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INFINITELY DIFFEREMTIABLE FUNCTIONS WITH PRESCRIBED DERIVATIVES AT A POINT

by

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RESUMEN. Usando series de Fourier y un teorema de Pólya, se da una nueva demostración de la existencia de funciones reales, infinitas veces diferenciables en una vecindad de 0, tales que sus derivadas $f^{(n)}(0)$, n = 0,1, 2,..., toman cualquier conjunto prefijado de valores.

ABSTRACT. The existence of functions mentioned in the title is proved via the existence of their Fourier series.

Starting with E. Borel [1] in 1895 and up until very recently [2] various proofs are given for the existence of an infinitely differentiable (real-valued) function defined on an interval of positive length containing, say, 0 such that f(0) and the n-th derivatives $f^{(n)}(0)$, for n = 1,2,3,... have prescribed values.

We give below a proof which is better motivated and therefore easier to remember. It is based in the following theorem of Polya.

THEOREM 1 [3], p.234. *Let us consider the following system of infinitely many linear equations:*

$$a_{j1}u_1+a_{j2}u_2+\dots+a_{jk}u_k+\dots = b_j \qquad (j=1,2,3,\dots) \qquad (1)$$

where the sequence $b_1,b_2,b_3,\dots$ *is arbitrary and the matrix* (a_{jk}) *satisfies the following two conditions:*

A. *For arbitrary* n *and* q*, the submatrix*

$$\begin{matrix} a_{1,q+1} & a_{1,q+2} & \dots \\ \dots & & \\ \dots & & \\ a_{n,q+1} & a_{n,q+2} & \dots \end{matrix}$$

formed by the entries that are in the first n *lines and not in the first* q *columns, has rank* n.

B. $\lim_{k\to\infty} a_{j-1,k}/a_{jk} = 0 \qquad (j=2,3,4,\dots)$.

Then, there exists a sequence $u_1,u_2,u_3,\dots$*, solution of the system* (1)*, such that the series at the left side of the equations converge absolutely.*

THEOREM. *Let* $c_0,c_1,c_2,\dots$ *be a sequence of real numbers. Then there exists an infinitely differentiable function* f *defined in the interval* $-\pi < x < \pi$ *such that*

$$f(0) = c_0 \quad \textit{and} \quad f^{(n)}(0) = c_n \quad \textit{for} \quad n=1,2,3,\dots \qquad (2)$$

Proof. We show that we can determine constants $a_0,a_1,a_2,\dots,b_1,b_2,b_3,\dots$ in such a way that the resulting trigonometric series

$$a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx) \tag{3}$$

satisfies the following equalities:

$$a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx) = c_0 \quad \text{at } x = 0 \tag{4}$$

and

$$\sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)^{(n)} = c_n \quad \text{at } x = 0 \tag{5}$$

for every $n \in \omega$, and such that the n-th (term by term) derivative $\sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)^{(n)}$ of $a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)$ and the latter are *uniformly* convergent for $-\pi < x < \pi$. But from this uniform convergence and (4), (5) it would follow that (3) is the Fourier series of functions f which satisfies (2).

Clearly, from (4) and (5) it follows that the (to be determined) constants $a_0, a_1, a_2, \ldots, b_1, b_2, b_3, \ldots$ must satisfy the following two infinite systems of linear equations each with infinitely many unknowns:

$$\begin{aligned}
a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \ldots &= c_0 \\
a_1 + 2^2 a_2 + 3^2 a_3 + 4^2 a_4 + 5^2 a_5 + 6^2 a_6 + \ldots &= -c_2 \\
a_1 + 2^4 a_2 + 3^4 a_3 + 4^4 a_4 + 5^4 a_5 + 6^4 a_6 + \ldots &= c_4 \\
\cdots\cdots\cdots \quad \ldots &= \ldots \\
\cdots\cdots\cdots \quad \ldots &= \ldots
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
b_1 + 2b_2 + 3b_3 + 4b_4 + 5b_5 + 6b_6 + \ldots &= c_1 \\
b_1 + 2^3 b_2 + 3^3 b_3 + 4^3 b_4 + 5^3 b_5 + 6^3 b_6 + \ldots &= -c_3 \\
b_1 + 2^5 b_2 + 3^5 b_3 + 4^5 b_4 + 5^5 b_5 + 6^5 b_6 + \ldots &= c_5 \\
\cdots\cdots\cdots \quad \ldots &= \ldots \\
\cdots\cdots\cdots \quad \ldots &= \ldots
\end{aligned} \tag{7}$$

It can be readily verified that the matrix of the coefficients of (6) as well as that of (7) satisfies conditions A and B of Pólya's theorem. Thus, there exist constants $a_0, a_1, a_2, \ldots, b_1, b_2, b_3, \ldots$ satisfying (6) and (7) and such that every infinite series appearing in (6) and (7) is *absolutely* convergent. But then from this absolute convergence if follows that each of the trigonometric series appearing in (4) and (5) is uniformly convergent for $-\pi < x < \pi$ which (as mentioned above) implies that (3) is the Fourier series of function f appearing in (2) whereby establishing the existence of f, as desired.

REMARK. From the above theorem it follows (cf. [2]) that every power series is a Taylor series. Indeed, without loss of generality, let $\sum_{n=0}^{\infty} c_n x^n$ be a power series. Then from (2) to (5) we see that (3) gives the Fourier series of a function f (defined for $-\pi < x < \pi$) whose Taylor series (about 0) is $\sum_{n=0}^{\infty} c_n x^n$. Clearly, if the radius of convergence of $\sum_{n=0}^{\infty} c_n x^n$ is positive then the power series itself also can be taken for f.

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