Revista Colombiana de Matemáticas
Vol. XVIII (1984), págs. 1 - 8

EXISTENCE OF UNIFORM BUNDLES

by

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RESUMEN. Se dan condiciones suficientes para garantizar la existencia de haces uniformes, a partir de una estructura uniforme y una familia de secciones.

ABSTRACT. Sufficient conditions for the existence of uniform bundles are given, under data provided by a uniform structure and a family of sections.

INTRODUCCION. For some time conditions have been known guaranteeing the existence of fields (bundles) of Banach spaces, in terms of the sections defining the field. Fell [3] postulated the continuity of the map $t \mapsto \|\sigma(t)\|$ for each section σ; later it turned out that the upper semicontinuity of these maps was a sufficient condition for the existence of these bundles [4]. In recent years it has been possible to decompose other classes of algebras and groups that do not quite fit in the context laid down by Fell. In the general framework of uniform fields introduced by Dauns and Hofmann [2], there are sufficient conditions depending on rather elaborate technical concepts that we replace here by a simple condition, appearing as a natural generalization of the semicontinuity of the norm postulated for bundles of Banach spaces; this condition is stated in Theorem 4.

Since every uniform structure is determined by a saturated family of pseudometrics [1], we formulate the order of closeness in the fibers by means of pseudometrics.

DEFINITION. Let $p:G \to T$ be a surjective function. A *selection* for p is a funcion $\sigma:Q \to G$ with $Q \subset T$ such that $p\sigma$ is the identity map of Q. If $Q = T$, σ is a *global selection.* If T is a topological space and Q is open, σ is a *local selection.* When both G and T are topological spaces, a continuous selection is called a *section.*

A *pseudometric* for p is a function $d:G\times G \to [0,+\infty]$ that satisfies the following conditions:

i) For every $u \in G$ and every $v \in G$, if $p(u) \neq p(v)$ then $d(u,v) = +\infty$.

ii) For every $u \in G$, $d(u,u) = 0$.

iii) For every $u \in G$ and every $v \in G$, $d(u,v) = d(v,u)$.

iv) For every $u \in G$, for every $v \in G$ and every $w \in G$, $d(u,v) \leqslant d(u,w) + d(w,v)$.

If d is a pseudometric for p and σ is a selection for p, $\mathcal{T}_\varepsilon(\sigma)$ denotes the set $\{u \in G \mid p(u) \in \text{dom}\,\sigma,\ d(u,\sigma(p(u))) < \varepsilon\}$ and it is called the *ε-tube around* σ.

The following definition of uniform bundle deviates somewhat from established terminology, [2], but is more compatible with recent trends in this theory, mainly in applications to Banach modules,[4],[5].

DEFINITION. Let G and T be topological spaces, $p:G \to T$ a surjective continuous function, and $(d_i)_{i \in I}$ a family of pseudometrics for p. Then (G,p,T) is called a *uniform bundle* provided that

i) For every $u \in G$, for every $\varepsilon > 0$, and for every $i \in I$ there exists a local section σ such that

$$u \in \mathcal{T}^i_\varepsilon(\sigma) = \{v \in G \mid d_i(v,\sigma(p(v))) < \varepsilon\}.$$

ii) The tubes $\mathcal{T}^i_\varepsilon(\sigma)$ where $i \in I$, $\varepsilon > 0$ and σ runs through the local sections for p, form a basis for the topology of G.

Hence, for each $u \in G$, a fundamental system of neighbourhoods consists of the tubes containing u, around local sections for p. The space T is called the *base space* of the bundle. For each $t \in T$, $p^{-1}(t)$ is the *fiber* above t. The space G is the *fiber space*.

DEFINITION. Let $p:G \to T$ be a surjective function and $(d_i)_{i \in I}$ a family of pseudometrics for p; the upper envelope of the family is clearly a pseudometric for p. The family $(d_i)_{i \in I}$ is called *saturated* if the upper envelope of any finite number of pseudometrics of the family also belongs to the family.

A given family of pseudometrics is always equivalent (in the sense that they define the same uniformity) to a saturated family of pseudometrics.

THEOREM. *Let* T *be a topological space and* $p:G \to T$

be a surjective function. Denote by Σ a set of local selections for p and let $(d_i)_{i\in I}$ be a saturated family of pseudometrics for p. We make the following assumptions:

a) For every $u \in G$, every $i \in I$ and $\varepsilon > 0$, there exists $\alpha \in \Sigma$ such that $u \in \mathcal{T}^i_\varepsilon(\alpha)$.

b) For every $i \in I$ and every $(\alpha,\beta) \in \Sigma\times\Sigma$ the function $s \mapsto d_i(\alpha(s),\beta(s)) : \mathrm{dom}\,\alpha \cap \mathrm{dom}\,\beta \to \bar{\mathbb{R}}$ is upper semicontinuous.

Then G can be equipped with a topology S such that:

1) S has a basis consisting of the sets of the form $\mathcal{T}^i_\varepsilon(\alpha_Q)$, where $i\in I$, $\varepsilon > 0$ and α_Q is the restriction to an open set $Q \subset \mathrm{dom}\,\alpha$ of an $\alpha \in \Sigma$.

2) Each $\alpha \in \Sigma$ is a section.

3) (G,p,T) is a uniform bundle.

Proof. We first show that the collection of all sets $\mathcal{T}^i_\varepsilon(\alpha_Q)$ with the specifications given in conclusion (1) is a basis for a topology S in G. Given two such tubes $\mathcal{T}^i_\varepsilon(\alpha_Q)$ and $\mathcal{T}^j_\delta(\beta_P)$, and

$$u \in \mathcal{T}^i_\varepsilon(\alpha_Q) \cap \mathcal{T}^j_\delta(\beta_P)$$

let

$$\rho = \min\{\tfrac{1}{4}(\varepsilon - d_i(u,\alpha(p(u)))), \tfrac{1}{4}(\delta - d_j(u,\beta(p(u))))\} .$$

Let $J = \{i,j\}$, $d_J = \sup\{d_i, d_j\}$ and $\xi \in \Sigma$ such that

$$u \in T^J_\rho(\xi) = \{v \in G \mid d_J(v,\xi(p(v))) < \rho\} ,$$

then $p(u) \in \{s \in T \mid d_i(\xi(s),\alpha(s)) < \varepsilon_i\}$ where

$$\varepsilon_i = \tfrac{1}{2}(d_i(u,\alpha(p(u))) + \varepsilon).$$

In fact, since $u \in \mathcal{T}^i_\varepsilon(\alpha_Q)$ it follows that $d_i(u,\alpha(p(u))) < \varepsilon$ and thus

$$d_i(u,\alpha(p(u))) < \frac{3}{4}d_i(u,\alpha(p(u))) + \frac{1}{4}\varepsilon \ ;$$

on the other hand the relation $u \in T^j_\rho(\xi)$ implies

$$d_i(u,\xi(p(u))) < \frac{1}{4}(\varepsilon - d_i(u,\alpha(p(u)))) \ ,$$

and so

$$d_i(\xi(p(u)),\ \alpha(p(u))) < \varepsilon_i \ .$$

Similarly, $p(u) \in \{s \in T \mid d_j(\xi(s),\beta(s)) < \delta_j\}$ where

$$\delta_j = \frac{1}{2}(d_j(u,\beta(p(u))) + \delta).$$

By the semicontinuity hypothesis the sets

$$\{s \in T \mid d_i(\xi(s),\alpha(s)) < \varepsilon_i\} \text{ and } \{s \in T \mid d_j(\xi(s),\beta(s)) < \delta\}$$

are open. It follows that

$$S = P \cap Q \cap \{s \in T \mid d_i(\xi(s),\alpha(s)) < \varepsilon_i\} \cap \{s \in T \mid d_j(\xi(s),\beta(s)) < \delta_j\}$$

is a neighbourhood of $p(u)$ in the space T and $\mathcal{T}^j_\rho(\xi_S) \subset \mathcal{T}^i_\varepsilon(\alpha)$; indeed the relation $v \in \mathcal{T}^j_\rho(\xi_S)$ implies

$$d_i(v,\xi(p(v))) < \rho < \frac{1}{2}(\varepsilon - d_i(u,\alpha(p(u))).$$

But $p(v) \in S$, therefore

$$d_i(\xi(p(v)),\alpha(p(v))) < \frac{1}{2}(d_i(u,\alpha(p(u))) + \varepsilon),$$

so $d_i(v,\alpha(p(v))) < \varepsilon$, thus $v \in \mathcal{T}^i_\varepsilon(\alpha_Q)$.

The inclusion $\mathcal{T}^{J}_{\rho}(\xi_S) \subset \mathcal{T}^{j}_{\delta}(\beta_p)$ is obtained in the same manner.

2) Let $\alpha \in \Sigma$ and $t \in \operatorname{dom} \alpha$. A fundamental neighbourhood of $\alpha(t)$ in G is of the form $\mathcal{T}^{i}_{\varepsilon}(\beta_Q)$, where $\beta \in \Sigma$, $Q \subset \operatorname{dom} \alpha$ is open in T, $\varepsilon > 0$, $i \in I$ and $\alpha(t) \in \mathcal{T}^{i}_{\varepsilon}(\beta_Q)$. By hypothesis (b), the set

$$\alpha^{-1}\mathcal{T}^{i}_{\varepsilon}(\beta_Q) = \{s \in Q \mid d_i(\alpha(s), \beta(s)) < \varepsilon\}$$

is open in T, therefore α is a section.

3) The tubes around arbitrary local sections are open, in fact, let $u \in G$ and let σ be a local section for p (not necessarily in Σ) such that $u \in \mathcal{T}^{i}_{\varepsilon}(\sigma)$; to prove that (G,p,T) is a uniform bundle, we must exhibit $\eta > 0$ and $\alpha \in \Sigma$ such that $u \subset \mathcal{T}^{i}_{\eta}(\alpha)$ and $\mathcal{T}^{i}_{\eta}(\alpha_p) \subset \mathcal{T}^{i}_{\varepsilon}(\sigma)$ for some neighborhood P of p(u) in T.

Let $\eta = \frac{1}{4}(\varepsilon - d_i(u,\sigma(p(u))))$ and let $\alpha \in \Sigma$ be such that $u \in \mathcal{T}^{i}_{\eta}(\sigma)$. Since $u \in \mathcal{T}^{i}_{\varepsilon}(\sigma)$ we have $d_i(u,\sigma(p(u))) < \varepsilon$, thus $d_i(u,\sigma(p(u)) < \frac{3}{4}d_i(u,\sigma(p(u))) + \frac{1}{4}\varepsilon$. On the other hand the relation $u \in \mathcal{T}^{i}_{\eta}(\alpha)$ implies

$$d_i(u,\alpha(p(u))) < \eta = \frac{1}{4}(\varepsilon - d_i(u,\sigma(p(u)))).$$

Therefore

$$d_i(\sigma(p(u)),\alpha(p(u))) < \frac{1}{2}d_i(u,\sigma(p(u))) + \frac{1}{2}\varepsilon,$$

so that $p(u) \in \sigma^{-1}(\mathcal{T}^{i}_{\varepsilon_i}(\alpha))$, where

$$\varepsilon_i = \frac{1}{2}(d_i(u,\sigma(p(u))) + \varepsilon).$$

Since σ is continuous, $\sigma^{-1}\mathcal{T}^{i}_{\varepsilon_i}(\alpha)$ is an open neighbourhood P of p(u). Then $v \in \mathcal{T}^{i}_{\eta}(\alpha_p)$ implies $p(v) \in P$ and hence

$$d_i(\alpha(p(v)),\ \sigma(p(v))) < \frac{1}{2}(d_i(u,\sigma(p(u))) + \varepsilon),$$

we also have

$$d_i(v,\alpha(p(v))) < \eta < \frac{1}{2}(\varepsilon - d_i(u,\sigma(p(u))),$$

thus $d_i(v,\sigma(p(v))) < \varepsilon$, that is

$$v \in \mathcal{T}^i_\varepsilon(\sigma). \quad \blacksquare$$

REMARK. When the family $(d_i)_{i\in I}$ reduces to a single element, call it d, and its restriction to each fiber is a metric, the above theorem establishes the existence of a bundle of metric spaces, provided $t \mapsto d(\alpha(t),\beta(t))$ is upper semicontinuous for $\alpha,\beta \in \Sigma$.

More particularly, assume that each fiber is a real or complex linear space and that the metric for p is given by means of a norm, i.e. by a map defined on G whose restriction to each fiber is a norm; if Σ is closed under pointwise addition the hypothesis (b) can be restated in terms of the semicontinuity of the maps, $t \mapsto \|\alpha(t)\|$, $\alpha \in \Sigma$.

REMARK. Consider a bundle of normed spaces (E,π,T). Denote by B_t the algebra of all bounded linear transformations of $E_t = \pi^{-1}(t)$. Let D_0 be the disjoint union of the family $\{B_t | t \in T\}$ and $\rho : D_0 \to T$ be defined by $\rho(u) = t$ if $u \in B_t$. Let Σ be a normed space consisting of a full set of bounded global sections of (E,π,T). Each $\sigma \in \Sigma$ gives rise to a pseudometric for ρ, namely

$$d_\sigma(u,v) = \|u\sigma(t) - v\sigma(t)\| \quad \text{where} \quad t = \rho(u) = \rho(v).$$

The fields (bundles) of automorphisms and derivations studied in [6] can be described in terms of this family of pseudometrics and our theorem can be used in both cases.

REMARK. If (G,p,T) is a uniform bundle and σ,τ are local sections for p with domain $D \subset T$, then for every pseudometric d, in the family defining the uniformity of G, the map

$$t \mapsto d(\sigma(t),\tau(t)) : D \to \bar{\mathbb{R}}$$

is upper semicontinuous. In fact, given $\varepsilon > 0$, $\{t \in D : d(\sigma(t),\tau(t)) < \varepsilon\} = \sigma^{-1}\mathcal{T}_\varepsilon(\tau)$ is open because $\mathcal{T}_\varepsilon(\tau)$ is open and σ is continuous.

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(Recibido en Octubre de 1983).