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ON PAIRWISE LINDELÖF SPACES

by

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ABSTRACT. In this paper we define pairwise Lindelöf spaces and study their properties and their relations with other topological spaces. We also study certain conditions by which a bitopological space will reduce to a single topology. Several examples are discussed and many well known theorems are generalized concerning Lindelöf spaces.

RESUMEN. En este artículo se definen espacios p-Lindelöf y se estudian sus propiedades y relaciones con otros tipos de espacios topológicos. También se estudian ciertas condiciones bajo las cuales un espacio bitopológico (con dos topologías) se reduce a uno con una sola topología. Se discuten varios ejemplos y se generalizan varios teoremas sobre espacios de Lindelöf.

1. <u>Introducción</u>. Kelly [6] introduced the notion of a bitopological space, i.e. a triple (X,τ_1,τ_2) where X is a set and τ_1, τ_2 are two topologies on X, he also defined pairwise Haus-

dorff, pairwise regular, pairwise normal spaces, and obtained generalizations of several standard results such as Urysohn's Lemma and the Tietze extension theorem. Several authors have since considered the problem of defining compactness for such spaces: see Kim [7], Fletcher, Hoyle and Patty [4], and Birsan [1]. Cooke and Reilly [2] have discussed the relations between these definitions.

In this paper we give a definition of pairwise Lindelöf bitopological spaces and derive some related results.

We will use p- , s- to denote *pairwise* and *semi-*, respectively, e.g. p- compact, s- compact stand for pairwise compact and semi-compact respectively.

The τ_i-closure, τ_i-interior of a set A will be denoted by Cl_iA and Int_iA respectively. The product topology of τ_1 and τ_2 will be denoted by $\tau_1 \times \tau_2$.

Let $\mathbb{R}$, $\mathbb{I}$, $\mathbb{N}$ denote the set of all real numbers, the interval $[0,1]$, and the natural numbers respectively. Let τ_d, τ_u, τ_c, τ_ℓ, τ_r denote the *discrete, usual, cocountable, left-ray* and *rigth-ray* topologies on $\mathbb{R}$ (or $\mathbb{I}$).

2. Pairwise Lindelöf Spaces.

Let us recall known definitions which are used in the sequel.

2.1 [4]. A cover $\mathcal{U}$ of the bitopological space (X,τ_1,τ_2) is called $\tau_1\tau_2$-*open* if $\mathcal{U} \subset \tau_1 \cup \tau_2$. If, in addition, $\mathcal{U}$ contains at least one non-empty member of τ_1 and at least one non-empty member of τ_2, it is called p-*open*.

2.2 [4]. A bitopological space is called p-*compact* if every p-open cover of the space has a finite subcover.

2.3 [3]. A bitopological space is calle s-*compact* if every $\tau_1\tau_2$-open cover of the space has a finite subcover.

2.4 [1]. A bitopological space (X,τ_1,τ_2) is called τ_1 *compact with respect to* τ_2 if for each τ_1-open cover of X there is a finite τ_2-open subcover.

2.5 [1]. A bitopological space (X, τ_1,τ_2) is called B-*compact* if it is τ_1 compact with respect to τ_2 and τ_2 compact with respecto to τ_1.

If we replace the word "finite" by the word "countable" in definitions 2.2, 2.3 and 2.4, then we obtain the definition of p-*Lindelöf*, s-*Lindelöf*, and (X,τ_1,τ_2) *is* τ_1 *Lindelöf with respect to* τ_2, respectively.

2.6 A bitopological space (X,τ_1,τ_2) is called B-*Lindelöf* if it is τ_1 Lindelöf with respecto to τ_2 and τ_2 Lindelöff with respecto to τ_1.

It is clear that (X,τ_1,τ_2) is s-Lindelöf if and only if (X,τ) is Lindelöf where τ is the least-upper-bound topology of τ_1 and τ_2. It is also clear that if (X,τ_1,τ_2) is B-Lindelöff then each (X,τ_i) must be a Lindelöf space for i = 1,2.

2.7 When we say that a bitopological space (X,τ_1,τ_2) has a particular topological property, without referring specially to τ_1 or τ_2, we shall then mean that both τ_1 and τ_2 have the property; for instance, (X,τ_1,τ_2) is said to be Hausdoff if both (X,τ_1) and (X,τ_2) are Hausdorff.

THEOREM 2.8. *The bitopological space* (X,τ_1,τ_2) *is* s-*Lin-*

delöf if and only if it is Lindelöf and p-*Lindelöf*.

Proof. Necessity follows inmediately from the observation that any p-open, τ_1-open or τ_2-open cover of (X,τ_1,τ_2) is $\tau_1\tau_2$-open. Conversely, if a $\tau_1\tau_2$-open cover of (X,τ_1,τ_2) is not p-open, then it is τ_1-open or τ_2-open.

EXAMPLE 2.9. The bitopological space $(\mathbb{R},\tau_d,\tau_c)$ is p-Lindelöf but is not s-Lindelöf.

EXAMPLE 2.10. Consider the two topologies τ_1,τ_2 on $\mathbb{R}$ defined by the basis

$$\mathcal{B}_1 = \{(-\infty,a):a > 0\} \cup \{\{x\}:x > 0\}, \text{ and}$$

$$\mathcal{B}_2 = \{(a, \infty):a < 0\} \cup \{\{x\}:x < 0\}.$$

Then $(\mathbb{R},\tau_1,\tau_2)$ is p-Lindelöf but is not Lindelöf. It is also clear that $(\mathbb{R},\tau_1,\tau_2)$ is not B-Lindelöf, for the τ_1-open cover $\{(-\infty,1)\} \cup \{\{x\}:x \geqslant 1\}$ of $\mathbb{R}$ has no countable τ_2-open subcover.

2.11 [8]. A bitopological space (X,τ_1,τ_2) is called p-*countably compact* if every countably p-open cover of X has a finite subcover.

2.12 A bitopological space (X,τ_1,τ_2) is called s-*countable compact* if every countably $\tau_1\tau_2$-open cover of X has a finite subcover.

2.13 A bitopological space (X,τ_1,τ_2) is called τ_1-*countably compact with respect to* τ_2 if for each countably τ_1-open cover of X there is a finite τ_2-open subcover.

2.14 A bitopological space (X,τ_1,τ_2) is called B-*countalby compact* if it is τ_1 countably compact with respect to τ_2 and τ_2 countably compact with respecto to τ_1.

The following fact is obvious:

THEOREM 2.15. (i) *Every* p(*resp.* s,B)-*compact space is* p(*resp.* s,B)-*countably compact and* p(*resp.* s,B)-*Lindelöf.*
(ii) *Every* p(*resp.* s,B)-*countably compact* p(*resp.* s,B)-*Lindelöf space is* p(*resp.* s,B)-*compact.*

EXAMPLE 2.16. The bitopological space $(\mathbb{R},\tau_d,\tau_c)$ is a p-Lindelöf space which is neither p-countably compact nor p-compact.

EXAMPLE 2.17. Let τ_s denotes the Sorgenfrey topology on $\mathbb{R}$. Then the bitopological space $(\mathbb{R},\tau_u,\tau_s)$ is s-Lindelof but is not B-Lindelöf, because the τ_s-open cover $\{[-n,n):n\in\mathbb{N}\}$ of $\mathbb{R}$ has no τ_u-open countable subcover. It is also clear that the space $(\mathbb{R},\tau_u,\tau_s)$ is neither s-countably compact nor s-compact.

EXAMPLE 2.18. It is clear that the bitopological space $(\mathbb{N},\tau_d,\tau_d)$ is B-Lindelöf but is neither B-countably compact nor B-compact.

THEOREM 2.19. *If* (X,τ_1,τ_2) *is a hereditary Lindelöf space then it is* s-*Lindelöf.*

Proof. Let $\mathcal{C} = \{U_\alpha : \alpha\in\Lambda\} \cup \{V_\beta : \beta\in\Gamma\}$ be a $\tau_1\tau_2$-open cover of X, where $U_\alpha\in\tau_1$ for each $\alpha\in\Lambda$ and $V_\beta\in\tau_2$ for each $\beta\in\Gamma$. Since $U = \cup\{U_\alpha\ \alpha\in\Lambda\}$ is τ_1-Lindelöf, there exists a countable set $\Lambda_1\subset\Lambda$ such that $U = \cup\{U_\alpha:\alpha\in\Lambda_1\}$. Similarly, since $V = \cup\{V_\beta:\beta\in\Gamma\}$ is τ_2-Lindelöf, there exists a countable set $\Gamma_1\subset\Gamma$ such that $V = \cup\{V_\beta:\beta\in\Gamma_1\}$. It is clear that $\{U_\alpha:\alpha\in\Lambda_1\}\cup\{V_\beta:\beta\in\Gamma_1\}$ is a countable subcover of $\mathcal{C}$ for X.

COROLLARY 2.20. *Every second countable bitopological space is* s-*Lindelöf.*

EXAMPLE 2.21. Let $X = \mathbb{R}\times I$ and $<$ be the lexicographical order on X. Let

$$\mathcal{B}_1 = \{[x,y) : x < y \ ; \ x,y \in X\} \quad \text{and} \quad \mathcal{B}_2 = \{(x,y] : x < y \ ; \ x,y \in X\}.$$

Let τ_1, τ_2 be the topologies on X which generated by the basis $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Then (X,τ_1,τ_2) is a Lindelöf space which is not p-Lindelöf, because the p-open cover

$$\{[(0,x),(1,x)),((0,x),(1,x)] : x \in \mathbb{R}\}$$

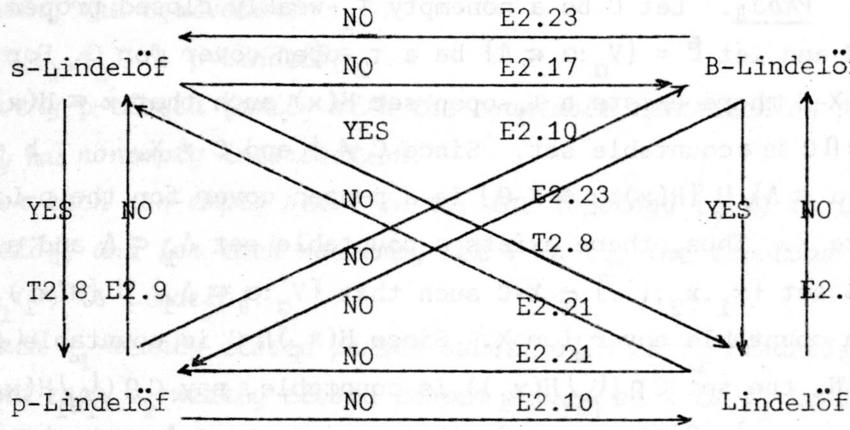
of X has no countable subcover. It is clear that (X,τ_1,τ_2) is neither s-Lindelöf nor B-Lindelöf.

EXAMPLE 2.22. Let X and τ_1 be the same as in example 2.21. Then the bitopological space (X,τ_1,τ_2) is not hereditary Lindelöf but it is s-Lindelöf.

EXAMPLE 2.23. Let $X = \mathbb{R}$, $\mathcal{B}_1 = \{X,\{x\} : x \in X-\{0\}\}$ and $\mathcal{B}_2 = \{X,\{x\} : x \in X-\{1\}\}$. Let τ_1,τ_2 be the topologies on X which are generated by the bases $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Then (X,τ_1,τ_2) is B-Lindelöf, for any τ_1-open cover of X or any τ_2-open cover of X must contain X as a member. However, (X,τ_1,τ_2) is not p-Lindelöf, for the p-open cover $\{\{x\} : x \in X\}$ of X has no countable subcover.

We may summarize some of the above examples and theorems by the diagram on the next page (T stands for theorem while E stands for example).

2.24 [7]. If τ is a topology on X and A is a non-empty subset of X then the *adjoint topology* (denoted by $\tau(A)$) is the topology on X defined by $\tau(A) = \{\phi,X\} \cup \{A\cup B : B \in \tau\}$.



2.25 A family $\mathcal{F}$ of nonvoid subsets of X is $\tau_1\tau_2$-closed if every member of $\mathcal{F}$ is τ_1-closed or τ_2-closed.

2.26 [8]. A family $\mathcal{F}$ of nonvoid τ_1- or τ_2-closed sets in X is p-*closed* if $\mathcal{F}$ contains members F_1 and F_2 such that F_1 is a τ_1-closed proper subset of X and F_2 is a τ_2-closed proper subset of X.

2.27 [6]. A set U in a topological space (X,τ) is called *weakly open* if for any $p \in U$ there exists an open set V containing p such that V-U is a countable set. A set F is called *weakly closed* if X-F is weakly open. If A is a subset of X and $p \in X$, then p is called a *weak-interior point* of A if there exists a weakly open set V containing p such that $V \subset A$. The set of all weak-interior points of a set A is denoted by WInt A.

It is clear that WInt A is the largest weakly open set contained in A. It is also clear that WInt A = A if and only if A is weakly open, and Int B $\subset$ WInt B for any set $B \subset X$.

LEMMA 2.28. *Let* (X, τ_1, τ_2) *be a* p-*Lindelöf space and* C *be a weakly closed proper subset in* (X,τ_1). *Then* C *is* τ_2-*Lindelöf*.

Proof. Let C be a nonempty τ_1-weakly closed proper subset of X and let $\mathcal{C} = \{V_\alpha : \alpha \in \Lambda\}$ be a τ_2-open cover for C. For each $x \in X-C$ there exists a τ_1-open set $H(x)$ such that $x \in H(x)$ and $H(x) \cap C$ is a countable set. Since $C \neq \phi$ and $C \neq X$, then $\{V_\alpha : \alpha \in \Lambda\} \cup \{H(x) : x \in X-C\}$ is a p-open cover for the p-Lindelöf space X. Thus, there exists a countable set $\Lambda_1 \subset \Lambda$ and a countable set $\{x_1, x_2, \ldots\} \subset X-C$ such that $\{V_\alpha : \alpha \in \Lambda_1\} \cup \{H(x_1), H(x_2), \ldots\}$ is a countable cover for X. Since $H(x_i) \cap C$ is countable for all $i \in \mathbb{N}$, the set $C \cap (\bigcup_{i=1}^{\infty} H(x_i))$ is countable, say $C \cap (\bigcup_{i=1}^{\infty} H(x_i)) = \{y_1, y_2, \ldots\}$. Since $y_i \in C$, there exists $\alpha_i \in \Lambda$ such that $y_i \in V_{\alpha_i}$. It is clear now that $\{V_\alpha : \alpha \in \Lambda_1\} \cup \{V_{\alpha_i} : i \in \mathbb{N}\}$ is a countable subcover for C. Hence C is τ_2-Lindelöf.

Since every closed set is weakly closed, we have the following corollary to Lemma 2.28.

COROLLARY 2.29. *A τ_i-closed proper subset of a* p-*Lindelöf space is τ_j-Lindelöf* $(i \neq j;\ i,j = 1,2)$.

Using a similar technique as above, we obtain the following:

COROLLARY 2.30. *A τ_i-closed proper subset of a* p-*compact space is τ_j-compact* $(i \neq j;\ i,j = 1,2)$.

It is important to note that the word "proper" in Lemma 2.28 can not be removed. For example, $\mathbb{R}$ is τ_c-closed but $\mathbb{R}$ is not τ_d-Lindelöf in example 2.16.

We now obtain four alternative characterizations of p-Lindelöf spaces.

THEOREM 2.31. *For the bitopological space* (X, τ_1, τ_2) *the* *following are equivalent:*

(a) (X,τ_1,τ_2) *is* p-*Lindelöf.*

(b) *Every* p-*closed family with the countable intersection property has nonempty intersection.*

(c) *For each non-empty set* V *in* τ_1, *the topology* $\tau_2(V)$ *is Lindelöf, and for each non-empy set* V *in* τ_2, *the topology* $\tau_1(V)$ *is Lindelöf.*

(e) *Each* τ_1*-weakly closed proper subset of* X *is* τ_2*-Lindelöf, and each* τ_2*-weakly closed proper subset of* X *is* τ_1*-Lindelöf.*

Proof. The fact that (a) is equivalent to (b) is obvious. The equivalence of (a), (c) and (d) can be obtained in an analogous way to the proof of [2,Theorem 2]. The fact that (a) implies (e) is due to Lemma 2.28. The fact that (e) implies (e) is obvious.

An easy characterization of s-Lindelöf spaces can be found in the following theorem.

THEOREM 2.32. *A bitopological space* (X,τ_1,τ_2) *is* s-*Lindelöf if and only if every* $\tau_1\tau_2$*-closed family with the countable intersection property has nonempty intersection.*

THEOREM 2.33. *Let* (X,τ_1,τ_2) *be* B-*compact and* (Y,τ_1',τ_2') *be* B-*Lindelöf. Then* $(X\times Y,\tau_1\times\tau_1',\tau_2\times\tau_2')$ *is* B-*Lindelöf.*

EXAMPLE 2.34. Let τ_f denote the cofinite topology on $\mathbb{R}$. Then $(\mathbb{R},\tau_f,\tau_d)$ is p-compact. However, the space $(\mathbb{R}^2,\tau_f\times\tau_f, \tau_d\times\tau_d)$ is not even p-Lindelöf, for the p-open cover $\{\mathbb{R}\times(\mathbb{R}-\{0\})\}\cup\{(x,0):x\in\mathbb{R}\}$ of $\mathbb{R}^2$ has no countable subcover.

2.35 [9]. A space (X,τ_1,τ_2) is said to be p-*Hausdorff* if, for any distinct points x and y, there is a τ_1-neighbourhood U of x and a τ_2-neighbourhood V of y such that $U \cap V = \phi$.

We observe that if (X,τ_1,τ_2) is p-Hausdorff, then both τ_1 and τ_2 are T_1-topologies. The following theorem characterizes p-Hausdorff spaces.

THEOREM 2.36. *The following properties are equivalent:*
(a) *The bitopological space* (X,τ_1,τ_2) *is* p-*Hausdorff*.
(b) *For each* $x \in X$,

$$\{x\} = \bigcap_{\alpha \in \Delta} \{Cl_1 U_\alpha : U_\alpha \text{ is a } \tau_2 \text{ neighbourhood of } x\}$$

and

$$\{x\} = \bigcap_{\alpha \in \Delta} \{Cl_2 U_\alpha : U_\alpha \text{ is a } \tau_1 \text{ neighbourhood of } x\}.$$

(c) *The diagonal* $D = \{(x,x) : x \in X\}$ *is a closed subset in each of the product topologies* $(X\times X, \tau_1\times\tau_2)$ *and* $(X\times X, \tau_2\times\tau_1)$.

Proof. (a) implies (b). Let $x \in X$ and $y \in X$ such that $y \neq x$. By (a) there exists a τ_1-open set V_1 and a τ_2-open set V_2 such that $y \in V_1$, $x \in V_2$ and $V_1 \cap V_2 = \phi$. This implies that $y \notin Cl_1 V_2$. This proves the first part of (b). The proof of the second part of (b) is similar to the one we just proved.

(b) implies (c). Let $(x,y) \in X\times X - D$. Then $x,y \in X$ and $x \neq y$. By the second part of (b), there exists a τ_1-open set U_1 containing x such that $y \in X - Cl_2 U_1$. Let $U_2 = X - Cl_2 U_1$. Then U_2 is a τ_2-open set and it is easy to check that $(x,y) \in U_1 \times U_2 \subset X\times X - D$. Hence D is a closed set in the topological space $(X\times X, \tau_1\times\tau_2)$. In a similar way we can prove that D is $\tau_2\times\tau_1$-closed subset of $X\times X$.

(c) implies (a). Let $x,y \in X$ such that $x \neq y$. Then $(x,y) \in X\times X - D$. Since D is a $\tau_1\times\tau_2$-closed set, there exists a τ_1-open set U_1 and a τ_2-open set U_2 such that $(x,y) \in U_1 \times U_2 \subset X\times X-D$. It is clear now that $x \in U_1$, $y \in U_2$ and $U_1 \cap U_2 = \phi$.

Recall that a space (X,τ) in which every countable intersection of open sets is open, is called a *P-space*.

COROLLARY 2.37. *Let* (X,τ_1,τ_2) *be a* p-*Hausdorff* P-*space. Then every* τ_i-*Lindelöf subset is* τ_j-*closed* $(i \neq j;\ i,j = 1,2)$.

Proof. Let A be a τ_i-Lindelöf subset and $x \in X-A$. By Theorem 2.36 we have

$$\{x\} = \bigcap_{\alpha\in\Delta} \{Cl_i U_\alpha : U_\alpha \text{ is a } \tau_j \text{ neighbourhood of } x\}$$

$(i \neq j;\ i,j = 1,2)$. Since $A \subset X-\{x\}$, therefore $\{X - Cl_i U_\alpha : \alpha \in \Delta\}$ is a τ_i-open cover of the τ_i-Lindelöf set A. Thus there exists a countable set $\Delta_1 \subset \Delta$ such that $\{X - Cl_i U_\alpha : \alpha \in \Delta_1\}$ is a cover for A, i.e. $A \subset \bigcup_{\alpha\in\Delta_1} X-Cl_i U_\alpha$. Let $U = \bigcap_{\alpha\in\Delta_1} U_\alpha$. Then U is a τ_j-open set, contains x and $U \subset X-A$. Hence A is τ_j-closed.

Using the same technique as above we obtain the following.

COROLLARY 2.38. *Let* (X,τ_1,τ_2) *be* p-*Hausdorff. Then every* τ_i-*compact subset is* τ_j-*closed* $(i \neq j;\ i,j = 1,2)$.

2.39 [6]. In a space (X,τ_1,τ_2), τ_1 *is said to be regular with respect to* τ_2 if, for each point x in X and each τ_1-closed set P such that $x \notin P$, there are a τ_1-open set U and a τ_2-open set V such that $x \in U$, $P \subset V$ and $U \cap V = \phi$.

(X,τ_1,τ_2) is p-*regular* if τ_1 is regular with respect to τ_2 and vice versa.

2.40 [9]. In a bitopological space (X,τ_1,τ_2), we say that

τ_1 *is coupled to* τ_2 iff for all $U \in \tau_1$, $Cl_1 U \subset Cl_2 U$.

It is interesting to note that if τ_1 is regular with respect to τ_2 and τ_2 is coupled to τ_1, then $\tau_1 \subset \tau_2$. Thus, if (X,τ_1,τ_2) is p-regular and τ_i is coupled to τ_j $(i \neq j,\ i,j = 1,2)$ then $\tau_1 = \tau_2$ and the resulting single topology is regular. It is also interesting to note that if τ_1' is coupled to τ_2' and τ_1' is regular with respect to τ_2' then (X,τ_1') is regular.

2.41 [6]. A space (X,τ_1,τ_2) is said to be p-*normal* if, given a τ_1-closed set C and a τ_2-closed set F such that $C \cap F = \phi$, there are a τ_1-open set G and a τ_2-open set V such that $F \subset G$, $C \subset V$ and $V \cap G = \phi$.

THEOREM 2.42. *Every* p-*regular,* p-*Lindelöf bitopological space* (X, τ_1,τ_2) *is* p-*normal.*

<u>*Proof*</u>. Let A be a nonempty τ_1-closed set and B be a nonempty τ_2-closed set with $A \cap B = \phi$. Since (X,τ_1,τ_2) is p-regular for each $a \in A$, there exist a τ_2-open set G_a and a τ_1-closed set F_a with $a \in G_a \subset F_a \subset X-B$. Also, for each $b \in B$, there exist a τ_1-open set C_b and a τ_2-closed set M_b with $b \in C_b \subset M_b \subset X-A$. Let $\mathcal{C} = \{C_b : b \in B\} \cup \{X-B\}$ and $\mathcal{G} = \{G_a : a \in A\} \cup \{X-A\}$. Since $\mathcal{C}$ and $\mathcal{G}$ are p-open covers for the p-Lindelöf space X, there exist countable subcollections $\{C_1,C_2,\ldots\}$ of $\mathcal{C}$ and $\{G_1,G_2,\ldots\}$ of $\mathcal{G}$ such that $A \subset \bigcup_{i=1}^{\infty} G_i$ and $B \subset \bigcup_{i=1}^{\infty} C_i$. Let $V_1 = C_1$ and, for each positive integer $n > 1$, let $V_n = C_n - \bigcup_{i=1}^{n-1} F_i$. For each positive integer n, let $H_n = G_n - \bigcup_{i=1}^{n} M_i$. Let $V = \bigcup_{n=1}^{\infty} V_n$ and $H = \bigcup_{n=1}^{\infty} H_n$. Then $V \in \tau_1$, $H \in \tau_2$, $A \subset H$ and $B \subset V$. Furthermore, $x \in H \cap V$, then $x \in H_m \cap V_n$ for some m and n, and so $x \in (G_m - \bigcup_{i=1}^{m} M_i) \cap (C_n - \bigcup_{i=1}^{n-1} F_i)$. Considering separately the cases $m > n$ and $m \leqslant n$ yields a contradiction and so $H \cap V = \phi$. Thus (X,τ_1,τ_2) is p-normal. ■

Let X be a fixed nonempty set, and

$\mathcal{B}_X = \{(X,\tau,\tau'): \tau$ and τ' are topologies on $X\}$.

Define the partial ordering $\leqslant$ on $\mathcal{B}_X$ by:

$(X,\tau_1,\tau_2) \leqslant (X,\tau_1',\tau_2')$ iff $\tau_1 \subset \tau_1'$ and $\tau_2 \subset \tau_2'$.

Then we have the following theorem.

THEOREM 2.43. *Let* $\mathcal{L} = \{(X,\tau,\tau') \in \mathcal{B}_X : (X,\tau,\tau')$ *is a* p-*Lindelöf* P-*space*$\}$ *and* $\mathcal{H} = \{(X,\tau,\tau') \in \mathcal{B}_X : (X,\tau,\tau')$ *is a* p-*Hausdorff* P-*space*$\}$. *If* $(X,\tau_1,\tau_2) \in \mathcal{L} \cap \mathcal{H}$, *then* (X,τ_1,τ_2) *is a minimal element of* $\mathcal{H}$ *and a maximal element of* $\mathcal{L}$.

Proof. Suppose $(X,\tau_1^*,\tau_2^*) \in \mathcal{H}$ such that $(X,\tau_1^*,\tau_2^*) \leqslant (X,\tau_1,\tau_2)$. Therefore $\tau_1^* \subset \tau_1$ and $\tau_2^* \subset \tau_2$. Let $G \in \tau_1 - \{\phi\}$. Then X-G is a τ_1-closed proper subset of X. Since (X,τ_1,τ_2) is p-Lindelöf, by Corollary 2.29, X-G is τ_2-Lindelöf. But $\tau_2^* \subset \tau_2$. Therefore X-G is τ_2^*-Lindelöf. Since (X,τ_1^*,τ_2^*) is p-Hausdorff P-space, by corollary 2.37, X-G is τ_1^*-closed. Hence $G \in \tau_1^*$. Consequently $\tau_1 = \tau_1^*$. In a similar way we can show that $\tau_2 = \tau_2^*$.

Now, let $(X,\tau_1',\tau_2') \in \mathcal{L}$ such that $(X,\tau_1,\tau_2) \leqslant (X,\tau_1',\tau_2')$. Then $\tau_1 \subset \tau_1'$ and $\tau_2 \subset \tau_2'$. Let $U \in \tau_1' - \{\phi\}$. Then X-U is a τ_1'-closed proper subset of X. Since (X,τ_1',τ_2') is p-Lindelöf, by Corollary 2.29, X-U is τ_2'-Lindelöf. But $\tau_2 \subset \tau_2'$. Therefore X-U is τ_2-Lindelöf. Since (X,τ_1,τ_2) is p-Hausdorff P-space, by corollary 2.37, X-U is τ_1-closed. Hence $U \in \tau_1$. Consequently $\tau_1' = \tau_1$. In a similar way we can show that $\tau_2' = \tau_2$.

Using the same technique as above we obtain the following theorem.

THEOREM 2.44. *Let* $\mathcal{C} = \{(X,\tau,\tau') \in \mathcal{B}_X : (X,\tau,\tau')$ *is* p-*compact*$\}$, *and* $\mathcal{H} = \{(X,\tau,\tau') \in \mathcal{B}_X : (X,\tau,\tau')$ *is* p-*Hausdorff*$\}$. *If* $(X,\tau_1,\tau_2) \in \mathcal{C} \cap \mathcal{H}$, *then* (X,τ_1,τ_2) *is a minimal element of* $\mathcal{H}$ *and a maximal element of* $\mathcal{C}$.

2.45 A function $f:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is p-*continuous* (*p-open*, *p-closed*, *p-homeomorphism*, respectively) iff $f:(X,\tau_1) \to (Y,\sigma_1)$ and $f:(X,\tau_2) \to (Y,\sigma_2)$ are continuous (open, closed, homeomorphism, respectively).

THEOREM 2.46. *Let* $f:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ *be a* p-*continuous onto map.*

(a) *If* (X,τ_1,τ_2) *is* p-*Lindelöf* (s-*Lindelöf*, B-*Lindelöf*, *respectively*), (Y,σ_1,σ_2) *is* p-*Lindelöf* (s-*Lindelöf*, B-*Lindelöf*, *respectively*).

(b) *If* (X,τ_1,τ_2) *is* p-*compact* (s-*compact*, B-*compact*, *respectively*), *then* (Y,σ_1,σ_2) *is* p-*compact* (s-*compact*, B-*compact*, *respectively*).

(c) *If* f *is one-to-one*, (Y, σ_1,σ_2) *is* p-*Hausdorff* P-*space and* (X,τ_1,τ_2) *is* p-*Lindelöf*, *then* f *is a homeomorphism.*

(d) *If* f *is one-to-one*, (Y,σ_1,σ_2) *is* p-*Hausdorff and* (X,τ_1,τ_2) *is* p-*compact*, *then* f *is a homeomorphism.*

Proof. (a) Let $\mathcal{E} = \{V_\alpha:\alpha \in \Delta\} \cup \{U_\alpha:\alpha \in \Delta\}$ be a p-open cover of V such that $V_\alpha \in \sigma_1$ and $U_\alpha \in \sigma_2$ $(\alpha \in \Delta)$. Then $\{f^{-1}(V_\alpha):\alpha \in \Delta\} \cup \{f^{-1}(U_\alpha):\alpha \in \Delta\}$ is a p-open cover of X because f is p-continuous and onto. Since (X,τ_1,τ_2) is p-Lindelöf, there exists a countable set $\Delta_1 \subset \Delta$ such that $\{f^{-1}(V_\alpha):\alpha \in \Delta_1\} \cup \{f^{-1}(U_\alpha):\alpha \in \Delta_1\}$ is a cover for X. Thus $\{V_\alpha:\alpha \in \Delta_1\} \cup \{U_\alpha:\alpha \in \Delta_1\}$ is a countable subcover of $\mathcal{E}$ for X. The remaining parts of the statement (a) are similarly proved.

(b) The proof is similar to that in (a).

(c) It suffices to show that f is p-closed. Let A be a τ_1-closed proper subset of X. Then, by Corollary 2.29, A is τ_2-Lindelöf. Hence f(A) is σ_2-Lindelöf because $f:(X,\tau_2) \to (Y,\sigma_2)$ is continuous. By Corollary 2.37, f(A) is σ_1-closed. Similarly, it can be shown that the image of every τ_2-closed subset of X is a σ_2-closed subset of Y. Hence f is p-closed.

(d) The proof is similar to that in (c).

3. Conditions under which a bitopological space is reduced to a single topology.

THEOREM 3.1. *Let* (X,τ_1,τ_2) *be a Hausdorff* p-*Lindelöf* p-*space. Then* $\tau_1 = \tau_2$.

Proof. Let $G \in \tau_1 - \{\phi\}$. Then X-G is a τ_1-closed proper subset of X. By Corollary 2.29, X-G is τ_2-Lindelöf. By Corollary 2.37 we have: "every Lindelöf subset of a Hausdorff P-space (X,τ) is closed". Thus X-G is τ_2-closed, i.e. $G \in \tau_2$. Hence $\tau_1 \subset \tau_2$. Similarly we have $\tau_2 \subset \tau_1$. Consequently $\tau_1 = \tau_2$.

THEOREM 3.2. *Let* (X,τ_1,τ_2) *be a compact* p-*Hausdorff space. Then* $\tau_1 = \tau_2$.

Proof. Let $G \in \tau_1$. Then X-G is a τ_1-closed subset of the compact space (X,τ_1). Therefore X-G is τ_1-compact. By Corollary 2.38, X-G is τ_2-closed, i.e. $G \in \tau_2$. Hence $\tau_1 \subset \tau_2$. Similarly we have $\tau_2 \subset \tau_1$. Thus $\tau_1 = \tau_2$.

LEMMA 3.3. *Let* (X,τ_1,τ_2) *be a* p-*Lindelöf space and let* F *be a* τ_1-*weakly closed set such that* $WInt_2(X-F) \neq \phi$. *Then* F *is* τ_1-*Lindelöf.*

Proof. Let $q \in WInt_2(X-F)$. Then there exists a τ_2-open set G containing q such that $G \cap F$ is a countable set. Let $\mathcal{C} = \{C_\alpha : \alpha \in \Delta\}$ be a τ_1-open cover for F. For each $x \in X-F$ there exists a τ_1-open set H(x) containing x such that $H(x) \cap F$ is a countable set. Since $\{C_\alpha : \alpha \in \Delta\} \cup \{G\} \cup \{H(x) : x \in X-F\}$ is p-open cover for the p-Lindelöf space X, there exist two countable sets $\Delta_1 \subset \Delta$ and $\{x_1 x_2, \dots\} \subset X-F$ such that

$$\{C_\alpha : \alpha \in \Delta_1\} \cup \{G\} \cup \{H(x_1), H(x_2), \dots\} \text{ is a cover for } X.$$

Since $H(x_i) \cap F$ is countable, the set $F \cap (\bigcup_{i=1}^{\infty} H(x_i))$ is countable, say $\{y_1, y_2, \ldots\}$. Let $\alpha_i \subset \Delta$ such that $y_i \in C_{\alpha_i}$ $(i \in \mathbb{N})$. Then $\{C_\alpha : \alpha \in \Delta_1\} \cup \{C_{\alpha_i} : i \in \mathbb{N}\}$ is a countable subcover of $\mathcal{C}$, i.e. F is τ_1-Lindelöf.

We can use the same technique as above to conclude the following theorem (We replace the word "countable" by the word "empty" in the proof of Lemma 3.3).

THEOREM 3.4. *Let* (X, τ_1, τ_2) *be a* p-*compact space and let* F *be a* τ_1-*closed set such that* $\mathrm{Int}_2(X-F) \neq \phi$. *Then* F *is* τ_1-*compact.*

EXAMPLE 3.5. In the p-Lindelöf space $(\mathbb{R}, \tau_d, \tau_c)$, $\mathbb{R}$ is a τ_d-closed set which is not τ_d-Lindelöf. This shows that "$\mathrm{WInt}_2(X-F) \neq \phi$" is a necessary condition in Lemma 3.3.

THEOREM 3.6. *Let* (X, τ_1, τ_2) *be a* p-*Hausdorff* p-*Lindelöf space, and let* U *be a* τ_1-*weakly open set containing a fixed point* p. *Then*

(a) *there exist* τ_1-*open sets* C_i $(i \in \mathbb{N})$ *and a* τ_1-*closed set* F *such that* $p \in \bigcap_{i=1}^{\infty} C_i \subset F \subset U$; *and;*

(b) *either* $p \in \mathrm{Cl}_2(X-U)$, *or, there exist* τ_2-*open sets* G_i $(i \in \mathbb{N})$ *and a* τ_1-*closed set* F *such that* $p \in \bigcap_{i=1}^{\infty} G_i \subset F \subset U$.

Proof. (a) Since $p \in U$ and U is τ_1-weakly open set, there exists a τ_1-open set A such that $p \in A$ and A-U is a countable set. For each $x \in X-U$ there exist a τ_1-open set B(x) and a τ_2-open set G(x) such that $x \in G(x)$, $p \in B(x)$ and $B(x) \cap G(x) = \phi$. Let $D(x) = A \cap B(x)$. Then $p \in D(x)$, $D(x) \in \tau_1$, $D(x) \cap G(x) = \phi$, and D(x)-U is a countable set. Since X-U is a τ_1-weakly closed proper subset of the p-Lindelöf space X, by Lemma 2.28, X-U is τ_2-Lindelöf. Therefore the τ_2-open cover $\{G(x) : x \in X-U\}$ has a countable subcover $\{G(x_1), G(x_2), \ldots\}$. Since $D(x_i)$-U $(i \in \mathbb{N})$ is a countable set, the set $\bigcup_{i=1}^{\infty} D(x_i)-U$ is countable, say $\{y_1, y_2, \ldots\}$. Let $C_i = D(x_i)-\{y_i\}$ $(i \subset \mathbb{N})$. Then C_i $(i \in \mathbb{N})$ is a τ_1-open set because (X,τ_1) is a T_1-space. Let $F = \bigcap_{i=1}^{\infty} X-G(x_i)$. Then $(\bigcap_{i=1}^{\infty} C_i) \cap G(x_j) = \phi$ for all $j \in \mathbb{N}$. Hence $(\bigcap_{i=1}^{\infty} C_i) \cap (\bigcup_{j=1}^{\infty} G(x_j)) = \phi$ i.e. $\bigcap_{i=1}^{\infty} C_i \subset F$. Since $\{G(x_j): j \in \mathbb{N}\}$ is a cover for $X-U$, then $F \subset U$. Hence $p \in \bigcap_{i=1}^{\infty} C_i \subset F \subset U$.

(b) If $p \in Cl_2(X-U)$, then we are done. Suppose $p \notin Cl_2(X-U)$. Therefore $p \in Int_2 U$, i.e. $Int_2(U) \neq \phi$. By Lemma 3.3, $X-U$ is τ_1-Lindelöf. For each $x \in X-U$ there exist a τ_1-open set $C(x)$ and a τ_2-open set $G(x)$ such that $p \in G(x)$, $x \in C(x)$ and $C(x) \cap G(x) = \phi$. The τ_1-open cover $\{C(x): x \in X-U\}$ has a countable subcover $\{C(x_1), C(x_2), \ldots\}$. Let $G_i = G(x_i)$ and $F = \bigcap_{i=1}^{\infty} X-C(x_i)$. Then $p \in \bigcap_{i=1}^{\infty} G_i \subset F \subset U$.

Using the same technique as in the proof of Theorem 3.6, we get the following theorem.

THEOREM 3.7. *Let (X,τ_1,τ_2) be a p-Hausdorff p-compact space and, let U be a τ_1-open set containing a fixed point p. Then*

(a) *there exist a τ_1-open set C and a τ_2-closed set F such that $p \in C \subset F \subset U$, and*

(b) *either $p \in Cl_2(X-U)$ or there exist a τ_2-open set G and a τ_1-closed set F such that $p \in G \subset F \subset U$.*

COROLLARY 3.8. *A p-Hausdorff p-compact space is p-regular (and hence, by Theorem 2.42, is p-normal).*

<u>*Proof*</u>. Use Theorem 3.7 (a).

COROLLARY 3.9. *A p-Hausdorff p-Lindelöf p-space is p-regular (and hence, by Theorem 2.42, is p-normal).*

Proof. Use Theorem 3.7 (a).

COROLLARY 3.10. *Let* (X,τ_1,τ_2) *be a* p-*Hausdorff* p-*compact space. If* $Int_2 U \neq \phi$ *for all* $U \in \tau_1 - \{X\}$, *then* $\tau_1 \subset \tau_2$.

Proof. Let $U \in \tau_1 - \{X\}$. Then X-U is a τ_1-closed set with $Int_2 U \neq \phi$. By Theorem 3.4 X-U is τ_1-compact. Hence, by Corollary 2.38, X-U is τ_2-closed, i.e. $U \in \tau_2$.

COROLLARY 3.11. *Let* (X,τ_1,τ_2) *be a* p-*Hausdorff* p-*compact space. If* $Int_2 U \neq \phi$ *for all* $U \in \tau_1 - \{X\}$, *and* $Int_1 V \neq \phi$ *for all* $V \in \tau_2 - \{X\}$. *Then* $\tau_1 = \tau_2$.

Proof. Use corollary 3.10.

It is interesting to note that Cooke and Reilly [2] obtained a theorem [2, Theorem 4] for B-compact, s-compact and bicompact spaces but did not get any analogous result for p-compact spaces. For this reason, Corollary 3.11 is an extension of the result [2, Theorem 4] .

COROLLARY 3.12. *Let* (X,τ_1,τ_2) *be a* p-*Hausdorff* p-*Lindelöf* P-*space. If* $Int_2 U \neq \phi$ *for all* $U \in \tau_1 - \{X\}$, *and* $Int_1 V \neq \phi$ *for all* $V \in \tau_2 - \{X\}$, *then* $\tau_1 = \tau_2$.

Proof. Let $U \in \tau_1 - \{X\}$. Then X-U is a τ_1-closed set with $Int_2 U \neq \phi$. By Lemma 3.3 X-U is τ_1-Lindelöf. Hence by Corollary 2.37, X-U is a τ_2-closed set, i.e. $U \in \tau_2$. Thus $\tau_1 \subset \tau_2$. Similarly we can prove $\tau_2 \subset \tau_1$. Thus $\tau_1 = \tau_2$.

THEOREM 3.13. *Let* (X,τ_1,τ_2) *be a* p-*Hausdorff space and* (X,τ_1) *a Lindelöf space. Let* U *be a* τ_1-*weakly open set and* $p \in U$. *Then there are* τ_2-*open sets* G_i $(i \in \mathbb{N})$ *and a* τ_1-*closed set* F *such that* $p \in \bigcap_{i=1}^{\infty} G_i \subset F \subset U$.

Proof. For each $x \in X-U$ there exist a τ_2-open set $G(x)$ and a τ_1-open set $H(x)$ such that $x \in H(x)$, $p \in G(x)$ and $G(x) \cap H(x) = \phi$. Since X-U is a τ_1-weakly closed set in the Lindelöf space (X,τ_1), therefore X-U is τ_1-Lindelöf. Thus the τ_1-open cover $\{H(x): x \in X-U\}$ has a countable subcover $\{H(x_1), H(x_2), \ldots\}$. Let $F = \bigcap_{i=1}^{\infty} X-H(x_i)$. Then F is τ_1-closed and $F \subset U$. Take $G_i = G(x_i)$. Then $p \in \bigcap_{i=1}^{\infty} G_i \subset F \subset U$.

Using a similar technique as above we can prove the following theorem.

THEOREM 3.14. *Let* (X,τ_1,τ_2) *be a* p-*Hausdorff space and* (x,τ_1) *a compact space. Let* U *be a* τ_1*-open set and* $p \in U$. *Then there are a* τ_2*-open set* G *and a* τ_1*-closed set* F *such that* $p \in G \subset F \subset U$.

COROLLARY 3.15. *If* (X,τ_1,τ_2) *is a Lindelöf* p-*Hausdorff* P-*space, then* $\tau_1 = \tau_2$.

Proof. Use Theorem 3.13.

Since every B-Lindelöf (s-Lindelöf) space is Lindelöf, we have the following corollary.

COROLLARY 3.16. *If* (X,τ_1,τ_2) *is* p-*Hausdorff* P-*space and either* B-*Lindelöf or* s-*Lindelöf, then* $\tau_1 = \tau_2$.

As a corollary to Theorem 3.14 we have the following result (see [2, Theorem 4]).

COROLLARY 3.17. *If* (X,τ_1,τ_2) *is* p-*Hausdorff and either* B-*compact or* s-*compact, then* $\tau_1 = \tau_2$.

4. Conclusion. As we noted, our results in this paper are generalizations of well-known classical theorems as well as extension of some theorems in the literature.

Naturally, any result stated in terms of τ_1 and τ_2 has a 'dual' in terms of τ_2 and τ_1. The definitions of separation and covering properties of two topologies τ_1 and τ_2, such as p-Hausdorff and p-Lindelöf, of course reduce to the usual separation and covering properties of one topology τ_1, such as Hausdorff when we take $\tau_1 = \tau_2$; and the theorems quoted above then yield as corollaries the classical results of which they are generalizations.

As an example of theorems which yield well known classical results are theorems 2.15, 2.33, 2.36, 2.42, 2.43, 2.44 and 2.46.

Theorem 2.8 is an analogue to [2, Theorem 1] while Theorem 2.3 (a,c,d) is an analogue to [2, Theorem 2]. We notice also that Corollary 3.11 is an extension of [2, Theorem 4]. Theorem 3.7 (a) implies the results in [4, Theorem 12 and 13] and [7, Theorem 2.18]. It is also clear that Corollary 2.30 is an analogue to [7, Theorem 2.9] and Corollary 2.38 is an analogue to [7, Lemma 2.11]. It is clear too that Theorems 2.8, 2.42 and Corollary 2.20 imply the result in [6, Lemma 3.2].

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