Revista Colombiana de Matemáticas
Vol. XVII (1983), págs. 105 - 116

I_3-∇ ALGEBRAS

by

Aldo V. FIGALLO

§1. **Introducción**. In this note we present an algebraic study of a fragment of the three-valued propositional calculus of Lukasiewicz, that is, we study from an algebraic standpoint the three-valued calculus where the characteristic matrix is given by the chain T = {0,1/2,1} and the connectives ↣ (Lukasiewicz implication) and ∇ (possibility operator are given by the tables:

↣	0	1/2	1
0	1	1	1
1/2	1/2	1	1
1	0	1/2	1

x	∇x
0	0
1/2	1
1	1

These algebras, called I_3-∇ algebras, play an analogous role to that of Boolean algebras for two-valued propositional calculus.

In 1968, A.Monteiro [16] introduced the notion of I_3

algebra as a system $(A, \rightarrowtail, 1)$ where A is a non empty set, 1 is an element of A, and $\rightarrowtail$ is a binary operation defined on A fulfilling the following conditions for all $x,y,z \in A$:

I1 $x \rightarrowtail (y \rightarrowtail x) = 1$

I2 $(x \rightarrowtail y) \rightarrowtail ((y \to z) \rightarrowtail (x \rightarrowtail z)) = 1$

I3 $(x \rightarrowtail y) \rightarrowtail y = (y \rightarrowtail x) \rightarrowtail x$

I4 $((x \rightarrowtail y) \rightarrowtail (y \rightarrowtail x)) \rightarrowtail (y \rightarrowtail x) = 1$

I5 $((x \rightarrowtail (x \rightarrowtail y)) \rightarrowtail x) \rightarrowtail x = 1$

I6 $1 \rightarrowtail x = x$.

The same author has proved that the following properties are true in any I_3 algebra:

I7 $x \rightarrowtail 1 = 1$

I8 $x \rightarrowtail x = 1$

I9 $(x \rightarrowtail y) \rightarrowtail ((z \rightarrowtail x) \rightarrowtail (z \rightarrowtail y)) = 1$

I10 $x \rightarrowtail (x \rightarrowtail (x \rightarrowtail y)) = x \rightarrowtail (x \rightarrowtail y)$

I11 $x \rightarrowtail (y \rightarrowtail z) = y \rightarrowtail (x \rightarrowtail z)$

I12 The relation $x \leqslant y$ if and only if $x \rightarrowtail y = 1$ is a partial ordering on A, and 1 is the greatest element of A.

I13 The element $x \vee y = (x \to y) \to y$ is the least upper bound of the elements x and y.

In addition, if we define a new binary operation $\to$, called *weak implication*, as follows [22]:

$$x \to y = x \rightarrowtail (x \rightarrowtail y)$$

this operation has the following properties:

C1 $x \to (y \to x) = 1$

C2 $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$

C3 $((x \to y) \to x) \to x = 1$

C4 (Modus Ponens) If $x = 1$ and $x \to y = 1$, then $x = 1$.

1.1. DEFINITION. An I_3-∇ *algebra* is a system $(A, \rightarrowtail, \nabla, 1)$ such that $(A, \rightarrowtail, 1)$ is an I_3 algebra and ∇ is a unary operation defined on A fulfilling the conditions:

$\nabla 1$ $x \rightarrowtail (x \rightarrowtail \nabla y) = \nabla(x \rightarrowtail y)$

$\nabla 2$ $\nabla(\nabla x \rightarrowtail \nabla y) = \nabla x \rightarrowtail \nabla y$

$\nabla 3$ $(x \rightarrowtail y) \rightarrowtail x = (\nabla x \rightarrowtail \nabla y) \rightarrowtail x$.

As usual, the I_3-∇ algebra $(A, \rightarrowtail, \nabla, 1)$ will be denote by the underlying set A.

The following result is a consequence of I1 to I13, $\nabla 1$. $\nabla 2$, and $\nabla 3$.

1.2. LEMMA. *The following properties are true in any* I_3-∇ *algebra:*

$\nabla 4$ $\nabla 1 = 1$

$\nabla 5$ $\nabla\nabla x = \nabla x$

$\nabla 6$ $\nabla x \rightarrowtail \nabla((x \rightarrowtail y) \rightarrowtail y) = 1$

$\nabla 7$ $\nabla x \rightarrowtail \nabla y = \nabla x \rightarrowtail (\nabla x \rightarrowtail \nabla y)$

$\nabla 8$ $\nabla x \rightarrowtail \nabla y = \nabla(\nabla x \rightarrowtail y)$

$\nabla 9$ $(x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail (\nabla x \rightarrowtail \nabla y)) = 1$

$\nabla 10$ $(\nabla x \rightarrowtail \nabla y) \rightarrowtail ((x \rightarrowtail (x \rightarrowtail y)) \rightarrowtail (x \rightarrowtail y)) = 1$

$\nabla 11$ $x \leqslant \nabla x$.

1.3. EXAMPLE. Consider $T = \{0, 1/2, 1\}$ with the operation defined in the introductory paragraph, then it is easy to see that $(T, \rightarrowtail, \nabla, 1)$ is an I_3-∇ algebra.

§2. Simple algebras.

If A, A' are I_3-∇ algebras, an $(I_3$-$\nabla)$-*homomorphism* h from A into A' is a mapping $h: A \rightarrow A'$ fulfilling the conditions:

$h(x \rightarrowtail y) = h(x) \rightarrowtail h(y)$

$h(\nabla x) = \nabla h(x)$

The *kernel* of an (I_3-∇)-homomorphism $h:A \to A'$ is the set $\text{Ker } h = \{x \in A : h(x) = 1\}$. It is easy to prove that if $D = \text{Ker } h$, then:

D1 $1 \in D$

D2 If x, $x \rightarrowtail y \in D$, then $y \in D$.

2.1. DEFINITION. A *deductive system* is a part D of an I_3-∇ algebra A that verifies D1 and D2.

It is well known that if D is a deductive system in an I_3 algebra A, then the relation $\equiv$ defined as $x \equiv y \pmod D$ if and only if $x \rightarrowtail y$, $y \rightarrowtail x \in D$, concerning the operation $\rightarrowtail$, defines a congruence in A [16]. Furthermore, from conditions ∇9 and D1, it is possible to conclude that $\equiv$ is a congruence in an I_3-∇ algebra A. Let A/D be the quotient algebra and $q:A \to A/D$ the canonical homomorphism, then D is the kernel of q. All the homomorphic images of A, up to isomorphisms, can be obtained in the above mentioned way.

Our next objective will be the determination of simple I_3-∇ algebras. To this end we shall need to study the maximal deductive systems.

2.2. DEFINITION. A deductive system D of an I_3-∇ algebra A is called *maximal* if (1) $D \neq A$, and (2) if $D \subseteq D' \subseteq A$ and D' is a deductive system, then $D = D'$ or $D' = A$.

2.3. REMARK. It is well know that a part D of and I_3 algebra A is a deductive system if D1 holds and D'2: if x, $x \rightarrowtail y \in D$ then $y \in D$. On the other hand, as the weak implication verifies C1, C2, C3 and C4, after [17], we can state that every deductive system of an I_3-∇ algebra is a meet of maximal deductive systems. In particular, [1] is the meet of all maximal deductive systems, that is, every I_3-∇ algebra is *deductively semisimple.*

2.4. DEFINITION. An I_3-∇ algebra A is said to be *simple* if:

(1) A is non trivial.

(2) The only homomorphic images of A are up to isomorphisms, the trivial ones, that is, A and the trivial algebra.

Taking into account that the homomorphic images of A are the algebras A/D, where D is a deductive system, we have that A/M is simple if and only if M is maximal. Also, 2.3 implies that every non trivial I_3-∇ algebra is a subdirect product of simple I_3-∇ algebras, and all the subdirectly irreducible I_3-∇ algebras are simple.

We have the following result which we shall need in the next theorem.

2.5. LEMMA. *If* M *is a deductive system in an* I_3-∇ *algebra* A, *we have*

(1) *If* $m \in M$, *then* $x \rightarrowtail m \in M$ *for every* $x \in A$

(2) M *is maximal if and only if for every* $x, y \notin M$, $x \rightarrowtail (x \rightarrowtail y) \in M$.

(3) *If* M *is maximal and* $\nabla x, \nabla y \notin M$, *then* $\nabla x \rightarrowtail \nabla y \in M$.

The proof of (1) and (2) can be found in [16], (3) is a consequence of (2) and ∇7.

It is clear that the algebra T of 1.3 is simple and $B = \{0,1\}$ and $L = \{1/2, 1\}$ are non isomorphic (I_3-∇)-subalgebras of T and therefore simple algebras. Moreover, the simple algebras are just the algebras T, B and L. Indeed:

2.6. THEOREM. *If* M *is a maximal deductive system of an* I_3-∇ *algebra* A, *then* $A/M \simeq T$ *or* $A/M \simeq B$ *or* $A/M \simeq L$.

Proof. Consider the sets $M_0 = \{x \notin M : \nabla x \notin M\}$ and $M_{1/2} = \{x \notin M : \nabla x \in M\}$. Then the mapping $h: A \to T$ defined by

$$h(x) = \begin{cases} 1 & \text{if } x \in M \\ 1/2 & \text{if } x \in M_{1/2} \\ 0 & \text{if } x \in M_0 \end{cases}$$

is an $(I_3$-$\nabla)$-homomorphism such that M = Ker h by Lemma 2.5. The theorem is proved if we observe that h(A) is an $(I_3$-$\nabla)$-subalgebra of T and $A/M \simeq h(A)$. ∎

From this theorem it follows that every non trivial I_3-∇ algebra is a subdirect product of copies of the algebras T, B and L.

§3. I_3-∇ algebras with a finite set of free generators.

The aim of this section is to determine the structure of the I_3-∇ algebras with n free generators L(n), where n is a finite positive cardinal number. Let $G = \{g_1, g_2, .., g_n\}$ be the set of free generators of L(n). If we note by T^G the set of all functions from G into T and Hom(L(n),T) the set of all homomorphisms from L(n) into T, it is clear that the application which maps each homomorphism $h: L(n) \to T$ into its restriction to G establishes a one-to-one correspondence between the sets Hom(L(n),T) and T^G. Hence, Hom(L(n),T) is finite.

3.1. LEMMA. *If $\mathfrak{M}$ is the family of all maximal deductive systems of* L(n), *then the application* φ:Hom(L(n),T) → $\mathfrak{M}$ *defined by* $\varphi(h)$ = Ker h, *is a bijection.*

Proof. Let $M \in \mathfrak{M}$, $q: L(n) \to L(n)/M$ the canonical $(I_3-\nabla)$-homomorphism, $i: L(n)/M \to T$ an $(I_3-\nabla)$-monomorphism, which exists by Theorem 2.6, then $h = i \circ q \in$ Hom(L(n),T) and $\varphi(h) = M$, therefore φ is onto. On the other hand, there exists only one automorphism in T, B or L, the automorphism $\alpha(x) = x$ for all x, and then, if $M \in \mathfrak{M}$, $\varphi^{-1}(M)$ has exactly one element and so φ is one-to-one. ∎

Since L(n) is a subdirect product of the finite algebras L(n)/M with $M \in \mathfrak{M}$, then from the above results it follows that:

3.2. COROLLARY. *The free* $I_3-\nabla$ *algebra* L(n), *where* n *is a finite positive cardinal number, is finite.*

We shall need the following result:

3.3. LEMMA. *The generators* g_i, $1 \leqslant i \leqslant n$, *are the minimal elements of* L(n).

Proof. Analogous to that of [10]. ∎

Consider the sets $G_i = \{x \in L(n) : g_1 \leqslant x\}$, $1 \leqslant i \leqslant n$, then $L(n) = \bigcup_{i=1}^{n} G_i$ and so $|L(n)| = |\bigcup_{i=1}^{n} G_i|$. Let $B_i^{(n)} = G_{i_1} \cap G_{i_2} \cap \ldots \cap G_{i_k}$, $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$, where $i = (i_1, i_2, \ldots, i_k)$. It is well known that $|L(n)| = \sum_i (-1)^{k+1} |B_i^{(n)}|$. Clearly, by symmetry, it is sufficient to determine $B_k = G_1 \cap G_2 \cap \ldots \cap G_k$, and then we will have

$$(1) \qquad |L(n)| = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} |B_k|.$$

3.4. LEMMA. *Let* $g_0 = g_1 \vee g_2 \vee \ldots \vee g_k$. *Then*

(1) $B_k = \{x \in L(n) : g_0 \leq x\}$.

(2) B_k *is a tree-valued Lukasiewicz algebra.*

Proof. (1) is obvious since g_0 is the least upper bound of $g_1, g_2, \ldots, g_k$; (2) is a consequence of the fact that B_k has least element g_0. ■

Let $\mathcal{M}_k$ be the family of all maximal deductive systems of B_k. Since B_k is finite, from the theory of three-valued Lukasiewicz algebras we know that

$$B_k \simeq \prod_{D \in \mathcal{M}_k} B_k/D .$$

We say that D is *three-valued* if $B_k/D \simeq T$, B-*two-valued* if $B_k/D \simeq B$ and L-*two-valued* if $B_k/D \simeq L$. Then, if we wish to determine $|B_k|$ we must compute the number of three-valued, B-two-valued and L-two-valued deductive systems of B_k.

The following result gives a characterization of the maximal deductive systems of B_k by means of the maximal deductive systems of L(n).

3.5. LEMMA. *If* D *is a deductive system of* B_k, *then* D *is maximal in* B_k *if and only if there exists a maximal deductive system of* L(n) *such that* $D = B_k \cap M$.

Proof. Suppose M a maximal deductive system of L(n) and consider $D = B_k \cap M$. Assume $D \neq B_k$. Since it is clear that D is a deductive system of B_k, we are going to prove that D is maximal. From 2.5 (2) it is sufficient to prove that $x \rightarrowtail (x \rightarrowtail y) \in D$ for all $x, y \in B_k - D$. Since $x, y \notin M$ and M is maximal, we get $x \rightarrowtail (x \rightarrowtail y) \in M$. But $x \rightarrowtail (x \rightarrowtail y) \in B_k$, so $x \rightarrowtail (x \rightarrowtail y) \in D$.

Comversely, let D be a maximal deductive system of B_k. Since B_k is finite, there exists $a \in B_k$ such that

$$D = D(a) = \{x \in B_k : a \rightarrow x = 1\} \qquad ([16],[10]).$$

Consider $D' = D'(a) = \{x \in L(n) : a \rightarrow x = 1\}$. Then $g_0 \notin D'$, because otherwise we would have $a \leq g_0$ and $g_0 \in D(a)$, which contradicts the fact that D(a) is proper. From 2.3 we can state that there exists a maximal deductive system M of L(n) such that $g_0 \notin M$ and $D'(a) \subseteq M$. Let us now prove that $M \cap B_k = D$. Clearly $D \subseteq M \cap B_k$ and $M \cap B_k$ is a proper deductive system of B_k and D is maximal, therefore $D = M \cap B_k$. The proof is now complete. ∎

Since every $f \in T^G$ can be extended to a unique homomorphism $h \in Hom(L(n),T)$ such that Ker h = M is a maximal deductive system of L(n), then $B_k \subseteq M$ or $B_k \cap M$ is a maximal deductive system of B_k.

Thus we must to determine the set $Hom^*(L(n),T)$ of all homomorphisms h from L(n) into T such that $B_k \nsubseteq$ Ker h.

3.6. LEMMA. *For every function* f *from* G *into* T, *the following conditions are equivalents:*

(1) *The extension* h *of* f *is an element of* $Hom^*(L(n),T)$.

(2) $f(g_i) \in \{0,1/2\}$, $1 \leq i \leq k$.

Proof. $B_k \subseteq$ Ker h if and only if $h(g_0) = 1$, but $h(g_0) = f(g_1) \vee f(g_2) \vee \ldots \vee f(g_k) = 1$ if and only if $f(g_i) = 1$ for some i since the ordering of T is total. ∎

Let f a function from G into T, h its extension and M = Ker h, then $M \cap B_k$ is B-two-valued if and only if $f(g_i) = 0$ for all i, $1 \leq i \leq k$, and $f(g_j) \in \{0,1\}$ for all $k+1 \leq j \leq n$.

Since there exist 2^{n-k} such functions, then there exist 2^{n-k} B-two-valued deductive systems of B_k.

If there exists g_i, $1 \leq i \leq k$, such that $f(g_i) = 1/2$, then $h(g_0) = 1/2$ and therefore $h(B_k) = \{1/2, 1\}$ and in that case $M \cap B_k$ is L-two-valued, and there exist $(2^k-1)3^{n-k}$ L-two-valued deductive systems of B_k.

On the other hand, M is three-valued if and only if $f(g_i) = 0$ for all i, $1 \leq i \leq k$, and there exists g_j such that $f(g_j) = 1/2$, $k+1 \leq j \leq n$. Therefore, we have $3^{n-k}-2^{n-k}$ three-valued deductive systems of B_k.

With the above results in hand we can write:

$$B_k \simeq [\prod_{i=1}^{2^{n-k}} B_i] \times [\prod^{(2^k-1)3^{n-k}} L_i] \times [\prod^{3^{n-k}-2^{n-k}} T_i]$$

Where $B_i = B$, $L_i = L$ and $T_i = T$, and taking into account (1)

$$|L(n)| = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{(2^{n-k}+(2^k-1)3^{n-k})} 3^{(3^{n-k}-2^{n-k})}.$$

Particular cases of this formula had been obtained by A. Monteiro and L. Iturrioz [9] for Tarski algebras, and by L. Iturrioz and O. Rueda [10] for I_3 algebras. In addition, we can state that the notion of $I_3-\nabla$ algebra is a generalization of the notion of $I_3-\Delta$ algebra ([6],[7]). In fact, if in an $I_3-\Delta$ algebra $(A,\rightarrowtail,\Delta,1)$ we define ∇ by means of $\nabla x = (x \rightarrowtail \Delta x) \rightarrowtail x$ the system $(A,\rightarrowtail,\nabla,1)$ is an $I_3-\nabla$ algebra.

*

REFERENCES

[1] Abad, M. and Monteiro, L., *Free symmetric Boolean algebras*, Revista de la Unión Matemática Argentina 27 (1976).

[2] Birkhoff, G., *On the structure of abstract algebras*, Proc. Camb. Phil. Soc. 31 (1935).

[3] ——————, *Lattice theory*, Amer. Math. Soc. Coll. Pub., 25, Providence, 1967.

[4] Cignoli, R., *Boolean elements in Lukasiewicz algebras I*. Proc. Japan Acad. 41 (1965).

[5] ——————, *Moisil algebras*, Notas de Lógica Matemática. Nº 27, Univ. Nac. del Sur, Bahía Blanca, 1970.

[6] Figallo, A., *Las álgebras* I_3-Δ, Revista de la Unión Matemática Argentina 30 (1981).

[7] ——————, *El operador de necesidad en las álgebras* I_3. To appear.

[8] Gastaminza, M.L. and Gastaminza, S., *Characterization of a De Morgan lattice in terms of implication and negation*, Proc. Japan Acad. 44, Nº 7 (1968).

[9] Iturrioz, L. and Monteiro, A., *Cálculo proposicional implicativo clásico con* n *variables proposicionales*, Revista de la Unión Matemática Argentina 23, Nº 3 (1965).

[10] Iturrioz, L. and Rueda, O., *Algèbres implicativas trivalentes de Lukasiewicz libres*, Discrete Mathematics 18 (1977).

[11] Mc Call, S. and Meyer, R.K., *Pure three-valued Lukasiewiczian implication*, Jour. Symb. Logic 31 (1966).

[12] Moisil, G.C., *Sur les ideaux des algebres Lukasiewicziennes trivalentes*, Analele Universitatii C.I. Parhon. Seria Acta logica 3 (1960).

[13] ——————, *Essais sur les logiques non-chrysippiennes*, Ed. Academiei Bucarest, 1972.

[14] Monteiro, A., *Sur la définition des algèbras de Lukasiewez trivalentes*, Bull Maht. Soc. Sc. Math. Phys., R.P. Roum. 7 (55) Nº1-2 (1963).

[15] ——————, *Seminario sobre las Algebras de Lukasiewicz trivalentes*, Lectures given al the Univ. Nac. del Sur, Bahía Blanca, Argentina, 1968.

[16] ——————, *Algebras implicativas trivalentes de Lukasiewicz*, Lectures given at the Univ. Na. del Sur, Bahía Blanca, Argentina, 1968.

[17] ——————, *La semi-simplicité des algèbres de Boole topologiques et les systemes deductifs*, Revista de la Unión Matemática Argentina 25 (1971).

[18] Monteiro, L., *Axiomes indepéndants pour les algèbres de Lukasiewicz trivalentes*, Bull. Math. de la Soc. Sci. Math. Phys. de la R.S. Roum. Tome 7 (55) N3 1-2.

[19] ——————, *Sur les Algèbres de Heyting trivalentes*, Notas de Lógica Matemática, Nº 19, Univ. Nac. del

Sur, Bahía Blanca, 1964.
[20] ———, *Les algèbres de Heyting et de Lukasiewicz trivalentes*, Notre Dame Journal of Formal Logic 11 (1970).
[21] ———, *Algèbres de Boole Monadiques libres*, Algebra Universalis 8 (1978).
[22] Rose, A., *Formalization du calcul propositionnel implicatif à m valeurs de Lukasiewicz*, C.R. Acad. Sci. Paris (1956).

*

Universidad Nacional de San Juan
Mitre N° 235 (Oeste)
5400 San Juan, ARGENTINA

(Recibido en Octubre de 1981,
versión revisada en Noviembre de 1983).