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HOMEOMORPHISMS, FUNCTIONAL EQUATIONS AND LINEAR INDEPENDENCE

by

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RESUMEN. Se estudian las soluciones f de la ecuación funcional $f = b(f \circ h)+g$, donde $b \in \mathbb{R}$, h es un homeomorfismo del intervalo $[0,1]$ en si mismo y $g \in C[0,1]$. El caso $|b| \neq 1$ se resuelve con toda generalidad. El caso $|b| = 1$ se resuelve bajo ciertas hipótesis de crecimiento para h, cuando g es un límite especial de funciones de la forma $\sum_{t=1}^{n} a_t h^t$, donde h^t es la iterada de h, t veces. Además se demuestra que si h es creciente y distinto de la identidad entonces los h^t son linealmente independientes.

In all that follows we shall assume that h is a homeomorphism of the number interval $[0,1]$ onto $[0,1]$, $\mathbb{R}$ is the set of real numbers, and $C_{[0,1]}$ is the real linear space of all continuous function on $[0,1]$, with sup norm denoted by $\|\cdot\|$.

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We here determine solutions to the functional equation

$$f = bf(h) = g$$

where $b \in \mathbb{R}$, and $g \in C_{[0,1]}$, see [1]. We then establish that if h is increasing, and distinct from the identity funcion, then $\{h^n: n \text{ is an integer}\}$ is linearly independent where h^n is the *nth* iterate of h.

THEOREM 1. *Suppose* $b \in \mathbb{R}$, $|b| \neq 1$ *and* $g \in C_{[0,1]}$. *There is a unique* $f \in C_{[0,1]}$ *such that* $f = bf(h) + g$.

Proof. $C_{[0,1]}$ is a Banach space. Let T denote the bounded linear transformation from $C_{[0,1]}$ into $C_{[0,1]}$ defined by

$$Tf = bf(h),$$

and observe that $|T| = |b|$. Suppose $|b| < 1$. Then $|T| < 1$ and hence $(I-T)^{-1}$ is a bounded linear trnasformation from $C_{[0,1]}$ into $C_{[0,1]}$, see [2]. Hence if $g \in C_{[0,1]}$ and $f = (I-T)^{-1}g$, then $f = bf(h) + g$. Suppose now that $|b| > 1$. Define $G = -(1/b)g(h^{-1})$. From the above it follows that there is a unique $f \in C_{[0,1]}$ such that $f = (1/b)\, f(h^{-1}) + G$ and hence $f = bf(h) + g$. ∎

Let us now turn our attention to the indeterminate case when $|b| = 1$, see [1]. We state without proof, three remarks:

REMARK 1. If each of f and g is in $C_{[0,1]}$ and $f = f(h) + g$ then for each number c, $(f+c) = (f+c)(h) + g$, and hence if there is a solution to $f = f(h) + g$, then there is a solution f such that $f(0) = 0$.

REMARK 2. If $h(x) < x$ for $0 < x < 1$, $f \in C_{[0,1]}$ and

$f = f(h)$, then $f \equiv f(0)$, and if $f = -f(h)$, $f \equiv 0$.

REMARK 3. If $h(x) < x$ for $0 < x < 1$ and each of f and g is in $C_{[0,1]}$ such that $f(0) = 0$ and $f = f(h) + g$, then

$$f = \sum_{t=0}^{\infty} g(h^t) \quad \text{on} \quad [0,1)$$

and

$$f = f(1) - \sum_{t=1}^{\infty} g(h^{-t}) \quad \text{on} \quad (0,1].$$

THEOREM 2. *Suppose* $g = \sum_{t=0}^{n} a_t h^t$ *where for each* t, $a_t \in \mathbb{R}$.

1. *If* $\sum_{t=0}^{n} a_t = 0$, $(g(1) = 0)$, *then there is* $f \in C_{[0,1]}$ *such that* $f = f(h) + g$.
2. *If* $\sum_{t=0}^{t} (-1)^t a_t = 0$, *then there is* $f \in C_{[0,1]}$ *such that* $f = -f(h) + g$.

Proof. For case 1, let

$$b_t = \sum_{i=0}^{t} a_i \qquad 0 \leqslant t \leqslant n-1,$$

and for case 2, let

$$c_t = (-1)^t \sum_{i=0}^{t} (-1)^i a_i \qquad 0 \leqslant t \leqslant n-1 .$$

Omitting the calculations we have the following. If

$$f = \sum_{t=0}^{n-1} b_t h^t ,$$

then $f = f(h) + g$, and if

$$f = \sum_{t=0}^{n-1} c_t h^t ,$$

then $f = -f(h) + g$. ■

We now extend the last result. Let

$$H = \{g : g(x) = \sum_{t=1}^{n} a_t h^t(x),\ a_t \in \mathbb{R},\ x \in [0,1],\ n = 1,2\ldots\},$$

where $h(x) < x$ if $0 < x < 1$.

THEOREM 3. *Suppose* $\lim_{x\to 0} h(x)/x = 0$, $1 < L < (1-h(x))/(1-x) < U$ *for* $\frac{1}{2} \leqslant x < 1$, $\{g_n\}_{n=1}^{\infty}$ *is a sequence in* H *which converges to a function* g, *and for each positive integer* n

$$g_n(x) = \sum_{t=1}^{n} {}_n a_t h^t(x)$$

so that

1. $\sum_{t=1}^{n} {}_n a_t = 0,$

2. $\sum_{t=1}^{n} |{}_n a_t| k^{-t} \leqslant A, \quad k > 1, \quad A > 0,$

3. $\left| \sum_{t=1}^{n} (n-t)\, {}_n a_t \right| \leqslant B, \quad B > 0,$ *and*

4. $\sum_{i=1}^{n-1} U^i \left| \sum_{t=i+1}^{n} {}_n a_t \right| < C, \quad C > 0.$

Then there is $f \in C_{[0,1]}$ *such that* $f = f(h) + g$.

<u>*Proof*</u>. From the last result we have for each positive integer n an $f_n \in C_{[0,1]}$ such that

$$f_n = f_n(h) + g_n \quad \text{and} \quad f_n(0) = 0.$$

Let $\varepsilon > 0$ and for each positive integer pair i, j

$$G_{ij} = g_i - g_j\,,$$

$$F_{ij} = f_i - f_j\,,$$

then

$$F_{ij} = F_{ij}(h) + G_{ij}$$

and by remark 3

$$F_{ij} = \sum_{t=0}^{\infty} G_{ij}(h^t) \quad \text{on} \quad [0,1).$$

Choose positive numbers ε_1, ε_2 where $\frac{1}{2} > \varepsilon_1$ and $1 > \varepsilon_2$, such that $\varepsilon > 4A\varepsilon_1$, and $\varepsilon_1 > k\varepsilon_2$. There is a $\delta > 0$ such that if $0 < x < \delta$, then

$$h(x) < x\varepsilon_2$$

and hence for each positive integer n

$$h^n(x) < x\varepsilon_2^n.$$

Suppose n is a positive integer and $0 < x < \delta$ then

$$|g_n(x)/x| \leq \sum_{t=1}^{n} |{}_na_t| h^t(x)/x$$

$$\leq \sum_{t=1}^{n} |{}_na_t| \varepsilon_2^t$$

$$\leq \sum_{t=1}^{n} |{}_na_t| k^{-t}(k\varepsilon_2)^t$$

$$\leq \sqrt{\sum_{t=1}^{n} |{}_na_t|^2 k^{-2t}} \sqrt{\sum_{t=1}^{n} (k\varepsilon_2)2t}$$

$$\leq A\sqrt{\sum_{t=1}^{n} \varepsilon_1^{2t}}$$

$$< A\varepsilon_1\sqrt{2} < \varepsilon/2.$$

Hence if each of i and j is a positive integer and $0 < x < \delta$, then

$$|G_{ij}(x)| < \varepsilon x.$$

Select $x_0 \in (0,1)$. There is an integer $N > 4$ such that if $n > N$, then $h^n(y_0) < \delta$ for all $0 \leqslant y_0 \leqslant x_0$. Recall that for any i, j

$$F_{ij}(y_0) = \sum_{t=0}^{\infty} G_{ij}(h^t(y_0))$$

and therefore

$$\begin{aligned} |F_{ij}(y_0)| &\leqslant \sum_{t=0}^{N} |G_{ij}(h^t(y_0))| + \sum_{t=N+1}^{\infty} |G_{ij}(h^t(y_0))| \\ &\leqslant \sum_{t=0}^{N} |G_{ij}(h^t(y_0))| + \varepsilon \sum_{t=N+1}^{\infty} h^t(y_0) \\ &\leqslant \sum_{t=0}^{N} |G_{ij}(h^t(y_0))| + \varepsilon y_0 \sum_{t=N+1}^{\infty} \varepsilon_2^t \\ &\leqslant \sum_{t=0}^{N} |G_{ij}(h^t(y_0))| + \varepsilon y_0 \varepsilon_2^{N+1}/(1-\varepsilon_2) \,. \end{aligned}$$

Recall that $\{g_n\}_{n=1}$ converges to g, hence there is an integer M such that if i and j are greater than M, then

$$\|G_{ij}\| < \varepsilon/(2(N+1)).$$

Hence

$$\begin{aligned} |F_{ij}(y_0)| &\leqslant (N+1)\varepsilon/(2(N+1)) + \varepsilon\varepsilon_2^{N+1}/(1-\varepsilon_2) \\ &\leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon \,. \end{aligned}$$

Thus we have that if $x_0 \in (0,1)$, then there is an integer M such that if each of i, j > M and $0 \leqslant y_0 \leqslant x_0$, then

$$|F_{ij}(y_0)| < \varepsilon.$$

It then follows that $\{f_i\}_{i=1}^{\infty}$ converges pointwise on $[0,1)$ to a function f, i.e.,

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad 0 \leqslant x < 1,$$

and since we also have uniform convergence on $[0,a)$ for any $0 < a < 1$ we also have that f is continuous on $[0,1)$. For each positive integer n,

$$f_n(x) = \sum_{t=0}^{\infty} g_n(h^t(x)) \qquad 0 \leqslant x < 1$$

and

$$g_n(x) = \sum_{t=1}^{n} {}_n a_t h^t(x) \qquad 0 \leqslant x \leqslant 1.$$

After routine calculations we have that

$$f_n(x) = \sum_{t=1}^{n-1} {}_n a_t \left[h^t + \ldots + h^{n-1}\right] \quad \text{if } 0 \leqslant x < 1.$$

Since, for each n, f_n is continuous on $[0,1]$,

$$f_n(1) = \sum_{t=1}^{n} (n-t)_n a_t$$

and therefore,

$$|f_n(1)| \leqslant B \qquad \text{for } n = 1,2,\ldots .$$

Hence we may select a subsequence $\{f_n(1)\}_{i=1}^{\infty}$ which has a limit which we shall denote by f(1), and then notice that

$$\lim_{i\to\infty} f_{n_i}(x) = f(x) \quad \text{for } x \in [0,1],$$

and recall that this is a pointwise limit. We need to establish that f is continuous at 1 and that $f = f(h) + g$. From remark 3, we have that

$$f_n(x) = f_n(1) - \sum_{i=1}^{\infty} g_n(h^{-i}(x)), \qquad 0 < x \leqslant 1,$$

and therefore

$$|f_n(x)-f_n(1)|/|1-x| = |\sum_{i=1}^{\infty} g_n(h^{-i}(x))/(1-x)|,$$

$$= |\sum_{i=1}^{\infty} \sum_{t=1}^{n} {}_na_t h^{t-i}(x)/(1-x)|, \quad 0 < x < 1.$$

Recall that $\sum_{t=1}^{n} {}_na_t = 0$, therefore

$$|f_n(x)-f_n(1)|/|1-x| = |\sum_{i=1}^{\infty} \sum_{t=1}^{n} {}_na_t(1-h^{t-i}(x))/(1-x)|$$

$$= |\sum_{i=1}^{n} \sum_{t=1}^{n} {}_na_t(1-h^{t-i}(x))/(1-x) + \sum_{i=n+1}^{\infty} \sum_{t=1}^{n} {}_na_t(1-h^{t-i}(x))/(1-x)|$$

$$= |{}_na_1 \sum_{t=0}^{\infty} (1-h^{-t}(x))/(1-x) + {}_na_2[(1-h(x))/(1-x) + \sum_{t=0}^{\infty} (1-h^{-t}(x))/(1-x)]$$

$$+\dots+{}_na_n[(1-h^{n-1}(x))/(1-x)+\dots+(1-h(x))/(1-x) + \sum_{t=0}^{\infty} (1-h^{-t}(x))/(1-x)]|$$

$$1 < L < (1-h(x))/(1-x) < U, \qquad 0 < x < 1$$

which implies that

$$(1-h^{i+1}(x))/(1-h^{i}(x)) < U \quad \text{for } i = 0,1,2,\dots$$

and hence

$$(1-h^{i}(x))/(1-x) < U^{i} \quad \text{for } i = 1,2,\dots .$$

Also $\qquad 1 < L < (1-h(x))/(1-x), \quad 0 < x < 1,$

and therefore

$$(1-x)/(1-h(x)) < 1/L < 1,$$

from which it follows that

$$(1-h^{-t}(x))/(1-x) < (1/L)^{t}.$$

Return now to

$$|f_n(x)-f_n(1)|/(1-x) = |[{}_na_1+\dots+{}_na_n] \sum_{t=0}^{\infty} (1-h^{-t}(x))/(1-x) +$$

$$+ {}_na_2[(1-h(x))/(1-x)]+\dots+{}_na_n[(1-h(x))/(1-x)+\dots+(1-h^{n-1}(x))/(1-x)]|$$

$$\leqslant |{}_na_2[1-h(x)]+\dots+{}_na_n[(1-h(x))/(1-x)+\dots+(1-h^{n-1}(x))/(1-x)]|$$

$$\leqslant |\sum_{t=2}^{n} {}_na_t[(1-h(x))/(1-x)] + \sum_{t=3}^{n} {}_na_t[(1-h^2(x))/(1-x)]$$

$$+\dots+{}_na_n[(1-h^{n-1}(x))/(1-x)]|$$

$$\leqslant |\sum_{i=1}^{n-1} (1-h^i(x))/(1-x) \sum_{t=i+1}^{n} {}_na_t|$$

$$\leqslant \sum_{i=1}^{n-1} U^i| \sum_{t=i+1}^{n} {}_na_t|$$

$$\leqslant C .$$

Hence we have that

$$|f_n(x)-f_n(1)| < C|1-x| \quad \text{if} \quad 0 < x \leqslant 1.$$

Recall that a subsequence $\{f_{n_i}\}_{i=1}^{\infty}$ was selected such that $\lim_{i\to\infty} f_{n_i}(1)$ existed.

To avoid complicating notation, and with no loss to the argument, let us assume that the original sequence converged pointwise on $[0,1]$ and uniformly on $[0,a)$ for each $0 \leqslant a < 1$.

$$|f(1)-f(x)| \leqslant |f(1)-f_n(1)|+|f_n(1)-f_n(x)|+|f_n(x)-f(x)|$$

$$< |f(1)-f_n(1)|+C|1-x|+|f_n(x)-f(x)| ,$$

in the limit,

$$|f(1)-f(x)| \leqslant C|1-x| ,$$

and hence f is continuous at 1. Let us now show that $f = f(h)+g$ on $[0,1]$. Now, $f(0) = 0$, $g(0) = 0$ and $f(1) = f(h(1))+g(1) =$

$f(1)+0 = f(1)$. Hence, suppose $0 < x < 1$:

$$|f(x)-f(h(x))-g(x)| = |f(x)-f_n(x)|+|f_n(h(x))-f(h(x))| + |g_n(x)-g(x)|+|f_n(x)-f_n(h(x))-g_n(x)|$$

$$\leqslant |f(x)-f_n(x)|+|f_n(h(x))-f(h(x))|+|g_n(x)-g(x)|.$$

Therefore $f(x) = f(h(x))+g(x)$ on $[0,1]$, which completes the argument. ■

THEOREM 4. *Suppose* $g \in C_{[0,1]}$, $G = g-g(h)$, *and* $h(x) < x$, *for* $0 < x < 1$. *If* $f \in C_{[0,1]}$, *then* $f = f(h(h))+G$ *and* $f(0) = 0$ *if and only if* $f = -f(h)+g-g(0)$.

Proof. If $f \in C_{[0,1]}$, $f(0) = 0$ and $f = f(h(h))+G$, then

$$f = -f(h)+f(h)+f(h(h))+g-g(h)$$
$$= -f(h)+g+[f+f(h)-g](h).$$

Let

$$L = f+f(h)-g$$

and observe that $L = L(h)$ and $L(0) = -g(0)$. By remark 2, $L \equiv -g(0)$ and hence

$$f = -f(h)+g-g(0).$$

Suppose now that

$$f = -f(h)+g-g(0).$$

Then

$$f(h) = -f(h(h))+g(h)-g(0)$$

and hence

$$f = f(h(h))+g-g(h)$$
$$= f(h(h))+G.$$

Since $f = -f(h)+g-g(0)$, it follows that $f(0) = -f(0)$ and hence $f(0) = 0$. ■

In Theorem 3, the set $H = \{g: g = \sum_{t=1} a_t h^t, a_t \in \mathbb{R}, n = 1,2..\}$ was considered, which raises the question as to linear independence of the set $\{h^n: n \text{ is an integer}\}$ when h(h) is not the identity function.

THEOREM 5. *If* h *is an increasing homeomorphism of* [0,1] *onto* [0,1] *distinct from the identity function, then the set* $M = \{h^n: n \text{ is an integer}\}$ *is linearly independent.*

Proof. Suppose that there is an increasing homeomorphis h of [0,1] onto [0,1] distinct from the identity function such that the set $M = \{h^i: i \text{ is an integer}\}$ is not linearly independent. It then follows that the set $P = \{h^i: i \text{ is is a nonnegative integer}\}$ is linearly dependent. Note that $\{h^0, h^1\}$ is a linearly independent set and therefore there is a least positive integer n such that $\{h^i: i = 0,1,...,n+1\}$ is linearly dependent. There is then a number sequence $\{A_i\}_{i=0}^{n+1}$ not all of whose elements are zero, such that

$$\sum_{i=0}^{n+1} A_i h^i = 0.$$

Since n is minimal, we have that $A_0 A_{n+1} \neq 0$. For j = 0,1,...,n let $B_j = -A_j/A_{n+1}$, then $B_0 \neq 0$ and

$$\sum_{j=0}^{n} B_j h^j = h^{n+1}$$

Moreover for each integer k

$$\sum_{j=0}^{n} B_j h^{j+k} = h^{n+1+k}$$

Since h(1) = 1 it follows that

$$\sum_{j=0}^{n} B_j = 1$$

and hence

$$B_n = 1 - \sum_{j=0}^{n-1} B_j .$$

For t = 0,1,...,n-1 let

$$C_t = -\sum_{j=0}^{t} B_j ,$$

and for each integer k let

$$\Delta_k = h^k - h^{k+1} .$$

Then we have that

$$\sum_{i=0}^{n-1} C_i \Delta_i = \Delta_n ,$$

and for each positive integer k

$$\sum_{j=0}^{k} \sum_{i=0}^{n-1} C_i \Delta_{i+j} = \sum_{j=0}^{k} \Delta_{n+j} .$$

The last expression telescopes to

$$\sum_{i=0}^{n-1} C_i (h^i - h^{i+1+k}) = h^n - h^{n+k+1}$$

and therefore

$$\sum_{i=0}^{n-1} C_i h^i - h^n = \sum_{i=0}^{n-1} C_i h^{i+1+k} - h^{n+k+1}$$

If there is a number a, $0 < a < 1$, such that $h(a) = a$, then there is a subinterval $[c,d]$ of $[0,1]$ such that either h or h^{-1} is bellow the identity between c and d. If it is h, then procede, if it is h^{-1} then let $k = h^{-1}$ and procede considering the set $\{k^n$: n is an integer$\}$ = $\{h^n$: n is an integer$\}$, in any event we would have a homeomorphism of $[c,d]$ onto $[c,d]$ which is below the identity between c and d. If now the set $\{h^n$: n is an integer$\}$

is linearly independent when restricted to $[c,d]$, then it is linearly independet on $[0,1]$. Hence the problem is to show that if $[c,d]$ is an interval and h is a homeomorphism of $[c,d]$ onto $[c,d]$ such that $h(x) < x$ for $c < x < d$, then $\{h^n : n \text{ is an integer}\}$ is linearly independent on $[c,d]$. Hence, without loss of generality assume $[c,d] = [0,1]$ and notice that

$$\lim_{j\to\infty} h^j(x) = 0 \quad \text{for} \quad 0 \leqslant x < 1,$$

and indeed, if $0 < a < 1$, then the convergence is uniform on $[0,a]$. Thus for $0 \leqslant x < 1$

$$\lim_{k\to\infty} \sum_{i=0}^{n-1} C_i h^{i+1+k}(x) - h^{n+k+1}(x) = 0 ,$$

and therefore

$$\sum_{i=0}^{n-1} C_i h^i(x) = h^n(x)$$

for $0 \leqslant x < 1$. Each of h^n and $\sum_{i=0}^{n-1} C_i h^i$ is continuous on $[0,1]$ and therefore

$$\sum_{i=0}^{n-1} C_i h^i = h^n$$

on $[0,1]$, which is a contradiction to the minimality of n, and hence the result is established.

REMARK 4. If H_t is a flow on $[0,1]$, see [3], then $\{H_t : t \text{ is a rational number}\}$ is linearly independent.

REMARK 5. If $h(x) = x^2$ for x in $[0,1]$, then $h^n(x) = x^{2n}$. Let $H = \{h^n : n \text{ is a positive integer}\}$. By Muntz' theorem [4] the linear subspace of $C_{[0,1]}$/constant functions generated by H is not dense in $C_{[0,1]}$/constant functions.

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