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A GENERALIZATION OF FUBINI'S THEOREM FOR BANACH ALGEBRA-VALUED MEASURES

by

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RESUMEN. Se demuestra una generalización del teorema de Fubini para medidas vectoriales en álgebras de Banach, en el caso en que la función a integrar toma también valores en el álgebra.

ABSTRACT. The present paper gives a generalization of Fubini's theorem when the function f and the vector measures μ_1 and μ_2 of bounded variation assume values in a Banach algebra.

Fubini's theorem for Bochner integrals with values in a Banach space has been known for a long time (see Hille and Phillips [5]). The object of the present work is to treat a generalization of this theorem when the function f and the

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vector measures μ_1 and μ_2 of bounded variation assume values in a Banach algebra. When the Banach algebra is commutative and the range of f is bounded, this generalization reduces to the theorem of Fubini for this case.

§1. **Preliminaries**. In this section we give some definitions and results from the literature on the theory of integration with respect to Banach algebra-valued measures of bounded variation.

$X \neq \{0\}$ will denote in the sequel a Banach algebra (real or complex) with norm $\|\cdot\|$, which is not assumed to have an identity. Unless otherwise mentioned, X is not commutative.

Let Σ be a σ-ring of subsets of a set $\Omega \neq \phi$, $\mu:\Sigma \to X$ is called a *measure* if μ is countably additive in Σ with respect to the norm topology of X. μ is called a measure *of bounded variation* when $\sup_{E\in\Sigma}|\mu|(E) < \infty$, where $|\mu|$ denotes the variation of μ. As $|\mu|$ is countably additive in the σ-ring Σ, μ is of bounded variation if and only if $|\mu|(E) < \infty$ for all $E \in \Sigma$.

If $\nu:\Sigma \to [0,\infty]$ is a positive measure, $\nu^*(E) = \inf\{\nu(F):E \subset F \in \Sigma\}$ is an outer measure on the hereditary σ-ring $H(\Sigma)$ generated by Σ. Let $M_\nu = \{E \in H(\Sigma): E$ is ν^*-measurable$\}$. Let $\tau(\nu) = \{E \subset \Omega: E \cap A \in M_\nu$ for every $A \in M_\nu\}$. The members of $\tau(\nu)$ are called ν-*measurable* sets. It is known that $\tau(\nu)$ is a σ-algebra containing M_ν and hence containing Σ (vide p.70 [2]).

The set function $\nu^*:\tau(\nu) \to [0,\infty]$, defined by

$$\nu^*(E) = \sup_{\substack{A\subset E \\ A\in M_\nu}} \nu^*(A)$$

is a positive measure and extends ν^* from M_ν to $\tau(\nu)$. The sets $E \in \tau(\nu)$ with $\nu^*(E) = 0$ are called *ν-negligible*. The notion of *almost everywhere* with respect to ν is defined en terms of ν-negligible. Also we shall denote ν^* by ν on $\tau(\nu)$.

A function $f:\Omega \to X$ is called *Σ-simple* if it admits a representation of the form

$$f(w) = \sum_{i=1}^{n} x_i \chi_{E_i}(w)$$

where $x_i \in X$, $E_i \in \Sigma$, $i = 1,2,\ldots,n$. It is true that

$$N(f) = \{w: f(w) \neq 0\} \in \Sigma.$$

(Vide Remarks p.83, [2]).

DEFINITION 1.1. A function $f:\Omega \to X$ is called *$|\mu|$-measurable*, where $\mu:\Sigma \to X$ is a measure of bounded variation, if there exists $N \in \tau(|\mu|)$, N $|\mu|$-negligible, such that there exists a sequence (s_n) of Σ-simple X-valued functions converging to f pointwise in $\Omega\setminus N$, i.e. $s_n \to f$ $|\mu|$-a.e. in Ω.

As μ is of bounded variation, $|\mu|^*$ is bounded in $H(\Sigma)$ and, hence, $|\mu|^*$ is bounded in $\tau(|\mu|)$. Therefore, Ω is $|\mu|$-integrable in the sense of Definition 6, p.75 of [2]. Consequently, by Theorem 2, p.99 of [2], a function $f:\Omega \to X$ which is $|\mu|$-measurable in the sense of Definition 4, p.89 of [2] is $|\mu|$-measurable in the sense of our Definition 1.1. Conversely, as a Σ-simple X-valued function s is clearly $|\mu|$-measurable in the sense of [2], by Theorem 1, p.94, of [2], we obtain that a function $f:\Omega \to X$ which is $|\mu|$-measurable in the sense of Definition 1.1 is $|\mu|$-measurable in the sense of [2]. Thus we have:

PROPOSITION 1.2. *Let $\mu:\Sigma \to X$ be a measure of bounded variation, $f:\Omega \to X$ is $|\mu|$-measurable in the sense of Definition 1.1 if and only if it is $|\mu|$-measurable in the sense of* [2].

The theory of integration in §8 of [2] can be simplified to some extent as we have μ defined on the μ-ring Σ.

For a Σ-simple function $f = \sum_i^n x_i \chi_{E_i}$, $x_i \in X$, $E_i \in \Sigma$, $i = 1,2,\ldots,n$ we define

$$\int_E f d\mu = \sum_{i=1}^n \mu(E_i \cap E) x_i, \quad E \in \Sigma \cup \{\Omega\}.$$

It is clear that

$$\left|\int_E f d\mu\right| \leq \int \|f\| d|\mu|. \tag{1}$$

DEFINITION 1.3. Let $\mu:\Sigma \to X$ be a measure of bounded variation. If $f:\Omega \to X$ is $|\mu|$-measurable, then we say that f is μ-*integrable* if there exists a sequence (s_n) of Σ-simple X-valued functions such that

i) $s_n \to f$ $|\mu|$-a.e in Ω ;

ii) $\int_\Omega \|s_n - s_m\| d|\mu| \to 0$ as $n,m \to \infty$

Then by (1), for $E \in \Sigma$, $\{\int_E s_n d\mu\}_{n=1}^\infty$ is a Cauchy sequence in X and it is therefore convergent in X. By Proposition 8 and 9, §7 of [2], $\int_E f d\mu = \lim_n \int_E s_n d\mu$ is well defined for $E \in \Sigma \cup \{\Omega\}$.

$\mathcal{L}(\mu,X)$ will denote the collection of all X-valued μ-integrable functions. From (i) and (ii) in Definition 1.3, it follows that $\|f\|$ in $|\mu|$-integrable (in classical sense) if f is so, and that

$$\left\|\int_\Omega f d\mu\right\| \leqslant \int_\Omega \|f\| d|\mu| < \infty ,$$

as $(\Omega,\Sigma,|\mu|)$ is a finite measure space.

Using the equivalence relation $f \sim g$ if $\{x: f(x) \neq g(x)\}$ is $|\mu|$-negligible, one sees that $L_1(\mu,X) = \mathcal{L}(\mu,X)/\sim$ is a Banach space under the norm

$$\|f\|_1 = \int_\Omega \|f\| d|\mu| .$$

PROPOSITION 1.4. *Let* $f:\Omega \to X$ *be* $|\mu|$*-measurable. If* $\|f\| \in \mathcal{L}(|\mu|,\mathbb{R})$, *then* $f \in \mathcal{L}(\mu,X)$ *and*

$$\left\|\int_\Omega f d\mu\right\| \leqslant \int_\Omega \|f\| d|\mu| .$$

Proof. It is obvious that $\|f\|$ is $|\mu|$-measurable. By Proposition 1.2 and by Theorem 2, p.99 of [2], there exists a sequence (s_n) of Σ-simple X-valued functions such that

i) $\|s_n(w)\| \leqslant \|f(w)\|$, $n \in \mathbb{N}$ and $w \in \Omega$

ii) $s_n \to f$ $|\mu|$-a.e.

Then by Theorem 3, p.136 of [2] (which applies here), $f \in \mathcal{L}(\mu,X)$ and $\left\|\int_\Omega f d\mu\right\| \leqslant \int_\Omega \|f\| d|\mu|$.

§2. **Product measures with values in** X. Throughout this section we shall assume that $\mu_i:\Sigma_i \to X$ are measures of bounded variation for $i = 1,2$, where Σ_i are σ-rings of subsets of $\Omega_i \neq \phi$, $i = 1,2$. Then $\sup_{E\in\Sigma_i} |\mu_i|(E) = M_i$ is finite for $i =$

1,2. In this section, using auxiliary functions h_E and h^E, $E \in \Sigma_1 \times \Sigma_2$, we prove the existence and uniqueness of the product measures $\mu_1 \times \mu_2$ and $(\mu_1 \times \mu_2)^t$ on $\Sigma_1 \times \Sigma_2$, such that

$$(\mu_1 \times \mu_2)(A \times B) = \mu_1(A)\mu_2(B)$$

and

$$(\mu_1 \times \mu_2)^t(A \times B) = \mu_2(B)\mu_1(A)$$

for $A \in \Sigma_1$ and $B \in \Sigma_2$. It is also true that $\mu_1 \times \mu_2$ and $(\mu_1 \times \mu_2)^t$ are of bounded variation in $\Sigma_1 \times \Sigma_2$.

DEFINITION 2.1. Let $E \in \Sigma_1 \times \Sigma_2$. We define the functions $h_E : \Omega_1 \to X$ and $h^E : \Omega_2 \to X$ as follows:

$$h_E(w_1) = \mu_2\{w_2 \in \Omega_2 : (w_1, w_2) \in E\}, \qquad w_1 \in \Omega_1$$

and

$$h^E(w_2) = \mu_1\{w_1 \in \Omega_1 : (w_1, w_2) \in E\}, \qquad w_2 \in \Omega_2.$$

Since $E_{w_1} = \{w_2 \in \Omega_2 : (w_1, w_2) \in E\} \in \Sigma_2$ and $E^{w_2} = \{w_1 \in \Sigma_1 : (w_1, w_2) \in E\} \in \Sigma_1$ because $E \in \Sigma_1 \times \Sigma_2$, the functions h_E and h^E are well defined.

LEMMA 2.2. *Let* $E \in \Sigma_1 \times \Sigma_2$. *Then* $h_E : \Omega_1 \to X$ *is* $|\mu_1|$*-measurable and* $h^E : \Omega_2 \to X$ *is* $|\mu_2|$*-measurable.*

Proof. We shall prove the result for h_E. In a similar manner the result for h^E can be proved.

Let $E = A \times B$, $A \in \Sigma_1$, $B \in \Sigma_2$. Then, in this case, $h_E = \mu_2(B)\chi_A$ which is clearly $|\mu_1|$-measurable. Consequently, if

$$E = \bigcup_{i=1}^{n} A_i \times B_i, \quad A_i \in \Sigma_1, \quad B_i \in \Sigma_2, \quad (A_i \times B_i) \cap (A_j \times B_j) = \phi,$$

for $i \neq j$,

then it is clear that

$$h_E(w_1) = \sum_{i=1}^{n} \mu_2(B_i)\chi_{A_i}(w_1)$$

which is a Σ_1-simple X-valued function and hence is $|\mu_1|$-measureble. Therefore, if R is the ring generated by $\{A\times B\colon A\in\Sigma_1, B\in\Sigma_2\}$, then h_E is $|\mu_1|$-measurable for each $E\in R$.

Let $M = \{E \in \Sigma_1\times\Sigma_2\colon h_E \text{ is } |\mu_1|\text{-measurable}\}$. Then by the foregoing argument, $R\subset M$. Let $\{E_n\}$ be a monotonic sequence in M with $E = \lim_n E_n$. Then $\{(E_n)_{w_1}\}$ is monotonic with $E_{w_1} = \lim_n (E_n)_{w_1}$. Since μ_2 is a vector measure, which is countably additive in Σ_2, it follows that

$$h_E = \mu_2(E_{w_1}) = \lim_n \mu_2((E_n)_{w_1}) = \lim_n h_{E_n}.$$

As $E_n \in M$, then h_{E_n} is $|\mu_1|$-measurable for $n\in\mathbb{N}$. Now by Proposition 1.2 and by Theorem 1, p.94 of [2], $\lim_n h_{E_n} = h_E$ is $|\mu_1|$-measurable. Hence $E\in M$ and consequently, by Theorem B, §8 of [4], we have that $M = S(R) = \Sigma_1\times\Sigma_2$. That is, h_E is $|\mu_1|$-measurable for each $E\in\Sigma_1\times\Sigma_2$.

COROLLARY 2.3. *Let* $\{E_i\}_{i=1}^n \subset \Sigma_1\times\Sigma_2$. *If* $\{x_i\}_1^n \subset X$ *and* $s\colon\Omega_1 \to X$ *is given by*

$$s(w_1) = \sum_{i=1}^{n} h_{E_i}(w_1)x_i$$

then s *is* $|\mu_1|$*-measurable. Similarly* $t\colon\Omega_2 \to X$, *given by*

$$t(w_2) = \sum_{i=1}^{n} h^{E_i}(w_2)x_i ,$$

is $|\mu_2|$*-measurable.*

Proof. Let $f_i(w_1) = h_{E_i}(w_1)x_i$, $w_1 \in \Omega_1$. As h_{E_i} is $|\mu_1|$-measurable, by Lemma 2.2, exists a sequence (s_n) of X-valued Σ_1-simple functions which converges to h_{E_i} $|\mu_1|$-a.e. in Ω_1. As $s_n x_i$ is also an X-valued Σ_1-simple function and as $s_n x_i \to f_i$ $|\mu_1|$-a.e. in Ω_1, it follows that f_i is $|\mu_1|$-measurable. Now by Proposition 1.2 and by corollary 1, p.101 of [2], s is $|\mu_1|$-measurable. By a similar argument we also have that t is $|\mu_2|$-measurable.

LEMMA 2.4. *For* $E \in \Sigma_1 \times \Sigma_2$, $h_E \in \mathcal{L}(\mu_1, X)$ *and* $h^E \in \mathcal{L}(\mu_2, X)$. *If we define*

$$\tau_1(E) = \int_{\Omega_1} h_E d\mu_1 \quad \textit{and} \quad \tau_2(E) = \int_{\Omega_2} h^E d\mu_2,$$

then τ_1 *and* τ_2 *are X-valued measures in* $\Sigma_1 \times \Sigma_2$. *Further,*

$$\tau_1(A \times B) = \mu_1(A)\mu_2(B) \quad \textit{and} \quad \tau_2(A \times B) = \mu_2(B)\mu_1(A)$$

for $A \in \Sigma_1$ *and* $B \in \Sigma_2$.

Proof. For $E \in \Sigma_1 \times \Sigma_2$, by Lemma 2.2., h_E is $|\mu_1|$-measurable and h^E is $|\mu_2|$- measurable. For $w_1 \in \Omega_1$

$$\|h_E(w_1)\| = \|\mu_2(E_{w_1})\| \leq |\mu_2|(E_{w_1}) \leq M_2$$

and hence $h_E : \Omega_1 \to X$ is bounded. As $\|h_E\|$ is $|\mu_1|$-measurable and bounded and $(\Omega_1, \Sigma_1, |\mu_1|)$ is a finite measure space it follows that $\|h_E\|$ is $|\mu_1|$-integrable and, consequently, by Proposition 1.4, h_E is μ_1-integrable. Similarly, h^E is μ_2-integrable.

Because of the similarity it is enough to prove that τ_1 is countably additive in $\Sigma_1 \times \Sigma_2$. Let $\{E_i\}_1^\infty \subset \Sigma_1 \times \Sigma_2$, $E = \bigcup_{i=1}^{\infty} E_i$, $E_i \cap E_j = \phi$, $i \neq j$. Then it is obvious that for $w_1 \in \Omega_1$,

$\{(E_i)_{w_1}\}$ is a disjoint sequence in Σ_2 and that $E_{w_1} = \bigcup_{i=1}^{\infty}(E_i)_{w_1}$. Further, for $w_1 \in \Omega_1$

$$h_E(w_1) = \mu_2(E_{w_1}) = \sum_{i=1}^{\infty} \mu_2((E_i)_{w_1}) = \lim_n \sum_{i=1}^{n} \mu_2((E_i)_{w_1})$$

$$= \lim_n h_{\left[\bigcup_{i=1}^{n} E_i\right]}(w_1). \tag{1}$$

Also we have that for $w_1 \in \Omega_1$

$$\begin{aligned} \|h_{\left[\bigcup_{i=1}^{n} E_i\right]}(w_1)\| &= \|\mu_2((\bigcup_{i=1}^{n} E_i)_{w_1})\| \\ &\leqslant |\mu_2|((\bigcup_{i=1}^{n} E_i)_{w_1}) \\ &\leqslant |\mu_2|(E_{w_1}) \leqslant M_2. \end{aligned} \tag{2}$$

If $\tilde{h}_E:\Omega_1 \to \mathbb{R}$ is given by $\tilde{h}_E(w_1) = |\mu_2|(E_{w_1})$, then from the theory of product measures in the case of positive measures (vide Berberian [1]), we have that $\tilde{h}_E$ is $|\mu_2|$-measurable and bounded by M_2. Therefore $\tilde{h}_E$ is $|\mu_2|$-integrable, as $|\mu_2|$ is a finite measure in Σ_2. We rewrite (2) in terms of $\tilde{h}_E$ as

$$\|h_{\left[\bigcup_1^n E_i\right]}(w_1)\| \leqslant \tilde{h}_E(w_1), \quad w_1 \in \Omega_1. \tag{2'}$$

From (1), (2') and the fact that $\tilde{h}_E \in \mathcal{L}(|\mu_1|,\mathbb{R})$, we obtain by Theorem 3, p.136 of [2], that

$$\int_{\Omega_1} h_E d\mu_1 = \lim_n \int_{\Omega_1} h_{\left[\bigcup_1^n E_i\right]} d\mu_1 = \sum_1^{\infty} \int_{\Omega_1} h_{E_i} d\mu_1.$$

i.e.

$$\tau_1(E) = \sum_1^{\infty} \tau_1(E_i).$$

Further, for $A \in \Sigma_1$ and $B \in \Sigma_2$, by the definition of the integral of an X-valued simple function, we have that

$$\tau_1(A\times B) = \int_{\Omega_1} h_{A\times B}d\mu_1 = \int_{\Omega_1} \mu_2(B)\chi_A d\mu_1 = \mu_1(A)\mu_2(B)$$

and similarly

$$\tau_2(A\times B) = \int_{\Omega_2} \mu_1(A)\chi_B d\mu_2 = \mu_2(B)\mu_1(A). \blacksquare$$

DEFINITION 2.5. Let R be the ring generated by the semi-ring $\{A\times B: A \in \Sigma_1, B \in \Sigma_2\}$. We define

$$\mu_1\times\mu_2 : R \to X \quad \text{by} \quad (\mu_1\times\mu_2)(E) = \sum_{i=1}^{n} \mu_1(A_i)\mu_2(B_i)$$

and

$$(\mu_1\times\mu_2)^t : R \to X \text{ by } (\mu_1\times\mu_2)^t(E) = \sum_{i=1}^{n} \mu_2(B_i)\mu_1(A_i),$$

where

$$E = \bigcup_{i=1}^{n}(A_i\times B_i), \quad (A_i\times B_i)\cap(A_j\times B_j) = \phi \text{ for } i \neq j, A_i \in \Sigma_1,$$
$$B_i \in \Sigma_2, \quad i = 1,2,\ldots,n.$$

THEOREM 2.6. $\mu_1\times\mu_2$ *and* $(\mu_1\times\mu_2)^t$ *are well defined in* R. *Moreover,* τ_1 *is the unique extension of* $\mu_1\times\mu_2$ *as an X-valued measure to* $\Sigma_1\times\Sigma_2$. *(The extension is also denoted by* $\mu_1\times\mu_2$ *and is called the product measure of* μ_1 *and* μ_2*). A similar result holds for* $(\mu_1\times\mu_2)^t$ *and its extension* τ_2 *(which is also denoted by* $(\mu_1\times\mu_2)^t$ *and called the transpose product measure of* μ_1 *and* μ_2*). Further,* $\mu_1\times\mu_2$ *and* $(\mu_1\times\mu_2)^t$ *are of bounded variation in* $\Sigma_1\times\Sigma_2$ *and satisfy:*

$$|\mu_1\times\mu_2|(E) \leqslant (|\mu_1|\times|\mu_2|)(E)$$

and

$$|(\mu_1\times\mu_2)^t|(E) \leqslant (|\mu_1|\times|\mu_2|)(E)$$

for $E \in \Sigma_1\times\Sigma_2$.

Proof. By Lemma 2.4, we have that $\tau_1(A\times B) = (\mu_1\times\mu_2)(A\times B)$ and $\tau_2(A\times B) = (\mu_1\times\mu_2)^t(A\times B)$. If R is the ring generated by $\{A\times B: A \in \Sigma_1, B \in \Sigma_2$, then $\tilde{\tau}_1 = \tau_1 \restriction R$ and $\tilde{\tau}_2 = \tau_2 \restriction R$ are countably additive in R and hence are finitely additive. Hence, if

$$E = \bigcup_{i=1}^{n}(A_i\times B_i), \quad (A_i\times B_i)\cap(A_j\times B_j) = \phi \text{ for } i \neq j,$$

$$A_i \in \Sigma_1, \quad B_i \in \Sigma_2, \quad i = 1,2,\ldots,n,$$

then

$$\tilde{\tau}_1(E) = \sum_{i=1}^{n}\tilde{\tau}_1(A_i\times B_i) = \sum_{i=1}^{n}\mu_1(A_i)\mu_2(B_i).$$

As $\tilde{\tau}_1$ is well defined on E and is independent of the representation as a finite disjoint union of measurable rectangles, it follows that $(\mu_1\times\mu_2)(E)$ is well defined for $E \in R$ and further, as $\mu_1\times\mu_2 = \tilde{\tau}_1$ in R, $\mu_1\times\mu_2$ is countably additive on R. Similarly $(\mu_1\times\mu_2)^t$ is well defined in R and $(\mu_1\times\mu_2)^t = \tilde{\tau}_2$ in R.

From Lemma 2.4, it follows that $\mu_1\times\mu_2$ has a countably additive extension τ_1 and $(\mu_1\times\mu_2)^t$ has a countably additive extension τ_2 in $\Sigma_1\times\Sigma_2$. We shall prove the uniqueness of τ_1. Similar arguments will prove the uniqueness of τ_2. If τ_1' is another countably additive X-valued extension of $\mu_1\times\mu_2$ in $\Sigma_1\times\Sigma_2$, then for $x^* \in X^*$, $x^*\tau_1(E) = x^*\tau_1'(E)$, $E \in \Sigma_1\times\Sigma_2$. In fact, for

$$E = \bigcup_{i=1}^{n}(A_i \times B_i), \quad (A_i \times B_i) \cap (A_j \times B_j) = \phi, \quad i \neq j,$$

$$A_i \in \Sigma_1, \quad B_i \in \Sigma_2,$$

we have

$$\|\tau_1(E)\| = \| \sum_{i=1}^{n} \mu_1(A_i)\mu_2(B_i)\|$$

$$\leq \sum_{i=1}^{n} \|\mu_1(A_i)\| \|\mu_2(B_i)\|$$

$$\leq \sum_{i=1}^{n} |\mu_1|(A_i)|\mu_2|(B_i)$$

$$= (|\mu_1| \times |\mu_2|)(E). \quad (1)$$

From the classical theory of product measures of finite positive measures, $|\mu_1| \times |\mu_2| : \Sigma_1 \times \Sigma_2 \to [0,\infty)$ is countably additive and finite (see Theorem 2, p.126 of [1]). As $\Sigma_1 \times \Sigma_2$ is a σ-ring, it follows that

$$\sup_{E \in \Sigma_1 \times \Sigma_2} (|\mu_1| \times |\mu_2|)(E) = M < \infty . \quad (2)$$

Therefore, from (1) and (2) we obtain that

$$\|\tau_1(E)\| \leq M \quad \text{for all } E \in \mathcal{R}.$$

Consequently, $x^*\tau_1$ is a bounded scalar measure in $\mathcal{R}$ and hence by the uniqueness of the Caratheodory-Hahn extension of bounded scalar measures, we have that $(x^*\tau_1)(E) = x^*\tau_1'(E)$, $E \in \Sigma_1 \times \Sigma_2$. As x^* is arbitrary in X^*, by the Hahn-Banach theorem

$$\tau_1(E) = \tau_1'(E), \quad E \in \Sigma_1 \times \Sigma_2 .$$

This completes the proof of existence and uniqueness of the extension of $\mu_1 \times \mu_2$ in $\Sigma_1 \times \Sigma_2$.

From (1), we have that

$$\mathcal{R} \subset \mathcal{M} = \{E \in \Sigma_1 \times \Sigma_2 : |\mu_1 \times \mu_2|(E) \leqslant (|\mu_1| \times |\mu_2|)(E)\} .$$

As $|\mu_1| \times |\mu_2|$ is a finite measure on $\Sigma_1 \times \Sigma_1$, by a known argument we see that $\mathcal{M}$ is a monotone class and hence by Theorem B, §8 of Halmos [4], $\mathcal{M} = S(\mathcal{R}) = \Sigma_1 \times \Sigma_2$. Therefore,

$$\sup_{E \in \Sigma_1 \times \Sigma_2} |\mu_1 \times \mu_2|(E) \leqslant M$$

and hence $\mu_1 \times \mu_2$ is of bounded variation in $\Sigma_1 \times \Sigma_2$. The corresponding result for $(\mu_1 \times \mu_2)^t$ is proved in a similar manner.

COROLLARY 2.7. *If* μ_1 *and* μ_2 *commute, in the sense that* $\mu_1(A)\mu_2(B) = \mu_2(A)\mu_1(B)$, *for* $A \in \Sigma_1$, $B \in \Sigma_2$, *then*

$$\tau_1 = \tau_2 = \mu_1 \times \mu_2 = (\mu_1 \times \mu_2)^t \quad \textit{in } \Sigma_1 \times \Sigma_2.$$

Proof. The result follows from Lemma 2.4 and the uniqueness part of Theorem 2.6.

§3. A generalized Fubini's theorem

§3. A generalized Fubini's theorem. Let Ω_i, Σ_i, μ_i, M_i, $i = 1,2$, be as in §2. Let $\mu_1 \times \mu_2$ and $(\mu_1 \times \mu_2)^t$ be the product and transpose product measures, respectively. In this section we obtain a generalized Fubini's theorem, giving the relation between the integral with respect to the product (transpose product) measure and a suitable iterated integral, when the function $f: \Omega_1 \times \Omega_2 \to X$ is $|\mu_1| \times |\mu_2|$-measurable and $\|f\|$ is $|\mu_1| \times |\mu_2|$-integrable. Further, when μ_1 and μ_2 commute and the range of f is bounded in X, this result reduces to the theorem of Fubini for such integrals.

Ω will denote $\Omega_1 \times \Omega_2$ in the sequel.

LEMMA 3.1. *Let* $f:\Omega \to X$ *be such that* f *is* $|\mu_1|\times|\mu_2|$*-measurable and* $\|f\| \in \mathcal{L}(|\mu_1|\times|\mu_2|, \mathbb{R})$. *Then* f *is* $(\mu_1 \times \mu_2)$ *and also* $(\mu_1\times\mu_2)^t$*-integrable.*

Proof. Let $f_{w_1}(w_2) = f(w_1, w_2) = f^{w_2}(w_1)$. As f is $|\mu_1|\times|\mu_2|$-measurable, there exist a sequence (s_n) of X-valued $\Sigma_1\times\Sigma_2$-simple functions and a set N, $(|\mu_1|\times|\mu_2|)$-negligible, such that $s_n \to f$ in $\Omega \setminus N$. As $|\mu_1\times\mu_2|(E) \leqslant |\mu_1|\times|\mu_2|(E)$ for $E \in \Sigma_1\times\Sigma_2$ by Theorem 2.6, it is clear that N is $|\mu_1\times\mu_2|$-negligible (vide Proposition 11, p.15 of [2]) and hence $s_n \to f$ $|\mu_1\times\mu_2|$-a.e. . Therefore, by Definition 1.1, f is $|\mu_1\times\mu_2|$-measurable. Consequently, $\|f\|$ is $|\mu_1\times\mu_2|$-measurable. Further,

$$\int_\Omega \|f\| d|\mu_1\times\mu_2| \leqslant \int_\Omega \|f\| d(|\mu_1|\times|\mu_2|) < \infty$$

and hence by Proposition 1.4, f is $(\mu_1\times\mu_2)$-integrable.

Since the inequality $|(\mu_1\times\mu_2)^t|(E) \leqslant |\mu_1|\times|\mu_2|(E)$, $E \in \Sigma_1\times\Sigma_2$ holds by Theorem 2.6, the above argument can be modified to prove that f is $(\mu_1\times\mu_2)^t$-integrable. ■

In the sequel we shall assume that $f:\Omega \to X$ is $|\mu_1|\times|\mu_2|$-measurable and that $\|f\|$ is $|\mu_1|\times|\mu_2|$-integrable.

LEMMA 3.2. *Let* N *be* $|\mu_1|\times|\mu_2|$*-negligible. Then there exist sets* A *and* B *such that* $\Omega\setminus N = A \cup B$, *where* $A \in \Sigma_1\times\Sigma_2$ *and* B *is* $|\mu_1|\times|\mu_2|$*-negligible. If the function* $g:\Omega \to X$ *satisfies the relation* $g(w_1,w_2) = f(w_1,w_2)\chi_A$, *then there exist* $C \subset \Omega_1$ *and* $D \subset \Omega_2$, $|\mu_1|$*-negligible and* $|\mu_2|$*-negligible, respectively,* *such that* g *is* $|\mu_1|\times|\mu_2|$*-measurable,* $g_{w_1} = f_{w_1}$ $|\mu_1|$-a.e. *and* $g^{w_2} = f^{w_2}$ $|\mu_1|$-a.e., *for* $w_1 \in \Omega_1 \setminus C$ *and* $w_2 \in \Omega_2 \setminus D$.

Proof. Since $\sup_{E\in\Sigma_1\times\Sigma_2} (|\mu_1|\times|\mu_2|(E) < \infty$, $\Omega\setminus N$ is a $|\mu_1|\times|\mu_2|$-integrable set in the sense of Definition 6. p.75 of |2|. Hence by Proposition 12, p.75 of [2], there exist sets $A \in \Sigma_1\times\Sigma_2$ and B, $|\mu_1|\times|\mu_2|$-negligible, such that $\Omega\setminus N = A \cup B$. Therefore, $g = f\chi_A$ is $|\mu_1|\times|\mu_2|$-measurable by Corollary 1, p.101 of [2].

Let h = f-g. Then $h(w_1,w_2) = 0$ for $(w_1,w_2) \in A$ and hence h = 0 $|\mu_1|\times|\mu_2|$-a.e. Now, by the classical Fubini's theorem, for $E \in \tau(|\mu_1|)$, $F \in \tau(|\mu_2|)$

$$0 = \int_{E\times F} \|h\| d(|\mu_1|\times|\mu_2|) = \int_E\left[\int_F\|h_{w_1}\| d|\mu_2|\right]d|\mu_1|$$

$$= \int_F\left[\int_E\|h^{w_2}\| d|\mu_1|\right]d|\mu_2|$$

where we consider the restrictions of $|\mu_1|$ and $|\mu_2|$ in

$$\tau_E(|\mu_1|) = \{G \in \tau(|\mu_1|): G \subset E\}$$

and

$$\tau_F(|\mu_2|) = \{G \in \tau(|\mu_2|): H \subset F\}$$

respectively. Fixing F and varying E we obtain that there exists $C \subset \Omega_1$, $|\mu_1|$-negligible, such that

$$\int_F\|h_{w_1}\| d|\mu_2| = 0, \quad \text{for} \quad w_1 \in \Omega_1 \setminus C$$

and now, varying F in $\tau(|\mu_2|)$, we deduce that $h_{w_1} = 0$ $|\mu_2|$-a.e. Similarly, we have that $h^{w_2} = 0$ $|\mu_1|$-a.e., for $w_2 \in \Omega\setminus D$, with D $|\mu_2|$-negligible. Hence $f_{w_1} = g_{w_1}$ $|\mu_1|$-a.e., and $f^{w_2} = g^{w_2}$ $|\mu_1|$-a.e., for such w_1 and w_2.

LEMA 3.3. *There exist sets* $N_i \subset \Omega_i$, $|\mu_i|$*-negligible,* $i = 1,2$ *such that*

(i) *for* $w_1 \in \Omega_1 \setminus N_1$, $\int_{\Omega_2} f_{w_1} d\mu_2$ *exists and*

(ii) *for* $w_2 \in \Omega \setminus N_2$, $\int_{\Omega_1} f^{w_2} d\mu_1$ *exists.*

Proof. As f is $|\mu_1|\times|\mu_2|$-measurable, there exist N $|\mu_1|\times|\mu_2|$-negligible and a sequence (s_n) of $\Sigma_1\times\Sigma_2$-simple X-valued functions such that $s_n \to f$ in $\Omega\setminus N$. By Lemma 3.3, there exist sets A and B, with $A \in \Sigma_1\times\Sigma_2$ and B $|\mu_1|\times|\mu_2|$-negligible, such that $\Omega\setminus N = A \cup B$. Let $g = f\chi_A$, then $s_n\chi_A \to g$ in Ω and $t_n = s_n\chi_A$ is $\Sigma_1\times\Sigma_2$-simple. Consequently,

$$g_{w_1}(w_2) = \lim_n (t_n)_{w_1}(w_2), \qquad w_2 \in \Omega_2$$

and hence g_{w_1} is $|\mu_2|$-measurable. Similarly, g^{w_2} is $|\mu_1|$-measurable.

Now from Lemma 3.2, it follows that $g_{w_1} = f_w$ $|\mu_2|$-a.e. for w_1 $|\mu_1|$-a.e., and $g^{w_2} = f^{w_2}$, $|\mu_1|$-a.e., for w_2 $|\mu_2|$-a.e. Consequently, by Proposition 1.2 and by Proposition 9, p.91 of [2], we have that f_{w_1} is $|\mu_2|$-measurable and f^{w_2} is $|\mu_1|$-measurable, for w_1 $|\mu_1|$-a.e. and w_2 $|\mu_2|$-a.e.

As $\|f\| \in \mathcal{L}(|\mu_1|\times|\mu_2|),\mathbb{R})$, applying the classical Fubini's theorem to $\|f\|$, we obtain that

$$\int_{\Omega_1} \|f^{w_2}\| d|\mu_1| < \infty$$

for $w_2 \in \Omega_2\setminus\tilde{N}_2$, with $\tilde{N}_2$ $|\mu_2|$-negligible, and

$$\int_{\Omega_2} \|f_{w_1}\| d|\mu_2| < \infty$$

for $w_1 \in \Omega \setminus \tilde{N}_1$, with $\tilde{N}_1$ $|\mu_1|$-negligible. Consequently, we have, by Proposition 1.4, that $\int_{\Omega_2} f_{w_1} d\mu_2$ exists for $w_1 \in \Omega_1 \setminus N_1$ and $\int_{\Omega_1} f^{w_2} d\mu_1$ exists for $w_2 \in \Omega_2 \setminus N_2$, where $N_1 = \tilde{N}_1 \cup C$ and $N_2 = \tilde{N}_2 \cup D$, C and D being those given in Lemma 3.2. ∎

LEMMA 3.4. *With* N_1, N_1 *as in Lemma 3.3, Let*

$$F(w_1) = \begin{cases} \int_{\Omega_1} f_{w_1} d\mu_2, & w_1 \in \Omega_1 \setminus N_1 \\ 0, & w_1 \in N_1 \ ; \end{cases}$$

and

$$G(w_2) = \begin{cases} \int_{\Omega_1} f^{w_2} d\mu_1, & w_2 \in \Omega_2 \setminus N_2 \\ 0, & w_2 \in N_2. \end{cases}$$

Then F *is* $|\mu_1|$*-measurable and* G *is* $|\mu_2|$*-measurable.*

Proof. As f is $|\mu_1|\times|\mu_2|$-integrable by Lemma 3.1, there exists a sequence (s_n) of X-valued $\Sigma_1\times\Sigma_2$-simple functions such that

$$\int_{\Omega} \|s_n - f\| d(|\mu_1|\times|\mu_2|) < \frac{1}{n}, \qquad n \in \mathbb{N} \tag{1}$$

and $s_n \to f$ $|\mu_1|\times|\mu_2|$-a.e. By the classical Fubini's theorem, there exists $\tilde{N}_n$, $|\mu_1|$-negligible, such that

$$F_n(w_1) = \int_{\Omega_2} \|(s_n)w_1 - f_{w_1}\| d|\mu_2| < \infty, \quad w_1 \in \Omega_1 \setminus \tilde{N}_n,$$

$F_n \in \mathcal{L}(|\mu_1|, \mathbb{R})$ and also

$$\int_{\Omega_1} F_n d|\mu_1| = \int_{\Omega} \|s_n - f\| d(|\mu_1|\times|\mu_2|) < \frac{1}{n}$$

by (1). Therefore

$$\lim_{n\to\infty} \int_{\Omega_1} F_n d|\mu_1| = 0.$$

That is, F_n tends to 0 in $|\mu_1|$-mean and hence by Proposition 1.4, p.130 of [2], there exists a subsequence (F_{n_k}) of (F_n) such that $F_{n_k} \to 0$ $|\mu_1|$-a.e. Therefore, there exists a set N_0, $|\mu_1|$-negligible, such that $F_{n_k}(w_1) \to 0$ for all $w_1 \in \Omega_1 \setminus N_0$. Let

$$\tilde{N} = N_1 \cup [\bigcup_{k=1}^{\infty} \tilde{N}_k] \cup N_0,$$

then $\tilde{N}$ is $|\mu_1|$-negligible. For $w_1 \in \Omega_1 \setminus \tilde{N}$,

$$\begin{aligned} \|F(w_1) - \int_{\Omega} (s_{n_k})_{w_1} d\mu_2\| &= \|\int_{\Omega_2} \{f_{w_1} - (s_{n_k})_{w_1}\} d\mu_2\| \\ &\leqslant \int_{\Omega_2} \|f_{w_1} - (s_{n_k})_{w_1}\| d|\mu_2| \\ &= F_{n_k}(w_1) \to 0 \end{aligned}$$

as $k \to \infty$. In other words, we have that

$$F(w_1) = \lim_{k} \int_{\Omega_2} (s_{n_k})_{w_1} d\mu_2, \quad |\mu_1|\text{-a.e.} \tag{2}$$

We observe that if

$$s = \sum_{i=1}^{n} x_i \chi_{E_i}, \quad x_i \in X, \quad E_i \in \Sigma_1 \times \Sigma_2, \quad i = 1,2,\ldots,n,$$

so that s is $\Sigma_1 \times \Sigma_2$-simple with values in X, then

$$\int_{\Omega_2} s_{w_1} d\mu_2 = \sum_{i=1}^{n} \mu_2((E_i)_{w_1}) x_i = \sum_{i=1}^{n} h_{E_i}(w_1) x_i.$$

Now, by Corollary 2.3, we obtain that $\int_{\Omega_2} s_{w_1} d\mu_2$ is $|\mu_1|$-measurable, and by (2) and Theorem 1, p.94 of [2] we have that F is $|\mu_1|$-measurable. The proof for G is similar and hence omitted.

THEOREM 3.5. (A Generalized Fubini's Theorem). *Let $f:\Omega \to X$ be $|\mu_1|\times|\mu_2|$-measurable with $\|f\|$ $|\mu_1|\times|\mu_2|$-integrable. Then there exist $N_i \in \Sigma_i$ with N_i $|\mu_i|$-negligible, $i = 1,2$, such that for $w_1 \in \Omega_1 \setminus N_1$ there exists the integral $\int_{\Omega_2} f_{w_1} d\mu_2$ and for $w_2 \in \Omega_2 \setminus N_2$, there exists the integral $\int_{\Omega_2} f^{w_2} d\mu_1$. If $F:\Omega_1 \to X$ and $G:\Omega_2 \to X$ are defined by*

$$F(w_1) = \int_{\Omega_2} f_{w_1} d\mu_2,\ w_1 \in \Omega_1 \setminus N_1 \quad , \quad G(w_2) = \int_{\Omega_1} f^{w_2} d\mu_1,\ w_2 \in \Omega_2 \setminus N_2,$$

then F is μ_1-integrable and G is μ_2-integrable. Further, we have that

$$\int_{\Omega} f d(\mu_1\times\mu_2) = \int_{\Omega_1} F d\mu_1 = \int_{\Omega_1}\int_{\Omega_2} f_{w_1} d\mu_2 d\mu_1$$

and

$$\int_{\Omega} f d(\mu_1\times\mu_2)^t = \int_{\Omega_2} G d\mu_2 = \int_{\Omega_2}\int_{\Omega_1} f^{w_2} d\mu_1 d\mu_2 .$$

Proof. By Lemma 3.4 and by Proposition 9, p.91 of [2], F is $|\mu_1|$-measurable and G is $|\mu_2|$-measurable, and the existence of N_i with specified properties is guaranteed by Lemma 3.3.

From (1) in the proof of Lemma 3.4, there exists a sequence (s_n) of X-valued $\Sigma_1\times\Sigma_2$-simple functions such that

$$\int_{\Omega} \|s_n - f\| d(|\mu_1|\times|\mu_2|) < \frac{1}{n} \tag{1}$$

for $n \in \mathbb{N}$. Let

$$s_n = \sum_{i=1}^{n} x_i \chi_{E_i}, \quad E_i \in \Sigma_1 \times \Sigma_2, \quad E_i \cap E_j = \phi \text{ for } i \neq j.$$

If

$$s(w_1) = \int_{\Omega_2} (s_n)_{w_1} d\mu_2 ,$$

then

$$s(w_1) = \sum_{i=1}^{n} h_{E_i}(w_i)x_i .$$

Then by Corollary 2.3, s is $|\mu_1|$-measurable and

$$\begin{aligned}
\|s(w_1)\| &\leqslant \sum_{i=1}^{n} \|h_{E_i}(w_1)\| \|x_i\| \\
&\leqslant (\max_{1 \leqslant i \leqslant n} \|x_i\|)(\sum_{i=1}^{n} \|\mu_2((E_i)_{w_1})\| \\
&\leqslant K(\sum_{i=1}^{n} |\mu_2|((E_i)_{w_1})) \\
&= K|\mu_2|((\bigcup_{i=1}^{n} E_i)_{w_1}) \\
&\leqslant KM_2 < \infty
\end{aligned}$$

where $K = \max_{1 \leqslant i \leqslant n} \|x_i\|$. Hence, $\|s\|$ is a bounded $|\mu_1|$-measurable function and so it is $|\mu_1|$-integrable. Consequently, by Proposition 1.4, s is $|\mu_1|$-integrable.

Now we shall prove that F is μ_1-integrable. It is clear that $\|F-s\|$ is $|\mu_1|$-measurable and

$$\int_{\Omega_1} \|F-s\| d|\mu_1| = \int_{\Omega_1} \|\int_{\Omega_2} (f_{w_1}-(s_n)_{w_1}) d\mu_2\| d|\mu_1|$$

$$\leq \int_{\Omega_1}\int_{\Omega_2} \| f_{w_1} - (s_n)_{w_1} \| d|\mu_2| d|\mu_1|$$

$$= \int_{\Omega} \| f - s_n \| d(|\mu_1| \times |\mu_2|)$$

$$< \frac{1}{n} \tag{2}$$

by the classical Fubini's theorem and by (1).

Let $t_n(w_1) = \int_{\Omega_2} (s_n)_{w_1} d\mu_2$. Then, by the foregoing, t_n is μ_1-integrable and consequently, from (2) it follows that

$$\int_{\Omega_1} \|F\| d|\mu_1| \leq \int_{\Omega_2} \|F - t_n\| d|\mu_1| + \int_{\Omega_1} \|t_n\| d|\mu_1| < \infty \quad .$$

Therefore, $\|F\|$ is $|\mu_1|$-integrable and hence by Proposition 1.4 F is μ_1-integrable.

By a similar argument it follows that G is μ_2-integrable.

AFFIRMATION 1. If

$$s = \sum_{i=1}^{n} x_i \chi_{E_i}, \quad E_i \cap E_j = \phi \text{ for } i \neq j, \quad E_i \subset \Sigma_1 \times \Sigma_2, \quad x_i \in X$$

then

$$\int_{\Omega} s \, d(\mu_1 \times \mu_2) = \int_{\Omega_1}\int_{\Omega_2} s_{w_1} d\mu_2 d\mu_1$$

In fact,

$$\int_{\Omega} s \, d(\mu_1 \times \mu_2) = \sum_{i=1}^{n} (\mu_1 \times \mu_2)(E_i) x_i = \sum_{i=1}^{n} \tau_1(E_i) x_i, \tag{3}$$

by Theorem 2.6.

We observe that if $g : \Omega_1 \to X$ is μ_1-integrable and $x \in X$ is a fixed vector in X, then gx is μ_1-integrable and

$$(\int_{\Omega_1} g d\mu_1)x = \int_{\Omega_1} gx\, d\mu_1.$$

Using this observation in (3), we have

$$\begin{aligned}\int_{\Omega} s\, d(\mu_1\times\mu_2) &= \sum_{i=1}^{n} (\int_{\Omega_1} h_{E_i} d\mu_1)x_i \\ &= \sum_{i=1}^{n} \int_{\Omega_1} h_{E_i} x_i d\mu_1 \\ &= \int_{\Omega_1} (\sum_{i=1}^{n} h_{E_i} x_i) d\mu_1 \\ &= \int_{\Omega_1} \{ \sum_{i=1}^{n} \mu_2((E_i)_{w_1})x_i\} d\mu_1 \\ &= \int_{\Omega_1} (\int_{\Omega_2} (\sum_{1}^{n} x_i \chi_{(E_i)_{w_1}}) d\mu_2) d\mu_1 \\ &= \int_{\Omega_1}\int_{\Omega_2} s_{w_1} d\mu_2 d\mu_1 .\end{aligned}$$

AFIRMATION 2. Let s be as in Affirmation 1. Then

$$\int_{\Omega} s\, d(\mu_1\times\mu_2)^t = \int_{\Omega_2}\int_{\Omega_1} s^{w_2} d\mu_1 d\mu_2.$$

In fact, using τ_2 of Lemma 2.4 in place of τ_1 in the proof of Affirmation 1, the present affirmation can be proved.

From (2), recalling that $s = t_n$, we have

$$\int_{\Omega_1} F\, d\mu_1 = \lim_n \int_{\Omega_1} t_n d\mu_1 = \lim_n \int_{\Omega_1}\int_{\Omega_2} (s_n)_{w_1} d\mu_2 d\mu_1. \qquad (4)$$

Using Affirmation 1 in (4), we obtain

$$\int_\Omega F\,d\mu_1 = \lim_n \int_\Omega s_n d(\mu_1\times\mu_2) = \int_\Omega f\, d(\mu_1\times\mu_2)$$

as

$$\left\| \int_\Omega f\, d(\mu_1\times\mu_2) - \int_\Omega s_n d(\mu_1\times\mu_2)\right\| \leqslant \int_\Omega \|f-s_n\| d(|\mu_1|\times|\mu_2|)$$

$$< \frac{1}{n} \to 0 \quad \text{as} \quad n\to\infty.$$

Similarly, using Affirmation 2 we have

$$\int_{\Omega_2} G d\mu_2 = \int_\Omega f d(\mu_1\times\mu_2)^t \ .$$

This completes the proof of the theorem.

COROLLARY 3.6 (Fubini's Theorem). *In addition, let* μ_1 *and* μ_2 *be commuting measures. If* $f:\Omega\to X$ *is a bounded* $|\mu_1|\times|\mu_2|$*-measurable function, then* f *is* $\mu_1\times\mu_2$*-integrable and*

$$\int_\Omega f d(\mu_1\times\mu_2) = \int_{\Omega_1}\int_{\Omega_2} f_{w_1} d\mu_2 d\mu_1 = \int_{\Omega_2}\int_{\Omega_1} f^{w_2} d\mu_1 d\mu_2 .$$

Proof. $\|f\|$ is $|\mu_1|\times|\mu_2|$-measurable and bounded. Hence, as $|\mu_1|\times|\mu_2|$ is a finite measure in $\Sigma_1\times\Sigma_2$, $\|f\|$ is $|\mu_1|\times|\mu_2|$-integrable and hence by Lemma 3.1, if is $\mu_1\times\mu_2$-integrable. Now by Theorem 3.5 and Corollary 2.7 the conclusion is obtained. ∎

It is known that for two complex measures μ_1 and μ_2 on σ-algebras Σ_1 and Σ_2 respectively, $|\mu_1\times\mu_2| = |\mu_1|\times|\mu_2|$ (vide Lemma III. 11.11 of [3]). But this is not generally true for X-valued measures as we see in the following counter example.

COUNTEREXAMPLE 3.7. Let X be a commutative Banach algebra with $x_1, x_2 \in X$ such that $\|x_1 x_2\| < \|x_1\|\|x_2\|$. Let Σ_1 and Σ_2 be σ-algebras of subsets of Ω_1 and Ω_2 respectively, with $\Omega_1 \neq \phi$ and $\Omega_2 \neq \phi$. Let $\nu_i : \Sigma_i \to [0,\infty)$ be non-negative non trivial measures, i = 1,2, then $\mu_i = x_i \nu_i$, i = 1,2, are X-valued measures on Σ_i with

$$|\mu_i| = \|x_i\|\nu_i \ , \quad i = 1,2,$$

$$|\mu_1 \times \mu_2| = \|x_i x_2\|(\nu_1 \times \nu_2)$$

$$|\mu_1| \times |\mu_2| = \|x_1\|\|x_2\|(\nu_1 \times \nu_2).$$

Consequently, $(\mu_1 \times \mu_2)(\Omega_1 \times \Omega_2) = \|x_1 x_2\|(\nu_1 \times \nu_2)(\Omega_1 \times \Omega_2)$

$= \|x_1 x_2\|\nu_1(\Omega_1)\nu_2(\Omega_2) < \|x_1\|\|x_2\|\nu_1(\Omega_1)\nu_2(\Omega_2) = (|\mu_1| \times |\mu_2|)(\Omega_1 \times \Omega_2)$.

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