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DIFFUSION LIMITS OF CONDITIONED CRITICAL GALTON-WATSON PROCESSES

by

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RESUMEN. Un proceso crítico de Galton-Watson con $[xn+o(n)]$ partículas iniciales se sujeta a la condición de tener $[yn+o(n)]$ partículas en el tiempo n, para ciertos valores $x,y \geqslant 0$. Se obtienen entonces leyes y difusiones en el límite cuando $n \to \infty$.

ABSTRACT. A critital Galton-Watson process initiated by $[xn+o(n)]$ particles is conditioned on having the size $[yn+o(n)]$ at time n, for some $x,y \geqslant 0$. As $n \to \infty$ limit laws and limiting diffusions are obtained.

1. Introduction. Let Z_n be the number of particles at time n in a critical Galton-Watson process (GW) and f the generating function of its offspring distribution. Assume $f''(1-) = 2\alpha < \infty$.

Lamperti and Ney (1968) have proved that the finite-di-

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mensional joint distributions of $\{\frac{1}{\alpha n}Z_{[nt]}\ ;\ 0 \leqslant t \leqslant 1\}$, conditioned on $Z_0 = [\alpha xn+o(n)]$ and $Z_n > 0$, converge, as $n \to \infty$, to those of a diffusion process $\{Y_{LN,x}(t)\ ;\ 0 \leqslant t \leqslant 1\}$ with initial state $x \geqslant 0$. Imposing the further condition $Z_{[nc]} = 0$, $c > 1$, i.e. conditioning on extinction in the interval $(n,cn]$, Esty (1976) obtained a limiting diffusion $\{Y_{E,x,c}(t)\ ;\ 0 \leqslant t \leqslant c\}$ which in turn has been shown to converge, as $c \searrow 1$, to another one, which we will denote by $\{Y_{E,x}(t)\ ;\ 0 \leqslant t \leqslant 1\}$. Letting $c \searrow 1$ might be thought of as conditioning on extinction 'at' time n.

In this paper we shall show that $\{\frac{1}{\alpha n}Z_{[nt]}\ ;\ 0 \leqslant t \leqslant 1\}$, conditioned on $Z_0 = [\alpha xn+o(n)]$ and $Z_n = [\alpha yn+o(n)]$, converges in finite-dimensional distributions, as $n \to \infty$, to a diffusion process $\{Y_{x,y}(t)\ ;\ 0 \leqslant t \leqslant 1\}$. It turns out that $\{Y_{E,x}(t)\}$ and $\{Y_{x,0}(t)\}$ coincide (in distribution) and that $\{Y_{LN,x}(t)\}$ may be obtained from $\{Y_{x,y}(t)\}$ by randomization, treating y as a random variable having the distribution of $Y_{LN,x}(1)$. Transition density, infinitesimal mean and variance of $\{Y_{x,y}(t)\}$ are given explicitly in Section 3.

In Section 4 we deduce some limit laws from the general results of Section 3. Finally, in the last section we treat the case of a critical Galton-Watson process with immigration (GWI).

To prove our results we apply local limit theorems (Kesten, Ney and Spitzer (1966), Mellein (1982)) and therefore we have to impose the hypotheses under which they hold. To be precise, we shall confine ourselves to a critical aperiodic GW $\{Z_n\}$ with $E\ Z_1^2 \log Z_1 < \infty$.

2. Preliminaries.

In what follows $x,y,\nu,t_0,t_1,\ldots$ are real numbers with $x \geqslant 0$, $y \geqslant 0$, $\nu > 0$ and $0 \leqslant t_0 \leqslant t_1 \leqslant \ldots$ We set $d_i = t_i - t_{i-1}$, $i = 1,2,\ldots$ and define for $k \in \mathbb{N}$ and $(x_1,\ldots,x_k) \in \mathbb{R}_k^+$

$= \{(y_1,\ldots,y_k) | y_1 > 0,\ldots,y_k > 0\}$,

$$ {}_\nu f_k(x_1,\ldots,x_k;t_1,\ldots,t_k;x,y;t_0,t_{k+1}) = (1) $$

$$
(1) = \begin{cases}
\dfrac{t_{k+1}-t_0}{d_1 d_{k+1}} \exp\{-\dfrac{x+x_1}{d_1} - \dfrac{y+x_k}{d_{k+1}} + \dfrac{x+y}{t_{k+1}-t_0}\}\left(I_{\nu-1}\{\dfrac{2}{t_{k+1}-t_0}\sqrt{xy}\}\right)^{-1} \\
\quad \cdot I_{\nu-1}\{\dfrac{2}{d_1}\sqrt{xx_1}\} I_{\nu-1}\{\dfrac{2}{d_{k+1}}\sqrt{yx_k}\} \prod_{r=2}^{k} h_\nu(x_{r-1},x_r;d_r), \\
\hfill \text{if } x > 0,\ y > 0. \\
\left(\dfrac{t_{k+1}-t_0}{d_1}\right)^{\nu} \dfrac{(\sqrt{x_1/y})^{\nu-1}}{d_{k+1}} I_{\nu-1}\{\dfrac{2}{d_{k+1}}\sqrt{yx_k}\} \prod_{r=2}^{k} h_\nu(x_{r-1},x_r;d_r) \\
\quad \cdot \exp\{-\dfrac{x_1}{d_1} - \dfrac{y+x_k}{d_{k+1}} + \dfrac{y}{t_{k+1}-t_0}\} \hfill \text{if } x = 0,\ y > 0. \\
\left(\dfrac{t_{k+1}-t_0}{d_{k+1}}\right)^{\nu} \dfrac{(\sqrt{x_k/x})^{\nu-1}}{d_1} I_{\nu-1}\{\dfrac{2}{d_1}\sqrt{xx_1}\} \prod_{r=2}^{k} h_\nu(x_{r-1},x_r;d_r) \\
\quad \cdot \exp\{-\dfrac{x+x_1}{d_1} - \dfrac{x_k}{d_{k+1}} + \dfrac{x}{t_{k+1}-t_0}\} \hfill \text{if } x > 0,\ y = 0. \\
\left(\dfrac{t_{k+1}-t_0}{d_1 d_{k+1}}\right)^{\nu} \dfrac{(\sqrt{x_1 x_k})^{\nu-1}}{\Gamma(\nu)} \exp\{-\dfrac{x_1}{d_1} - \dfrac{x_k}{d_{k+1}}\} \prod_{r=2}^{k} h_\nu(x_{r-1},x_r;d_r), \\
\hfill \text{if } x = 0,\ y = 0,
\end{cases}
$$

where $h_\nu(v,w;t) = \frac{1}{t}\exp\{-\frac{v+w}{t}\} I_{\nu-1}(\frac{2}{t}\sqrt{vw})$, $v,w,t > 0$ and I_d denotes the Bessel function of order d with purely imaginary argument.

REMARK 1. The function $f_k(\cdot,\ldots,\cdot;t_1,\ldots,t_k;x,y,t_0,t_{k+1})$ is a k-dimensional density on $\mathbb{R}_k^+$. This is easily seen by an induction argument (consult e.g. Watson (1952) for the evaluation of the integral arising for k = 1). The densities which result in the cases k = 1, xy = 0, are well-known: ${}_{\nu}f_1(\cdot,t_1;0,0;t_0,t_2)$ is a gamma density with parameters $((t_2-t_0)/d_1d_2,\nu)$ and ${}_{\nu}f_1(\cdot,t_1;0,y;t_0,t_2)$ (as well as ${}_{\nu}f_1(\cdot,t_1;x,0;t_0,t_2)$) a randomized gamma density (see. e.g. Feller (1971, p.58); start with a gamma density with parameters $(\frac{t_2-t_0}{d_1d_2},\nu+m)$ and take m as an integer-valued random variable subject to a Poisson distribution with parameter $\frac{yd_1}{d_2(t_2-t_0)}$ to get the stated density).

We introduce some more notation. For $0 \leqslant s < 1$, $0 < t < 1-s$ let

$$
{}_{\nu}g_y(v,w;s,t) =
\begin{cases}
\left(\frac{1-s}{t(1-s-t)}\right)^{\nu}\frac{w^{\nu-1}}{\Gamma(\nu)}\exp\{-\frac{w(1-s)}{t(1-s-t)}\}, & \text{if } v = y = 0. \\
\left(\frac{1-s}{t}\right)^{\nu}\frac{1}{1-s-t}(\frac{w}{y})^{(\nu-1)/2}I_{\nu-1}\{\frac{2}{1-s-t}\sqrt{yw}\}\exp\{-\frac{yt^2+w(1-s)^2}{t(1-s)(1-s-t)}\}, & \text{if } v = 0,\ y > 0. \\
\left(\frac{1-s}{1-s-t}\right)^{\nu}\frac{1}{t}(\frac{w}{v})^{(\nu-1)/2}I_{\nu-1}(\frac{2}{t}\sqrt{vw})\exp\{-\frac{v(1-s-t)^2+w(1-s)^2}{t(1-s)(1-s-t)}\}, & \text{if } v > 0,\ y = 0 \\
\frac{1-s}{t(1-s-t)}\ \frac{I_{\nu-1}(\frac{2}{1-s-t}\sqrt{yw})I_{\nu-1}(\frac{2}{t}\sqrt{vw})}{I_{\nu-1}(\frac{2}{1-s}\sqrt{yv})}\exp\{-\frac{yt^2+v(1-s-t)^2+w(1-s)^2}{t(1-s)(1-s-t)}\} & \text{if } y > 0,\ v > 0,
\end{cases}
$$

and

$$
{}_{\nu}c_y(s,z) =
\begin{cases}
2-\nu-\frac{2z}{1-s}+\frac{2\sqrt{yz}}{1-s}\frac{I_{\nu-2}(\frac{2\sqrt{zy}}{1-s})}{I_{\nu-1}(\frac{2\sqrt{zy}}{1-s})} & \text{if } y > 0,\ z > 0. \\
\nu-\frac{2z}{1-s} & \text{if } y = 0,\ z \geqslant 0. \\
\nu & \text{if } y > 0,\ z = 0.
\end{cases}
$$

For further reference we state the following lemma. The cumbersome but routine calculations involved in its proof are omitted.

LEMMA 1. *Let* $0 \leq s < 1$, $0 < t < 1-s$. *Then for* $v \geq 0$

$$\lim_{t\searrow 0}\frac{1}{t}\int_{|w-v|>\delta,\,w\geq 0}(w-v)^2\,{}_{\nu}g_y(v,w;s,t)\,dw = 0 \quad \textit{for all}\ \delta > 0$$

$$\lim_{t\searrow 0}\frac{1}{t}\int_0^{\infty}(w-v)\,{}_{\nu}g_y(v,w;s,t)\,dw = {}_{\nu}c_y(s,v),$$

$$\lim_{t\searrow 0}\frac{1}{t}\int_0^{\infty}(w-v)^2\,{}_{\nu}g_y(v,w;s,t)\,dw = 2v.$$

3. <u>The limiting diffusion in the GW case.</u> Let $Q_n(i,j)$ be the n-step transition probability of $\{Z_n\}$ and assume $Q(1,1) < 1$, $\sum_{j=2}^{\infty} Q(1,j)j^2\log j < \infty$ and g.c.d.$\{k\in\mathbb{N}\,|\,Q(1,k) > 0\} = 1$, $Q = Q_1$. Then (see Mellein (1982)).

$$Q_n(i,j) \underset{\substack{j,n\to\infty,\ i\geq 1\\ \frac{j}{n},\frac{i}{n}\ \text{remain bounded}}}{\sim} \frac{1}{\alpha n}\exp\left\{-\frac{i+j}{\alpha n}\right\}\sqrt{i/j}\; I_1\left(\frac{2}{\alpha n}\sqrt{ij}\right) . \tag{2}$$

This local limit theorem will be the basic tool in the proof of Lemma 2, together with the following estimates which are due to Kesten, Ney and Spitzer (1966)

$$\sup_{j\geq 1} Q_n(i,j) \leq M i n^{-2} \tag{3}$$

$$\sup_{j\geq 1} Q_n(i,j) \leq M'(in)^{-\frac{1}{2}} \tag{4}$$

To facilitate the statement of Lemma 2 we set $Y_n(t) = \frac{1}{\alpha n}Z_{[nt]}$,

$n \in \mathbb{N}$, $t \geq 0$, denote by ${}_{\nu}F_k$ the k-dimensional distribution function corresponding to the density ${}_{\nu}f_k$ defined in (1) and put $S = \{k \in \mathbb{N} | Q_n(1,k) > 0$ for some $k = k(n) \in \mathbb{N}\}$. We remark that $S = \mathbb{N}$ if $Q(1,1) > 0$ and that by the aperiodicity of $\{Z_n\}$ there exists an integer m such that $S \supset \{m, m+1, \ldots\}$.

LEMMA 2. *Let* r *and* s *be integervalued functions with* $r(n) = o(n)$; $s(n) = o(n)$ *as* $n \to \infty$ *and* $s(n) \geq 1$ *if* $x = 0$, $r(n) \in S$ *if* $y = 0$. *Then for all* $k \in \mathbb{N}$ *and all* $(z_1, \ldots, z_k) \in \mathbb{R}_k^+$

$$F_{k,n}(z_1, \ldots, z_k; t_1, \ldots, t_k; x, y; t_0, t_{k+1})$$

$$= P(Y_n(t_1) \leq z_1, \ldots, Y_n(t_k) \leq z_k | Z_{[nt_0]} = [\alpha xn] + s(n),\ Z_{[nt_{k+1}]} = [\alpha yn] + r(n))$$

$$\xrightarrow[n\to\infty]{} {}_2F_k(z_1, \ldots, z_k; t_1, \ldots, t_k; x, y; t_0, t_{k+1}).$$

Proof. For the sake of simplicity we shall drop from here on (sometimes) the square brackets with the understanding that zn, etc. means [zn], etc. Let $0 < \varepsilon < \min(z_1, \ldots, z_k)$, put

$$S_n(z_1, \ldots, z_k; \varepsilon)$$

$$= \sum_{j_k = \varepsilon n}^{\alpha n z_k} \cdots \sum_{j_1 = \varepsilon n}^{\alpha n z_1} P(Z_{nt_1} = j_1, \ldots, Z_{nt_k} = j_k | Z_{nt_0} = \alpha xn + s(n), Z_{nt_{k+1}} = \alpha yn + r(n)),$$

$$T_n(z_1, \ldots, z_k; \varepsilon)$$

$$= F_{k,n}(z_1, \ldots, z_k; t_1, \ldots, t_k; x, y; t_0, t_{k+1}) - S_n(z_1, \ldots, z_k; \varepsilon),$$

$$R_n(j) = \frac{Q_{nd_1}(\alpha xn+s(n),j)}{Q_{n(t_{k+1}-t_0)}(\alpha xn+s(n),\alpha yn+r(n))} \tag{5}$$

and write

$$F_{k,n}(z_1,\ldots,z_k;t_1,\ldots,t_k;x,y;t_0,t_{k+1})$$

$$= \sum_{j_k=1}^{\alpha nz_k} \cdots \sum_{j_1=1}^{\alpha nz_1} R_n(j_1)\prod_{r=2}^{k} Q_{nd_r}(j_{r-1},j_r)Q_{nd_{k+1}}(j_k,\alpha yn+r(n)). \tag{6}$$

We break the proof into several steps.

(a) There are constants L and N such that $R_n(j) < L$ for all $j \geq 1$, $n \geq N$.

Applying (3) we get an upper bound for the numerator on the right side of (5) while lower bounds for the denominator in (5) may be obtained by inspection of its asymptotic behavior. This may be deduced from (2) in the cases $x \geq 0$, $y > 0$; $x \geq 0$, $y = 0$, $r(n) \to \infty$ as $n \to \infty$. Concerning the remaining case, i.e. $x \geq 0$, $y = 0$, there is a subsequence of $r(n)$ which remains bounded, we observe that

$$Q_n(i,j) \underset{\substack{n\to\infty \\ S\ni j \text{ fixed} \\ \frac{i}{n} \text{ remains bounded}}}{\sim} \frac{i}{\alpha n^2}\theta_j \exp\{-\frac{i}{\alpha n}\}$$

with $\{\theta_k | k \in \mathbb{N}\}$ the normalized ($\sum_{k=1}^{\infty} \theta_k (Q(1,0))^k = 1$) stationary measure of $\{Z_n\}$ which is bounded away from zero on S, i.e. $\theta_j \geq b > 0$ for all $j \in S$ and b appropiately chosen. The last fact is an immediate consequence of well-known results of Kesten, Ney and Spitzer (1966).

(b) For sufficiently large N, as $\varepsilon \searrow 0$,

$$F_{k,n}(\varepsilon,\ldots,\varepsilon;t_1,\ldots,t_k;x,y;t_0,t_{k+1}) = o(1),$$

uniformly in $n \geqslant N$:

Substituting estimates for $Q_{nd_r}(j_{r-1},j_r)$, $r = 2,\ldots,k$ and $Q_{nd_{k+1}}(j_k,\alpha yn+r(n))$ provided by (4), into (6) (with $z_1=\ldots=z_k=\varepsilon$) and using (a) gives

$$F_{k,n}(\varepsilon,\ldots,\varepsilon;t_1,\ldots,t_k;x,y;t_0,t_{k+1}) \leqslant M''(\sqrt{\varepsilon})^k$$

for sufficiently large n.

(c) For sufficiently large N, as $\varepsilon \searrow 0$,

$$T_n(z_1,\ldots,z_k;\varepsilon) = o(1),$$

uniformly in $n \geqslant N$:

$T_n(z_1,\ldots,z_k;\varepsilon)$ may be written as a sum of the 2^k-1 terms having the form

$$\sum_{j_k=r_kn}^{\alpha ns_k}\ldots\sum_{j_1=r_1n}^{\alpha ns_1} R_n(j_1)\prod_{r=2}^{k} Q_{nd_r}(j_{r-1},j_r)Q_{nd_{k+1}}(j_k,\alpha yn+r(n)) \qquad (7)$$

with $(r_i,s_i) = (0,\varepsilon)$ or $(r_i,s_i) = (\varepsilon,z_i)$, $i = 1,2,\ldots,k$ and $(r_1,\ldots,r_k;s_1,\ldots,s_k) \neq (\varepsilon,\ldots,\varepsilon;z_1,\ldots,z_k)$. The asymptotic behaviour ($\varepsilon \searrow 0$, $n \to \infty$) of one of the summands appearing in (7) (that with $(r_1,\ldots,r_k;s_1,\ldots,s_k) = (0,\ldots,0;\varepsilon,\ldots,\varepsilon)$) is dealt with in (b), that of the remaining terms is checked quite similarly.

(d) $S_n(z_1,\ldots,z_k;\varepsilon)$ may be identified as a Riemann sum, approximated ($n \to \infty$) by the integral

$$\int_{\varepsilon}^{z_k}\ldots\int_{\varepsilon}^{z_1} {}_2f_k(x_1,\ldots,x_k;t_1,\ldots,t_k;x,y;t_0,t_{k+1})dx_1\ldots dx_k. \qquad (8)$$

We confine ourselves to the case $xy > 0$. The cases $xy = 0$

are handled similarly. With the aid of (2) we get for the sum defining $S_n(z_1,\dots,z_k;\varepsilon)$, as $n \to \infty$

$$(1+o(1)) \sum_{j_k=\varepsilon n}^{\alpha n z_k} \dots \sum_{j_1=\varepsilon n}^{\alpha n z_1} \frac{1}{\alpha n d_1}\sqrt{(\alpha xn+s(n))/j_1}\exp\{-\frac{1}{\alpha n d_1}(\alpha xn+s(n)+j_1)\}$$

$$\cdot\ I_1(\frac{2}{\alpha n d_1}\sqrt{j_1(\alpha xn+s(n))})\prod_{r=2}^{k}\{\frac{1}{\alpha n d_r}\sqrt{j_{r-1}/j_r}\exp\{-\frac{1}{\alpha n d_r}(j_{r-1}+j_r)\}$$

$$\cdot I_1(\frac{2}{\alpha n d_r}\sqrt{j_{r-1}j_r})\}\frac{I_1(\frac{2}{\alpha n d_{k+1}}\sqrt{j_k(\alpha yn+r(n))})}{I_1(\frac{2}{\alpha n(t_{k+1}-t_0)}\sqrt{(\alpha xn+s(n))(\alpha yn+r(n))})}\frac{t_{k+1}-t_0}{d_{k+1}}$$

$$\cdot\ \sqrt{j_k/(\alpha xn+s(n))}\exp\{-\frac{1}{\alpha n d_{k+1}}(j_k+\alpha yn+r(n))+\frac{\alpha xn+s(n)+\alpha yn+r(n)}{\alpha n(t_{k+1}-t_0)}$$

$$= (1+o(1)) \sum_{j_k=\varepsilon n}^{\alpha n z k} \dots \sum_{j_1=\varepsilon n}^{\alpha n z_1} \frac{t_{k+1}-t_0}{d_1 d_{k+1}}(\alpha n)^{-k}\frac{I_1(\frac{2}{d_1}\sqrt{xj_1/\alpha n})I_1(\frac{2}{d_{k+1}}\sqrt{yj_k/\alpha n})}{I_1(\frac{2}{t_{k+1}-t_0}\sqrt{xy})}$$

$$\cdot\ \exp\{-\frac{1}{d_1}(x+\frac{j_1}{\alpha n})-\frac{1}{d_{k+1}}(y+\frac{j_k}{\alpha n})+\frac{x+y}{t_{k+1}-t_0}\}$$

$$\cdot\ \prod_{r=2}^{k}\frac{1}{d_r}\exp\{-\frac{1}{d_r}(\frac{j_{r-1}}{\alpha n}+\frac{j_r}{\alpha n})\}\ I_1(\frac{2}{d_r}\sqrt{j_{r-1}j_r/(\alpha n)^2}),$$

the last expression being a Riemann sum for the integral in (8). The theorem follows from (c) and (d).

We now state our main theorem.

THEOREM 1. *Assume that the hypotheses of Lemma 2 hold. Then* $\{Y_n(t) = \frac{1}{\alpha n} Z_{[nt]};\ 0 \leq t \leq 1\}$, *conditioned on* $Z_0 = [\alpha nx]+s(n)$ *and* $Z_n = [\alpha ny]+r(n)$ *converges in finite-dimensional distributions to a (Markovian) diffusion process*

$\{Y_{x,y}(t);\ 0 \leq t \leq 1\}$ *with* $Y_{x,y}(0) = x$, $Y_{x,y}(1) = y$. *Its respective transition densities are*

$$f_y(w,s+t|v,s) = {}_2g_y(v,w;s,t), \qquad 0 \leq s < 1, \quad 0 < t < 1-s$$

$$f_x(w,s-t|v,s) = {}_2g_x(v,w;1-s,t), \quad 0 < s \leq 1, \quad 0 < t < s.$$

For its infinitesimal means $a_y^+(s,z)$ *and* $a_x^-(s,z)$ *and its infinitesimal variances* $b_y^+(s,z)$ *and* $b_x^-(s,z)$ *hold the relations*

$$\lim_{t\searrow 0} \frac{1}{t} \int_0^\infty (w-z)f_y(w,s+t|z,s)dw = a_y^+(s,z) = {}_2c_y(s,z), \qquad 0 \leq s < 1,$$

$$\lim_{t\searrow 0} \frac{1}{t} \int_0^\infty (w-z)f_x(w,s-t|z,s)dw = a_x^-(s,z) = {}_2c_x(1-s,z), \quad 0 < s \leq 1,$$

$$b_y^+(s,z) = 2z = b_x^-(s,z).$$

The joint density of $(Y_{x,y}(t_1),\ldots,Y_{x,y}(t_k))$, $k \in \mathbb{N}$, $t_k < 1$, *is given by* (1) *with* $\nu = 2$, $t_0 = 0$, $t_{k+1} = 1$.

Proof. The family of distribution functions ${}_2F_k$ gotten in Lemma 2 satisfies the consistency conditions in Kolmogorov's existence theorem and hence determines a stochastic process. Its properties are immediate from Lemmas 1 and 2 and the special factorized form of the ${}_2f_k$.

REMARK 2. A simple comparison shows that $\{Y_{E,x}(t)\}$ and $\{Y_{x,0}(t)\}$ are equal in distribution, i.e. conditioning on "extinction *at* time n" and on "there are o(n) $\geq$ 1 particles at time n" lead to the same diffusion process.

To establish a relation with the process $\{Y_{LN,x}(t)\}$ obtained by Lamperti and Ney (1968) we introduce the following Laplace transforms. Let $0 < t \leq 1$ and r and s as in Lemma 2.

$$L_{LN}(\theta,t|x) = \lim_{n\to\infty} E(\exp\{-\frac{\theta}{\alpha n} Z_{[nt]}\} | Z_0 = [\alpha x n + s(n)]\ ,\ Z_n > 0)$$

$$L(\theta,t|x,y) = \lim_{n\to\infty} E(\exp\{-\frac{\theta}{\alpha n} Z_{[nt]}\}|Z_0=[\alpha xn+s(n))],\ Z_n=[\alpha yn+r(n)]).$$

LEMMA 3. *The Laplace transform* $L_{LN}(\theta,1|x)$ *has associated density*

$$w_x(z) = \begin{cases} \exp\{-z\}, & x = 0 \\ & \qquad z > 0. \\ \sqrt{x/z}\,I_1(2\sqrt{xz})\dfrac{\exp\{-(x+z)\}}{1-\exp\{-x\}}, & x > 0 \end{cases}$$

Proof. See Lamperti and Ney (1968) or check it directly.

THEOREM 2. *Let* $0 < t \leqslant 1$.

$$L_{LN}(\theta,t|x) = \int_0^\infty L(\theta,t|x,y)w_x(y)dy.$$

Proof. A direct verification is possible (see e.g. Abramowitz and Stegun (1965) for the necessary integral formulas) but there is a nice probabilistic argument. It is

$$E(\exp\{-\frac{\theta}{\alpha n} Z_{nt}\}|Z_0 = \alpha xn+s(n),\ Z_n > 0)$$

$$= E\{E(\exp\{-\frac{\theta}{\alpha n} Z_{nt}\}|Z_0 = \alpha xn+s(n),\ Z_n > 0)|\frac{Z_n}{\alpha n}\}$$

$$= \int_0^\infty E(\exp\{-\frac{\theta}{\alpha n} Z_{nt}\}|Z_0 = \alpha xn+s(n),\ Z_n = \alpha yn,\ Z_n > 0)$$

$$\cdot\ dP(\frac{1}{\alpha n} Z_n \leqslant y|Z_0 = \alpha xn+s(n),\ Z_n > 0),$$

and the theorem follows from Lemma 3.

We mention that the previous statement may easily be generalizaed to k-dimensional Laplace transforms.

In the following theorem we list some asymptotic properties of the infinitesimal means of the process $\{Y_{x,y}(t)\}$ which follow immediately from the corresponding behavior of the Bessel functions involved.

THEOREM 3.

(a) $\lim_{y\to\infty} a_y^+(s,y) = 0$

(b) $a_y^+(s,z) \sim \frac{2\sqrt{z}}{1-s}\sqrt{y} \quad y \to \infty,\ z > 0$

(c) $a_y^+(s,z) \sim -\frac{2z}{1-s} \quad z \to \infty$

(d) $a_y^+(s,z) \sim \frac{2}{1-s}(\sqrt{zy}-z) \quad s \nearrow 1$

(e) $a_y^+(s,z) = a_y^-(1-s,z)$.

REMARK 3. The relations in Theorem 3 may be read like this; e.g. (a) "Large populations which will not vary in size in the long run stagnate instantaneously (on the average)" and (c) "Large populations which will be comparatively small in the future exhibit (on the average) an (instantaneous) linear decrease".

4. Limit laws. In this section we state some limit laws which are special cases of Theorem 2.

THEOREM 4. *Let* $0 < t < 1$. *Then* $L(\theta,t|x,y) =$

$$
\begin{cases}
(1+\theta t(1-t))^{-2} & \text{if } x = y = 0\ , \\
(1+\theta t(1-t))^{-2}\exp\{\frac{x}{t}(1-t)(\frac{1}{1+\theta t(1-t)}-1)\} & \text{if } x > 0,\ y = 0, \\
(1+\theta t(1-t))^{-2}\exp\{\frac{yt}{1-t}(\frac{1}{1+\theta t(1-t)}-1)\} & \text{if } x = 0,\ y > 0, \\
(1+\theta t(1-t))^{-1}\exp\{\frac{yt^2+x(1-t)^2}{t(1-t)}(\frac{1}{1+\theta t(1-t)}-1)\} & \\
\cdot\ I_1\{2\sqrt{xy}(1+\theta t(1-t))^{-1}\}/I_1(2\sqrt{xy}) & \text{if } x > 0,\ y > 0.
\end{cases}
$$

REMARK 4. This means that $\frac{1}{\alpha n}Z_{[nt]}$, conditioned on

$Z_0 = [\alpha xn]+s(n)$ and $Z_n = [\alpha yn]+r(n)$, converges in distribution to a random variable Z which may be represented by means of the following *independent* random variables as

$$Z = \begin{cases} E + E' & \text{if } x = y = 0, \\ E + E' + E_1 + \ldots + E_N & \text{if } x > 0,\ y = 0, \\ E + E' + E_1 + \ldots + E_M & \text{if } x = 0,\ y > 0, \\ E + E_1 + \ldots + E_S + E_1' + \ldots + E_T' & \text{if } x > 0,\ y > 0, \end{cases}$$

where $E, E', E_1, E_2, \ldots, E_1', \ldots$ are exponential random variables with mean $t(1-t)$, N, M and S are Poisson random variables with parameters $\frac{x}{t}(1-t)$, $\frac{y\,t}{1-t}$ and $\frac{yt^2+x(1-t)^2}{t(1-t)}$ respectively, and T has the probability generating function $b_{x,y}(z) = I_1(2z\sqrt{xy})/I_1(2\sqrt{xy})$ with the notable properties

$$b_{x,y}^{(2r)}(0) = 0,\ r = 0,1,2,\ldots \text{ and } \lim_{x\searrow 0} b_{x,y}(z) = \lim_{y\searrow 0} b_{x,y}(z) = z.$$

5. <u>The limiting diffusion in the GWI case.</u>

Let $\{X_n\}$ be a critical GWI and $P_n(i,j)$ its n-step transition probability function. Assume that its immigration distribution $\{P(0,k)|k = 0,1,2,\ldots\}$ satisfies $0 < P(0,0) < 1$ and $\sum_{k=2}^{\infty} P(0,k)k\log k < \infty$. As to its offspring distribution $\{Q(1,k)|k = 0,1,2,\ldots\}$ we suppose that the hypotheses at the beginning of Section 3 hold. Under these assumptions Mellein (1982) obtained a local limit theorem for $P_n(i,j)$, corresponding to (2), which may be used to prove the following theorem.

As in Section 3 we define $S' = \{k \in \mathbb{N} \cup \{0\} | P_n(0,k) > 0$ for some $n = n(k) \in \mathbb{N}\}$ which obviously satisfies $S' \supset \{0,m,m+1,..\}$ for some $m \in \mathbb{N}$.

Let $\alpha = \frac{1}{2}\sum_{k=2}^{\infty} k(k-1)Q(1,k)$ (as in the previous sections) and $\gamma = \frac{1}{\alpha}\sum_{k=1}^{\infty} kP(0,k)$.

THEOREM 5. *Let* s *and* r *be integervalued functions with* $s(n) = o(n)$; $r(n) = o(n)$, $n \to \infty$, $s(n) \geqslant 0$ *if* $x = 0$ *and* $r(n) \in S'$ *if* $y = 0$. *Then the following statements hold. The finite-dimensional distributions of* $\{\frac{1}{\alpha n} X_{[nt]};\ 0 \leqslant t \leqslant 1\}$, *conditioned on* $X_0 = [\alpha xn]+s(n)$ *and* $X_n = [\alpha yn]+r(n)$ *converge to those of a (Markovian) diffusion process* $\{W_{x,y}(t);\ 0 \leqslant t \leqslant 1\}$ *with* $W_{x,y}(0) = x$, $W_{x,y}(1) = y$,

infinitesimal means

$$_I a_y^{+}(s,z) = {}_\gamma \dot{c}_y(s,z)\ , \quad 0 \leqslant s < 1,$$

$$_I a_y^{-}(s,z) = {}_\gamma c_x(1-s,z), \quad 0 < s \leqslant 1,$$

infinitesimal variances

$$_I b_y^{+}(s,z) = 2z = {}_I b_x^{-}(s,z) \quad 0 < s < 1$$

and transition densities

$$_I f_y(w,s+t|v,s) = {}_\gamma g_y(v,w;s,t)\ , \quad 0 \leqslant s < 1,\ 0 < t < 1-s$$

$$_I f_x(w,s-t|v,s) = {}_\gamma g_x(v,w;1-s,t), \quad 0 < s \leqslant 1,\ 0 < t < s.$$

The joint density of $(W_{x,y}(t_1),\ldots,W_{x,y}(t_k))$, $k \in \mathbb{N}$, $t_k < 1$ *is given by* (1) *with* $\nu = \gamma$, $t_0 = 0$, $t_{k+1} = 1$.

REMARK 5. It is easy to deduce from the foregoing theorem the following limit law. Let $xy > 0$. Conditional on $X_0 = [\alpha xn]+s(n)$ and $X_n = [\alpha yn]+r(n)$, $\frac{1}{\alpha n}X_{[nt]}$ converges in distribution to the random variable

$$G + E_1 + \ldots + E_N + E_1' + \ldots + E_M'$$

with

G a gamma r.v. with parameters $(\frac{t}{1-t},\gamma)$,

N a Poisson r.v. with parameter $\frac{yt^2+x(1-t)^2}{t(1-t)}$,

M a r.v. with probability generating function

$$b_{\gamma,x,y}(z) = \frac{1}{I_{\gamma-1}(2\sqrt{xy})} \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\gamma+k)}(\sqrt{xy})^{2k+\gamma-1} z^{2k},$$

$E_1,E_2,\ldots,E_1',E_2',\ldots$ exponential random variables with mean $t(1-t)$ and all previous random variables being independent.

REMARK 6. a) It is well-known that the limit distribution $(n \to \infty)$ of $\frac{1}{\alpha n}Z_{[nt]}$, conditioned on $Z_0 = 1$, $Z_n > 0$ and of $\frac{1}{\alpha n}X_{[nt]}$ coindice in the case $t = 1$, $\gamma = 1$, i.e. the effect (on the limit law of $Z_n/\alpha n$) of conditioning on non-extinction is the same as immigrating at an average rate of α. But this is no longer true if $t < 1$.

$$\lim_{n\to\infty} E(\exp\{-\frac{\theta}{\alpha n}Z_{[nt]}\}|Z_0 = 1, Z_n > 0) = ((1+\theta t)(1+\theta t(1-t)))^{-1}$$

$$\lim_{n\to\infty} E(\exp\{-\frac{\theta}{\alpha n}X_{[nt]}\}|X_0 = 1) = (1+\theta t)^{-\gamma} .$$

b) A comparison of Theorem 1 and 5 shows that the case $\gamma = 2$ plays an outstanding rôle. If $\{X_n\}$ and $\{Z_n\}$ are initiated in the same manner and behave similarly at time n then they are indistinguishable (in distribution) at intermedium times if the immigration rate equals 2α $(n \to \infty)$.

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