*Revista Colombiana de Matemáticas*
*Vol. XVIII (1984), págs. 33 - 40*

# M-IDEALS IN BANACH SPACES

by

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ABSTRACT. Let X and Y be given Banach spaces, and L(X,Y) be the space of bounded linear operators from X into Y. Compact operators are denoted by K(X,Y). It is shown that under certain conditions if K(X,Y) is an M-ideal of L(X,Y), then Y is an M-ideal of $Y^{**}$. Further it is shown that if X and Y are reflexive and K(Y,Y) is an M-ideal of L(Y,Y), then $K(X,Y)^{**}$ is isometric to L(X,Y).

RESUMEN. Sean X y Y espacios de Banach y L(X,Y) el espacio de operadores lineales acotados de X en Y. El subespacio de operadores compactos se denota K(X,Y). Se demuestra que bajo ciertas condiciones, si K(X,Y) es un M-ideal de L(X,Y) entonces Y es un M-ideal de $Y^{**}$. Además, si X y Y son reflexivos y K(Y,Y) es un M-ideal de L(Y,Y), entonces $K(X,Y)^{**}$ es isométrico a L(X,Y). Esto generaliza resultados análogos de A. Lima y P. Harmand.

§0. **Introducción**. Let X and Y be given Banach spaces. The space of bounded linear operators from X into Y is denoted by

$L(X,Y)$. We let $K(X,Y)$ be the compact elements in $L(X,Y)$. Lima, [5], showed that if $K(X,X)$ is an M-ideal of $L(X,X)$, then $X$ is an M-ideal in $X^{**}$. In a subsequent paper, Harmand and Lima, [4], have shown that if $X$ is reflexive and $K(X,X)$ is an M-ideal of $L(X,X)$, then $K(X,X)^{**}$ is isometric to $L(X,X)$.

The object of this paper is to generalize the above mentioned results to $L(X,Y)$. In section 2 we show that if the pair $(X,Y)$ satisfies the so called E-property, and $K(X,Y)$ is an M-ideal of $L(X,Y)$, then $Y$ is an M-ideal of $Y^{**}$. Further, we show that if $X$ and $Y$ are reflexive, and $K(Y,Y)$ is an M-ideal of $L(Y,Y)$, then $K(X,Y)^{**}$ is isometric to $L(X,Y)$. Some other results are given. All Banach spaces are assumed to be real.

## §1. Preliminaries on M-ideals.

A closed subspace $J$ of a Banach space $X$ is called an *L-summand* of $X$ if there exists a closed subspace $J' \subseteq X$ such that $X = J \oplus J'$ and for $x_1 \in J$ and $x_2 \in J'$ one has $\|x_1+x_2\| = \|x_1\|+\|x_2\|$. The subspace $J$ is called an *M-ideal* of $X$ if $J^{\perp}$ is an L-summand of $X^*$, where $J^{\perp} = \{\psi \in X^*: \psi(J) = 0\}$.

Another equivalent definition of M-ideals is given via the intersection properties of balls: let $B(x,r) = \{y \in X: \|x-y\| \leq r\}$. Then $J$ is an M-ideal of $X$ if and only if given any three balls $B(a_i,r_i)$, $i = 1,2,3$, in $X$ such that

$$\bigcap_{i=1}^{3} B(a_i,r_i) \neq \phi \quad \text{and} \quad J \cap B(a_i,r_i) \neq \phi, \quad i = 1,2,3,$$

then

$$J \cap \left(\bigcap_{i=1}^{3} B(a_i,r_i)\right) \neq \phi.$$

We refer to [1] and [6] for more on M-ideals in Banach spaces.

§2. **M-Ideals and compact operators**. A pair of Banach spaces $(X,Y)$ is said to satisfy the E-*property* if for every $y \in Y^{**}$, there exists a non-compact $T \in L(X,Y)$ and an $x \in X^{**}$ such that $\|T\| \leq 1$, $\|x\| \leq 1$ and $T^{**}(x) = y$.

The pair $(X,X)$ clearly satisfies the E-property, by taking $T = I$ = identity operator. However, since $L(\ell^p,\ell^q) = K(\ell^p,\ell^q)$, $1 \leq q < p < \infty$, [7], the pair $(\ell^p,\ell^q)$ does not satisfy the E-property.

THEOREM 2.1. *Let* $X$ *and* $Y$ *be given Banach spaces such that the pair* $(X,Y)$ *satisfies the* E-*property. If* $K(X,Y)$ *is an* M-*ideal of* $L(X,Y)$, *then* $Y$ *is an* M-*ideal of* $Y^{**}$.

*Proof.* By Lima [6], it is enough to prove that for every $y \in Y^{**}$ and $y_1,y_2,y_3$ in $Y$ with $\|y\| = 1$ and $\|y_i\| \leq 1$, and for every $\varepsilon > 0$, there exists $z \in Y$ such that

$$\|y+y_i-z\| \leq 1+\varepsilon, \qquad i = 1,2,3.$$

Thus, let $\varepsilon,y,y_1,y_2,y_3$ be given as above. Since $(X,Y)$ satisfied the E-property, there exists a non-compact operator $T \in L(X,Y)$ with $\|T\| \leq 1$ and an $x \in X^{**}$, $\|x\| \leq 1$ such that $T^{**}(x) = y$. Choose $x^* \in X^*$ such that $1-\varepsilon \leq x^*(x) < 1$. Define the compact operators $S_i \in K(X,Y)$:

$$S_i = x^* \otimes y_i, \qquad i = 1,2,3.$$

Since $K(X,Y)$ is an M-ideal in $L(X,Y)$, there exists $U \in K(X,Y)$ such that

$$\|T+S_i-U\| \leq 1+\varepsilon, \qquad i = 1,2,3.$$

Hence

$$\|(T+S_i-U)^{**}\| = \|T^{**}+S_i^{**}-U^{**}\| \leqslant 1+\varepsilon,$$

for i = 1,2,3. Consequently

$$\|(T^{**}+S_i^{**}-U^{**})(x)\| \leqslant 1+\varepsilon.$$

Since U is compact, then, [3,p.624], $U^{**}(x) = z \in Y$. Thus

$$\|y+x^*(x)y_i-z\| \leqslant 1+\varepsilon.$$

But $1-\varepsilon \leqslant x^*(x) \leqslant 1$. Hence

$$\|y+y_i-z\| \leqslant 1+2\varepsilon.$$

This completes the proof of the theorem.

COROLLARY 2.2. *If the pair* (X,Y) *satisfies the* E-*property, and* K(X,Y) *is an* M-*ideal of* L(X,Y), *then* Y* *has the Radon-Nikodym property.*

*Proof.* This follows from [4, Theorem 2.6] and the previous theorem. ■

For the Banach spaces X and Y, let $X \hat{\otimes} Y$ be the complete projective tensor product of X with Y, [8]. Let Y* or X** have the Radon-Nikodym property. Collins and Ruess, [2], showed that the map

$$V : X^{**} \hat{\otimes} Y^* \rightarrow K(X,Y)^*$$

defined by

$$\langle V(\phi),T\rangle = \sum_{i=1}^{\infty} \langle T^{**}(x_i^{**}),y_i^*\rangle ,$$

for every $\phi = \sum_{i=1}^{\infty} x_i^{**} \otimes y_i^*$ in $X^{**} \hat{\otimes} Y^*$, is a quotient map. Hence,

$$K(X,Y)^* \simeq X^{**} \hat{\otimes} Y^*/N ,$$

where $N = \ker V$. Consequently,

$$K(X,Y)^{**} \simeq N^{\perp} = \{Q \in (X^{**} \hat{\otimes} Y^*)^* : Q(N) = 0\}.$$

**THEOREM 2.3.** *Let* $K(Y,Y)$ *be an* M-*ideal of* $L(Y,Y)$. *Then* $L(X,Y) \subseteq K(X,Y)^{**} \subseteq L(X^{**},Y^{**})$.

*Proof.* It is enough to show that if $T \in L(X,Y)$ then $T^{**} \in (\ker V)^{\perp} = N^{\perp}$. Since $K(Y,Y)$ is an M-ideal of $L(Y,Y)$, then, [4], there exists a net $(T_\alpha)$ in $K(Y,Y)$ such that

(i) $\|T_\alpha\| \leqslant 1$ for all $\alpha$

(ii) $\|T_\alpha^*(y^*)-y^*\| \to 0$ for all $y^* \in Y^*$.

(iii) $\|I-T_\alpha\| \to 1$.

Now, let $T \in L(X,Y)$, and $\phi \in \ker V$, with $\phi = \sum_{i=1}^{\infty} x_i^{**} \otimes y_i^*$. Clearly $T_\alpha T \in K(X,Y)$ and

$$0 = \langle T_\alpha T, V(\phi)\rangle$$

$$= \sum_{i=1}^{\infty} \langle T_\alpha^{**} T^{**}(x_i^{**}), y_i^*\rangle$$

$$= \sum_{i=1}^{\infty} \langle T^{**}(x_i^{**}), T_\alpha^*(y_i^*)\rangle.$$

By property (ii) of the net $(T_\alpha)$ and for $\phi \in X^{**} \hat{\otimes} Y^*$, we have

$$0 = \langle T_\alpha T, V(\phi)\rangle \underset{\alpha}{\to} \langle T, V(\phi)\rangle$$

$$= \sum_{i=1}^{\infty} \langle T^{**}(x_i^{**}), y_i^*\rangle$$

However, it is well know that $L(X^{**},Y^{**}) \simeq (X^{**} \hat{\otimes} Y^*)$ via the trace functional. Consequently $T^{**} \in N^{\perp} \simeq (K(X,Y))^{**}$. This completes the proof of the theorem. ∎

As a corollary to the previous theorem we get:

**THEOREM 2.4.** *Let* $X$ *and* $Y$ *be reflexive Banach spaces and* $K(Y,Y)$ *be an* M*-ideal of* $L(Y,Y)$. *Then* $K(X,Y)^{**} \simeq L(X,Y)$.

*Proof.* By Theorem 2.3, $L(X,Y) \subseteq K(X,Y)^{**} \subseteq L(X^{**},Y^{**})$. Since $X^{**} = X$, $Y^{**} = Y$, the result follows.

**THEOREM 2.5.** *Let* $K(X,Y^*)$ *be an* M*-ideal of* $K(X,Y^*)^{**}$. *Then* $X$ *and* $Y$ *are reflexive.*

*Proof.* The Banach spaces $X^*$ and $Y^*$ can be embedded isometrically in $K(X,Y^*)$. But then the result follows from [4, Corollary 3.7.]

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(Recibido en Septiembre de 1983).