot)):\Omega \to \mathbb{R}$ is measurable.*

<u>*Proof.*</u> Lemma 6 in [6].

THEOREM 10. *Let (W,A,μ) be a σ-finite measure space, X a real separable Banach space, $C:W \to A(X)$ separable , $T:Gr(C) \to KC(X)$ a u.s.c. stochastic operator, H like in propo-*

sition 8, *and* $H(w) := \{x \in C(w) \; ; \; x \in H(w,x)\} \neq \emptyset$ *for all* $w \in W$. *Then*,

(a) *there exists a stochastic fixed-point* $x(\cdot):W \to X$ *of* H *and* T;

(b) *if in addition* (W,A,μ) *is a complete measure space then* $H:W \to A(X)$ *is measurable, and there exists a stochastic fixed-point* $x(\cdot):W \to X$ *which fulfills* $x(w) \in H(w,x(w)) \subset T(w,x(w))$ *for all* $w \in W$.

Proof. For $(w,x) \in Gr(C)$ define $\tilde{x}(w,x) := x$. It is easily verified that this function $\tilde{x}(\cdot,\cdot):Gr(C) \to X$ is $(A\times\mathcal{B},\mathcal{B})$-measurable. By proposition 8(e) and lemma 9 the function $N(w,x) := d(\tilde{x}(w,x),H(w,x)) = d(x,H(w,x))$ is $(A\times\mathcal{B},\mathcal{B}(\mathbb{R}))$-measurable. Furthermore

$$\begin{aligned} Gr(H) &= \{(w,x) \; ; \; x \in C(w), \; x \in H(w,x)\} \\ &= \{(w,x) \in Gr(C) \; ; \; d(x,H(w,x)) = 0\} \\ &= N^{-1}(0) \in A\times\mathcal{B}. \end{aligned}$$

(a) Apply the theorem of Aumann (Theorem 5.2 in [9]) to the multifunction H.

(b) By proposition 8 (d) the multifunction $H(w,\cdot)$ is u.s.c. This implies $H(w) = \overline{H(w)}$, and so the measurability of H by theorem 3.5 in [9]. Now we apply the theorem of Kuratowski, Ryll-Nardzewski (Theorem 5.1 in [9]).

Now we will show the existence of a random fixed-point when T is a continuous stochastic operator. The second part of the following theorem has already been proven in Theorem 6 of [7].

THEOREM 11. *Let* (W,A,μ) *be a σ-finite measure space,* X *a real separable Banach space,* $C:W \to A(X)$ *separable,* $T:Gr(C) \to AB(X)$ *a continuous stochastic operator, that is* $\lim_{n\to\infty} D(T(w,x_n),T(w,x)) = 0$ *for each* $w \in W$ *and for each sequence* $x_n \in C(w)$ *with* $\lim_{n\to\infty} x_n = x$. *Then it follows that*

(a) T *is* $(A\times\mathcal{B},\mathcal{B})$*-measurable.*

(b) *There exists a random fixed-point of* T *if*

$T(w) := \{x \in C(w) \ ; \ x \in T(w,x)\} \neq \emptyset$ *for all* $w \in W$.

<u>*Proof*</u>. (a) For $u \in X$ we have $C_u := \{w \in W \ ; \ u \in C(w)\} \in A$, and $T(\cdot,u): C_u \to AB(X)$ measurable. Thus for fixed $u,x \in X$ the function $d(x,T(\cdot,u))$ is $(A \cap C_u, \mathcal{B}(\mathbb{R}))$-measurable by theorem III.9 of [3]. Using moreover the inequality $|d(x,T(w,v))-d(x,T(w,\bar{v}))| \leqslant D(T(w,\bar{v})), T(w,\bar{v}))$, we obtain for fixed $x \in X$, $r \geqslant 0$.

$$\{(w,y) \in Gr(C) \ ; \ d(x,T(w,y)) \leqslant r\}$$

$$= \bigcap_{n\in\mathbb{N}} \bigcup_{z\in Z} (\{w \in C_z \ ; \ d(x,T(w,z)) < r+\tfrac{1}{n}\} \times U_{1/n}(z)) \cap Gr(C)$$

$$\in (A \times \mathcal{B}) \cap Gr(C).$$

Let Z be choosen like in Definition 2(c). Now apply theorem III. 9 of [3] once more.

(b) Exactly like in theorem 10 we show $Gr(T) \in A \times \mathcal{B}$ using part (a) of our theorem 11. Then we apply the theorem of Aumann (theorem 5.2 in [9]).

<u>§3. Random fixed-point theorems for ultimately compact stochastic operators.</u>

DEFINITION 12. Let $D \neq \emptyset$ be a closed subset of the Banach space X, $f: D \to P(X)$ a multifuncion. We denote by α, β, δ ordinal numbers. By trasnfinite induction we define the sets

$f_0 := \overline{\text{conv}}\, f(D)$

$f_\alpha := \overline{\text{conv}}\, f(D \cap f\alpha-1)$ if $\alpha-1$ exists

$f_\alpha := \bigcap_{\beta<\alpha} f_\beta$ if $\alpha-1$ does not exist,

which have the following well-known properties:

(a) each f_α is closed and convex

(b) $f_\alpha \subset f_\beta$ if $\alpha \geq \beta$, and hence $f(D \cap f_\beta) \subset \overline{\text{conv}} f(D \cap f_\beta) = f_{\beta+1} \subset f_\beta$.

(c) There exists an ordinal number δ such that $f_\beta = f_\delta$ for all $\beta \geq \delta$. We denote this *limit set* f_δ by f_∞. Thus $f(D \cap f_\infty) \subset \overline{\text{conv}} f(D \cap f_\infty) = f_{\delta+1} = f_\delta = f_\infty$, and we have $f_\infty = \emptyset$ if and only if $D \cap f_\infty = \emptyset$.

(d) If $x \in f(x)$ then $x \in f_\infty$.

If in addition the mapping f is u.s.c. on D and if $f(D \cap f_\infty)$ is relatively compact, then f is called *ultimately compact*. This means that the limit set f_∞ is compact.

LEMMA 13. *Let* $D \neq \emptyset$ *be a closed subset of the Banach space* X; $g,f: D \to A(X)$ *u.s.c. multifunctions with* $g(x) \subset f(x)$ *for all* $x \in D$ *and* f *ultimately compact. Then,* (a) g *is ultimately compact and* $g_\infty \subset f_\infty$, (b) $\emptyset \neq g_\infty \subset D$ *if* $g(D) \subset D$ *and* $f_\infty \neq \emptyset$.

Proof. (a) Obvious; for details of the demonstration see for example the proof of theorem 14(b).

(b) We have $\emptyset \neq D \cap f_\infty$ compact. Because $g(D \cap f_\infty) \subset f(D \cap f_\infty)$ and lemma 5(b)(e), the set $g(D \cap g_\infty)$ is compact, moreover $\emptyset \neq g(D \cap f_\infty) \subset D$. We defined $Q_0 := g(D \cap f_\infty) \subset f_\infty \cap D$, $Q_{n+1} := g(Q_n)$ for all $n \in \mathbb{N}$. This is a decreasing sequence of sets where each Q_n is nonempty and compact by lemma 5(b), (e). Hence $Q := \bigcap_{n=1}^\infty Q_n$ is compact and nonempty. We will show now $Q \subset g(Q)$; let $x \in Q = \bigcap_{n=1}^\infty g(Q_{n-1})$, hence $x = g(q_n)$ for some $q_n \in Q_n$ for all $n \in \mathbb{N}$. We may assume $\lim_{n\to\infty} q_n = q$ for some $q \in Q_0$ by the compactness of Q_0. Obviously $q \in Q$. From lemma 5(b) it follows $x \in g(q) \subset g(Q)$. A usual conclusion with transfinite induction gives us $Q \subset g_\alpha$ for each ordinal number α, and so $\emptyset \neq Q \subset g_\infty$.

THEOREM 14. *Assume that* $D \neq \emptyset$ *is an open subset of the Banach space* X, $A := \bar{D}$, $g,f: A \to KC(X)$ *u.s.c. multifunc-*

tions, $g(x)\subset f(x)$ *for all* $x\in A$, f *ultimately compact, and* $h(t,x) := tg(x)+(1-t)f(x)$ *for* $(t,x)\in[0,1]\times A$. *Then*

(a) $h:[0,1]\times A \to KC(X)$ *is a u.s.c. multifunction,*

(b) h *is ultimately compact,* $h_\infty\subset f_\infty$,

(c) $\deg(J-g,D,0) = \deg(J-f,D,0)$ *if* $0\notin(J-f)(\partial D)$, *where* $\deg(\cdot,\cdot,\cdot)$ *is the generalization of the Leray-Schauder degree introduced by Petryshyn and Fitzpatrick in* [15].

Proof. (a) $h(t,x)$ is a convex, compact set for fixed $(t,x)\in[0,1]\times A$. For an arbitrary compact set $L\subset[0,1]\times A$ we verify the compactness of $h(L)$. Let $w_n\in h(L)$ with $w_n\in h(t_n,x_n) = t_ng(x_n)+(1-t_n)f(x_n)$ for some $(t_n,x_n)\in L$ which implies $w_n = t_nu_n+(1-t_n)v_n$ for some $u_n\in g(x_n)$, $v_n\in f(x_n)$. We may take $(t_n,x_n)\to(t,x)\in L$ and by lemma 5(g) also $u_n\to u$, $v_n\to v$. Applying lemma 5(b) we get $u\in g(x)$, $v\in f(x)$. Hence $w_n = t_nu_n+(1-t_n)v_n\to tu+(1-t)v\in tg(x)+(1-t)f(x) = h(t,x)$. We now show that h is closed. Therefore let $(t_n,x_n)\in[0,1]\times A$, $(t_n,x_n)\to(t,x)$, $w_n\in h(t_n,x_n)$, $w_n\to w$. Since $L = \{(t_n,x_n),(t,x)\ ;\ n\in\mathbb{N}\}$ is a compact subset of $[0,1]\times A$, one proves as before that $w\in h(t,x)$. Finally, from lemma 5(c) the desired result follows.

(b) It suffices to show $h_\alpha\subset f_\alpha$ for each ordinal number. At first $h(t,x) = tg(x)+(1-t)f(x)\subset f(x)$ for all $(t,x)\in[0,1]\times A$. Therefore $h_0 = \overline{\operatorname{conv}}h([0,1]\times A)\subset\overline{\operatorname{conv}}f(A) = f_0$, and taking $h_\beta\subset f_\beta$ for all ordinals $\beta<\alpha$ we deduce $h_\alpha = \overline{\operatorname{conv}}h([0,1]\times(A\cap h_{\alpha-1}))\subset \overline{\operatorname{conv}}h([0,1]\times[A\cap f_{\alpha-1}])\subset\overline{\operatorname{conv}}f(A\cap f_{\alpha-1}) = f_\alpha$ if $\alpha-1$ exists, and $h_\alpha = \bigcap_{\beta<\alpha}h_\beta\subset\bigcap_{\beta<\alpha}f_\beta = f_\alpha$ if $\alpha-1$ does not exist.

(c) $0\notin(J-f)(\partial D)$ is equivalent to $x\notin f(x)$ for all $x\in\partial D$. So $x\notin h(t,x)$ for all $(t,x)\in[0,1]\times\partial D$. So $\deg(J-h(t,\cdot),D,0)$ is well defined for each $t\in[0,1]$ and independent of t by (a), (b) and theorem 2.2 of [15].

REMARK 15. (a) If in the above theorem we have

$\deg(J-f,D,0) \neq 0$, then $f_\infty \neq \emptyset$ and also $g_\infty \neq \emptyset$.

(b) In the article [16], Sadovski gives on pages 137, 138 the example of two ultimately compact functions f,g which satisfy for $h(t,x) = tg(x)+(1-t)f(x)$ the condition $x \notin h(t,x)$ for $x \in \partial D$, $t \in [0,1]$, but for which nevertheless $\deg(J-f,D,0) \neq \deg(J-g,D,0)$. Therefore, in contrast to the case of compact or condensing operators, our theorem 14 is not evident.

(c) For definition 12, lemma 13, and theorem 14 we obviously do not need separability of X.

DEFINITION 16. Let $C:W \to P(X)$ be measurable and $T(\cdot,\cdot):Gr(\bar{C}) \to P(X)$ a multifunction. T is called an *ultimately compact stochastic operator* if and only if:

(i) T is a stochastic operator,

(ii) $T(w,\cdot):\bar{C}(w) \to P(X)$ is ultimately compact for each $w \in W$.

THEOREM 17. *Let be* $C:W \to O(X)$ *measurable,* $T:Gr(\bar{C}) \to KC(X)$ *an ultimately compact stochastic operator,* $x \notin T(w,x)$ *for all* $x \in \partial C(w)$, $w \in W$, *and finally* $\deg(J-T(w,\cdot),C(w),0) \neq 0$ *for all* $w \in W$. *Then there exists a random fixed-point of* T.

<u>*Proof*</u>. By lemma 3(d) the multifunction $\bar{C}$ is separable. We apply proposition 8 and obtain for each $w \in W$ the u.s.c. multifunction $H(w,\cdot):\bar{C}(w) \to KC(X)$, $H(w,x) \subset T(w,x)$. From theorem 14, it follows $\deg(J-H(w,\cdot),C(w),0) \neq 0$ for all $w \in W$. Thus there exists for all $w \in W$ an element $x(w) \in C(w)$, $x(w) \in H(w,x(w))$. Using the notation of theorem 10, we have $H(w) \neq \emptyset$ for all $w \in W$, and so there exists a random fixed-point of T by the same theorem 10.

COROLLARY 18. *Let* $C:W \to O(X)$ *be measurable, each* $C(w)$ *a symmetric neighborhood of the origin,* $T:Gr(\bar{C}) \to KC(X)$ *an odd ultimately compact stochastic operator, and* $x \notin T(w,x)$ *for all*

$x \in \partial C(w)$, $w \in W$. *Then there exists a stochastic fixed point of* T.

Proof. For each $x \in \bar{C}(w)$ the operator T satisfies the condition $T(w,x) = -T(w,-x)$. Hence, by theorem 2.4 of [15], $\deg(J-T(w,\cdot),C(w),0)$ is an odd integer for all $w \in W$. Now we use the above theorem 17.

THEOREM 19. *Let* $C:W \to AC(X)$ *be separable, and* $T:Gr(C) \to KC(X)$ *an ultimately compact stochastic operator with* $T(w,C(w)) \in C(w)$ *and* $T_\infty(w,\cdot) \neq \emptyset$ *for all* $w \in W$. *Then* T *has a stochastic fixed-point.*

Proof. Passing to the operator H in proposition 8, we get $H:Gr(C) \to KC(X)$ as ultimately compact stochastic operator with $H(w,C(w)) \subset C(w)$ and $H_\infty(w,\cdot) \neq \emptyset$ for all $w \in W$, by lemma 13. Theorem 3.6 of [15] guarantees $\mathcal{H}(w) \neq \emptyset$, and so by theorem 10 there exists a random fixed-point of T.

§4. Special cases.

DEFINITION 20. (a) For a bounded subset B of the Banach space X we define the *Kuratowski-measure* of noncompactness:

$$\chi(B) := \inf\{\varepsilon > 0 \; ; \; B \text{ admits a finite covering by sets of diameter } \leq \varepsilon\},$$

and the *Hausdorff-measure* of noncompactness:

$$\gamma(B) := \inf\{\varepsilon > 0 \; ; \; B \text{ admits a finite } \varepsilon\text{-ball covering}\}.$$

(Fundamental properties of χ, γ may be found in [5, p.19]).

(b) A multifunction $f:D \to K(X)$, $\emptyset \neq D$ closed subset of X, is said to be χ-*condensing* if and only if f is u.s.c. on D, maps bounded sets to bounded sets, and satisfies $\chi(f(B)) < \chi(B)$ for each bounded $B \subset D$ which is not relatively compact. A corresponding definition holds for γ-*condensing* multifunctions.

(c) A multifunction $f:D \to K(X)$, $\emptyset \neq D$ closed subset of X, is said to be *compact* if and only if f is u.s.c. on D and maps bounded subsets of D to relatively compact sets.

(d) A multifunction $f:D \to K(X)$, $\emptyset \neq D$ closed subset of X, is said to be a *contraction* (with constant k) if and only if there exists a $k \in (0,1)$ such that $D(f(x),f(y)) \leq k\|x-y\|$ for all $x,y \in D$.

We now list up some well-known relations between these properties.

LEMMA 21. *Let* D *be a nonvoid closed subset of the Banach space* X *and* $f,g:D \to KC(X)$ *multifunctions.*

(a) *If* f *is compact, then* f *is* χ-*condensing and* γ-*condensing.*

(b) *If* f *is* χ-*condensing or* γ-*condensing, then* f *is ultimately compact.*

(c) *Let* D *be bounded,* $f:D \to KC(X)$ *compact,* $g:X \to KC(X)$ *a contraction. Then the sum* $f+g:D \to KC(X)$ *is* γ-*condensing.*

(d) *Let* D *be bounded,* $f:D \to KC(X)$ *compact,* $g:D \to X$ *a single-valued contraction. Then* $f+g:D \to KC(X)$ *is* χ-*condensing.*

(e) *Let* D *be bounded,* $f:D \to KC(X)$ *compact,* $g:D \to KC(X)$ *a contraction with* $k < \frac{1}{2}$. *Then* $f+g:D \to KC(X)$ *is* χ-*condensing.*

Proof. (a) Obvious, (b) see lema 3.2 in [15], (c), (d), (e) see remark 3.9 in [15].

DEFINITION 22. Let $C:W \to P(X)$ be measurable and $T(\cdot,\cdot):Gr(\bar{C}) \to K(X)$ a multifunction. Exactly the same as before we may define now a χ-condensing, γ-condensing, compact or contrative stochastic operator T.

THEOREM 23. (Type Kakatani, Schauder, Rothe). *Let*

$C:W \to AC(X)$ *be separable,* $T:Gr(C) \to KC(X)$ *a* χ *or* γ*-condensing random operator with* $T(w,\partial C(w)) \subset C(w)$ *for all* $w \in W$. *Then* T *has a random fixed-point.*

Proof. The operator H, constructed in proposition 8, satisfies $H:Gr(C) \to KC(X)$, $H(w,x) \subset T(w,x)$, $H(w,\partial C(w)) \subset C(w)$ for all $w \in W$, and is a χ or γ-condensing stochastic operator, too. Suppose that $\operatorname{int}C(w_0) = \emptyset$ for $w_0 \in W$. This implies $\partial C(w_0) = C(w_0)$ and so $H(w_0,C(w_0)) \subset C(w_0)$. By corollary 3.5 of [15] there exists a $x_0 \in C(w_0)$ with $x_0 \in H(w_0,x_0)$, which signifies $H(w_0) \neq \emptyset$. Otherwise, $D := \operatorname{int}C(w_1) \neq \emptyset$ for $w_1 \subset W$, then $C(w_1) = \bar{D}$, $\partial D = \partial C(w_1)$, D convex, $H(w_1,\partial D) \subset \bar{D}$. We apply now corollary 3.4 of [15], thus obtaining an element $x_1 \in C(w_1)$ with $x_1 \in H(w_1,x_1)$, that is $H(w_1) \neq \emptyset$. Now use theorem 10. Finally, we add in passing that one can easily generalize the corollary 3.4 in [15] to arbitrary open convex sets, so that it might indeed be applied to our slightly more general situation here.

THEOREM 24. *Let* $C:W \to O(X)$ *be measurable,* $0 \in C(w)$ *for all* $w \in W$; $T:Gr(\bar{C}) \to KC(X)$ *a* χ *or* γ*-condensing stochastic operator with* $\lambda x \notin T(w,x)$ *for all* $x \in \partial C(w)$, $\lambda \geq 1$, $w \in W$. *Then* T *has a stochastic fixed-point.*

Proof. By theorem 3.2 in [15] we have $\deg(J-T(w,\cdot),C(w),0) = 1$ for all $w \in W$. Because of lemma 21(b) and theorem 17 there eixsts a random fixed-point of T.

COROLLARY 25. (type Krasnoselski). *Let* $C:W \to AC(X)$ *be separable and each* $C(w)$ *bounded. Let* $G(\cdot,\cdot):Gr(C) \to KC(X)$ *be a compact random operator, let* $S(\cdot,\cdot):Gr(C) \to X$ *be single-valued contractive (or* $S(\cdot,\cdot):Gr(C) \to KC(X)$ *be contrative with* $k < \frac{1}{2}$, *or* $S(\cdot,\cdot):W \times X \to KC(X)$ *be a contrative random operator) and let* $T := S+G$ *fulfill the condition* $T(w,\partial C(w)) \subset C(w)$ *for all* $w \in W$.

Then T *has a stochastic fixed-point.*

Proof. By parts (c), (d), (e) of lemma 21 the stochastic operator T is χ or γ-condensign. Applying theorem 23 gives the desired result.

FINAL REMARKS. (a) In [7], theorem 23 was proven for compact random operators and corollary 25 for a single-valued stochastic operator $T = S+G$ where $S:Gr(C) \to X$ is contractive and $G:Gr(C) \to X$ compact.

(b) One may deduce another corollary of Krasnoselski type based on theorem 24, replacing the condition $T(w,\partial C(w)) \subset C(w)$ of corollary 25 by $\lambda x \notin T(w,x)$ for all $x \in \partial C(w)$, $\lambda \geqslant 1$, $w \in W$.

(c) If the Banach space X satisfies the condition of Opial (see [14]), if each C(w) is weakly compact, and if we require that S is nonexpansive, G completely continuous, then we can derive a further result similar to that of corollary 25. This generalizes corollary 18 in [6]. We will not present the details, a proof is obvious after observing corollary 3.9 in [15].

(d) In the preceeding sections we have paid attention only to those fixed-point theorems which are consequence of the Leray-Schauder degree. Clearly there are also other fixed-point theorems which have more in common with modifications of the Banach fixed-point principle, and which have stochastic versions, too. For a survey see the article of Ivanov [10].

ADDENDUM.

(1) A special case of this article above was also treated in the publication of S. Itoh: Measurable and condensing multivalued mappings and random fixed-point theorems, Kodai Math. J. 2(1979) 3, 293-299.

(2) The notion of a separable multifunction (see definition 2c) is closely related to the notion of a almost uniformly

separable set which appears in the work of K. Deimling: A caratheodory theory for systems of integral equations, Annali di Mat. Pura Appl. (IV), vol. LXXXVI (1970) 217-260.

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