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A MAXIMALITY PRINCIPLE ON ORDERED METRIC SPACES

by

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RESUMEN. Se presenta un resultado sobre existencia de elementos maximales en espacios métricos ordenados el cual no requiere para su demostración el lemma de Zorn, basta usar el axioma de selecciones dependientes. Como aplicación se obtienen generalizaciones de resultados de Brezisy Browder, Bishop y Phelps, Ekeland y otros.

ABSTRACT. A maximal element result on a class of order complete metric spaces, as well as an order type extension of a Bishop-Phelps-Brøndsted theorem about semicontinuous functions are presented.

A fundamental result that may be formulated about maximal elements in an arbitrary (partially) ordered set is the celebrated Zorn's theorem (see, e.g., J.Kelley [10,p.33]) and its subsequent variants. However, in many concrete situations, the ambient ordered set is endowed with some sypplementary structures, making the above maximality principle to be, technically speaking, unnecessary. It is the main aim of the present note to illustrate this assertion in case of an additional metric structure. Moreover, it should be noted that, since in our reasonings only an ordinary induction argument is required, our maximal element result (stated below) might be considered at the same time as a metric version of a general ordering principle due to H. Brezis and F.E. Browder [2] (see also I. Ekeland [8]).

Let (X,d) be a metric space and let $\leqslant$ be an *ordering* on X (i.e., a reflexive, antisymmetric and transtitive relation on X). For every $x \in X$, let $X(x,\leqslant)$ denote the set of all $y \in X$ with $x \leqslant y$. A sequence $(x_n : n \in \mathbb{N}) \subset X$ is said to be *monotone* if and only if $x_i \leqslant x_j$ whenever $i \leqslant j$, $i,j \in \mathbb{N}$, and *asymptotic* if and only if $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. A subset $Y \subset X$ is called *order-closed* if and only if for every monotone sequence $(x_n : n \in \mathbb{N}) \subset Y$ and every element $x \in X$ with $x_n \to x$ as $n \to \infty$, we have $x \in Y$. In this context, the initial ordering $\leqslant$ is said to be *self-closed* if and only if for every $x \in X$, $X(x,\leqslant)$ is order-closed. Finally, the ambient metric space (X,d) is said to be *order-asymptotic* if and only if every monotone sequence is asymptotic, and *order-complete* if and only if every monotone Cauchy sequence converges (in X). A satisfactory motivation for the introduction of these notions will be offered later on (see also, in this direction, the author's paper [15]); for the moment, we shall specify only that these notions are particularly involved in the statement of the main (maximal element) result of this note, a result that may be formulated as follows.

THEOREM 1. *Under the above notational conventions, suppose that*

(i) $\leqslant$ *is a self-closed ordering*

(ii) (X,d) *is order-asymptotic*

(iii) (X,d) *is order-complete*·

Then, for every $x \in X$ *there is a maximal element* $z \in X$ *with* $x \leqslant z$ *(that is, for every* $y \in X$, $y \neq z$, *the relation* $z \leqslant y$ *does not hold).**

<u>*Proof*</u>. Firstly, we claim that every $x \in X$ has the property (P): *for every* $\varepsilon > 0$ *there is an element* $y \geqslant x$ *such that* $d(y,z) < \varepsilon$ *for all* $z \geqslant y$.

Indeed, suppose there is an element $x \in X$ for which (P) does not hold. Then, there must be a positive $\varepsilon > 0$ such that for every $y \geqslant x$ there is an element $z \geqslant y$ with $d(y,z) \geqslant \varepsilon$. In this case, we immediately construct a monotone sequence $(y_n : n \in \mathbb{N}) \subset X$ with $d(y_n, y_{n+1}) \geqslant \varepsilon$, for all $n \in \mathbb{N}$, that is, non-asymptotic, contradicting (ii) and proving our assertion. Now, let $x \in X$ be arbitrarily fixed. Put $x_0 = x$ and $\varepsilon_0 = (\frac{1}{2})^0 = 1$. By (P), an element $x_1 \geqslant x_0$ may be found, with $d(x_1, y) < 1$ for all $y \geqslant x_1$. Furthermore, for $x_1 \in X$ and $\varepsilon_1 = (\frac{1}{2})^1 = \frac{1}{2}$ an element $x_2 \geqslant x_1$ may be chosen so that $d(x_2, y) < \frac{1}{2}$, for all $y \geqslant x_2$, and so on. By induction, we get a monotone sequence $(x_n : n \in \mathbb{N}) \subset X$, satisfying condition.

$$d(x_{n+1}, y) < (\tfrac{1}{2})^n, \quad \text{all} \quad y \geqslant x_{n+1}, \quad n \in \mathbb{N}. \tag{1}$$

As an inmediate consequence of (1), $(x_n : n \in \mathbb{N})$ is a monotone Cauchy sequence, so that, by (iii), $x_n \to z$ as $n \to \infty$, for some $z \in X$. We claim that z is our desired element. Indeed, from (i) coupled with

$$x = x_0 \leqslant x_1 \leqslant \ldots ; \quad x_n \to z \quad \text{as} \quad n \to \infty$$

we get $x_n \leqslant z$, for all $n \in \mathbb{N}$, and in particular $x \leqslant z$. On the other hand, let y X be such that $z \leqslant y$. Then $x_{n+1} \leqslant y$, for all

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* In the presence of (ii), conditions (i) and (iii) are equivalent to the single condition: every monotone (Cauchy) sequence is bounded (N.E.).

$n \in \mathbb{N}$, and combining with (1):

$$d(x_{n+1}, y) < (\tfrac{1}{2})^n, \quad \text{all} \quad n \in \mathbb{N},$$

that is, $x_n \to y$ as $n \to \infty$ and therefore, by the uniqueness of the limit, $z = y$, completing the proof. Q.E.D.

REMARK 1. It should be noted that in case $\leqslant$ is a *semi-closed* ordering in L. Nachbin's sense [12,p.100] the above result coincides with theorem 3.1 of the author [15] (see also, theorem 3.1 of [14]) proved by a direct application of Zorn's theorem.

A first (partial) indication concerning the power of the main result follows from the considerations below. Again let (X,d) be a metric space and let $\leqslant$ be an ordering on X. A function $f: X \to \mathbb{R}$ is said to be *order-lsc* if and only if for every $t \in \mathbb{R}$, the subset of elements $x \in Y$ with $f(x) \leqslant t$ is order-closed. Of course, every *lsc* function is also order-lsc and moreover, if we suppose $\leqslant$ is a self-closed ordering, every *decreasing* function is necessarily order-lsc for, if the monotone sequence $(x_n : n \in \mathbb{N}) \subset X$, the real number $t \in \mathbb{R}$ and the element $x \in X$ are as above, then $x_n \leqslant x$, $n \in \mathbb{N}$, so that $f(x) \leqslant f(x_n) \leqslant t$, $n \in \mathbb{N}$ proving our claim. This shows that the notion of order-lsc function is effectively more general that of lsc function (indeed, it suffices to consider the metric space $X = \mathbb{R}$, endowed with the usual ordering and the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \text{sign}(-x)$, $x \in \mathbb{R}$, which is clearly decreasing (hence, order-lsc) but not lsc, as it can be readily verified). Of course, in an analogous way one can introduce the notion of *order-usc* function.

Now as an important application of theorem 1, we may state and prove the following result about order-lsc functions.

THEOREM 2. *Let* (X,d) *and* $\leqslant$ *be such that condition* (i)-

(iii) *hold and let* $f:X \to \mathbb{R}$ *be a function satisfying*: (iv) f *is order-lsc. Then for every* $x \in X$ *there is an element* $z \in X$ *such that*

(a) $x \preccurlyeq z$ *and* $d(x,z) \leq f(x)-f(z)$,

(b) *for every* $y \in X$, $y \neq z$, *either* $z \preccurlyeq y$ *does not hold, or* $d(z,y) > f(z)-f(y)$, *in case* $z \preccurlyeq y$.

Proof. Let $\lesssim$ be an ordering on X defined by

$$x \lesssim y \text{ if and only if } x \preccurlyeq y \text{ and } d(x,y) \leq f(x)-f(y). \quad (2)$$

Since, evidently, $\lesssim$ is finer than $\preccurlyeq$ (i.e., $x \lesssim y$ implies $x \preccurlyeq y$) it immediately follows that (X,d) is order-complete and order-asymptotic, with $\preccurlyeq$ replaced by $\lesssim$. We claim that moreover, $\lesssim$ is selfclosed. Indeed, let $x \in X$, $(y_n : n \in \mathbb{N}) \subset X$ and $y \in X$ be such that

$$x \lesssim y_n,\ n \in \mathbb{N}\ ;\ y_n \lesssim y_m,\ n \leq m\ ;\ y_n \to y \text{ as } n \to \infty. \quad (3)$$

From this fact, it easily follows by (2)

$$x \preccurlyeq y_n,\ n \in \mathbb{N}\ ;\ y_n \preccurlyeq y_m,\ n \leq m\ ;\ y_n \to y \text{ as } n \to \infty,$$

which gives, by (i), $x \preccurlyeq y$. On the other hand, from the first part of (3), we get

$$d(x,y_n) \leq f(x)-f(y_n)\ ,\quad n \in \mathbb{N},$$

and therefore, letting $n \to \infty$ and taking into account (iv) plus the evident fact that the sum of two order-lsc functions is also order-lsc, we obtain $d(x,y) \leq f(x)-f(y)$, proving that $x \lesssim y$ and establishing our claim. So, theorem 1 applies with $\preccurlyeq$ replaced by $\lesssim$, and this completes the proof. Q.E.D.

REMARK 2. A very important example of ordering satisfying the requirement (i) and (ii) is that offered by

$$x \preccurlyeq y \text{ if and only if } d(x,y) \leq g(x)-g(y), \quad (4)$$

$g:X \to \mathbb{R}$ being a function from X into $\mathbb{R}$ satisfying (i)' g is lsc and bounded below.

In this case, a sufficient condition for (iii) is

(iii)' (X,d) is complete;

hence, under the additional assumption f = g the corresponding statement of theorem 2 coincides with similar maximal element results established by A. Brøndsted [3], [4] (see also, E.Bishop and R.R. Phelps [1], M. Turinici [14], J.D. Weston [16]).

Now as a second indication about the generality of our main result, let us consider the following situation. Suppose (V,d) is a given metric space and let $F:V \to \mathbb{R}\cup\{+\infty\}$ be an extended real valued function, not identically $+\infty$. F is said to be a *self-lsc* function if and only if for every sequence $(x_n : n\in\mathbb{N}) \subset X$, every real number $t\in\mathbb{R}$ and every element $x\in X$, with $F(x_n)\leqslant t$ for $n\in\mathbb{N}$, $(F(x_n) : n\in\mathbb{N})$ descending and $x_n \to x$ as $n\to\infty$, we have $F(x) \leqslant t$. Or course, every lsc function from V into $\mathbb{R}\cup\{+\infty\}$ is also self-lsc but the converse is not in general true, since, e.g. the function $F(x) = x+[x]$, $x\in\mathbb{R}$ (here $[x]$ denotes the integral part of x) is clearly self-lsc but not lsc. This shows that the notion of self-lsc function is effectively more general than the notion of lsc function. By a dual procedure one can also introduce the notion of self-usc function.

As another important consequence of theorem 1 we may state and prove a generalization of a theorem of I. Ekeland [8], who proves it for lsc functions.

THEOREM 3. *Let* (V,d) *be a complete metric space and suppose* $F:V\to\mathbb{R}\cup\{+\infty\}$ *is a self-lsc function not identically* $+\infty$, *bounded from below. Let* $\varepsilon > 0$ *and a point* $u\in V$ *with* $F(u) \leqslant \inf F(v)+\varepsilon$ *be given. Then there is a point* $v\in V$ *such that*

(c) $F(v) \leqslant F(u)$ *and* $d(u,v) \leqslant 1$,

(d) *for every* $w \neq v$, $F(w) > F(v)-\varepsilon d(v,w)$.

Proof. Let X be the subset of all elements $v \in V$ with $F(v) \leqslant \inf F(v)+\varepsilon$ (clearly, X is not empty, since $u \in X$). Define an ordering $\leqslant$ on X by the convention

$$x \leqslant y \text{ if and only if } d(x,y) \leqslant F(x)-F(y). \tag{5}$$

Firstly we claim that X is closed with respect to (convergent) monotone sequences that is, if the monotone sequence $(x_n : n \in \mathbb{N}) \subset X$ converges to $x \in V$ then $x \in X$; indeed in this case (denoting for simplicity $t = \inf F(V)+\varepsilon$), $F(x_n) \leqslant t$, $n \in \mathbb{N}$, and by (5) $(F(x_n) : n \in \mathbb{N})$ is descending so that by our hypothesis about F $F(x) \leqslant t$, i.e., $x \in X$. It immediately follows that x is order-complete with respect to the induced metric. Secondly, by a classical argument (see, e.g., author's paper [15]) X is also order-asymptotic. Finally the argumento used in the proof of theorem 2 may be also employed here to draw the conclusion that the ordering is self-closed. Therefore theorem 1 again applies and this completes the proof. Q.E.D.

REMARK 3. It is not hard to see that, under the conditions stated in theorem 3, the restriction $F \restriction X$ appears as an order-lsc function with respect to the class of orderings $\leqslant$ on X given by (5) and hence, theorem 3 may be considered at the same time as being a consequence of theorem 2.

REMARK 4. Suppose F is a lsc function in the classical sense. Then theorem 3 coincides with an important variational type result established by I. Ekeland [7], [8,th.1]. Moreover, it was shown in the above quoted Ekeland's papers that this last result extends J. Caristi's theorem [6] (see also, F.E. Browder [5], S. Kasahara [9], W.A. Kirk [11], J. Siegel [13], M. Turinici [14], C.S. Wong [17]) and hence, our theorem 3 may be considered also as an extension of these results.

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