§1. Introduction. The essentials of the method of topological localization, was presented in [1] for the first time, and a few years later in Hofmann's survey article [2]. If \( p : G \to T \) is a surjection, the following data are given:

a) A uniformity on \( G \)

b) A topology on \( T \)

c) A family \( \Sigma \) of selections for \( p \).

One seeks to establish the continuity of each \( \alpha \in \Sigma \), for an appropriate topology on \( G \); but this in general can not be secured, unless \( G \) is modified in a drastic manner. The process was described in terms of the entourages of the given uniformity and the neighborhood filters of the base space \( T \). A family of modified stalks is obtained and their disjoint union provides
a new space $\hat{G}$ over $T$.

Recently, K.H. Hofmann [3], [4], gave a very elegant presentation of this localization process in terms of directed colimits, valid for bundles of Banach spaces. A feature of this presentation worth to mention is the giving of the data in the form of a presheaf.

The purpose of this paper is to give a metric version of Hofmann's localization method, generalizing the Banach bundle situation. The construction provides a universal arrow from the given presheaf to the functor that assigns to each bundle of metric spaces the presheaf of its bounded local section.

§2. Directed colimits.

2.1. Let $X$, $Y$ be metric spaces and $f:X \to Y$. The map $f$ is said to be contractive if $d(f(a), f(b)) \leq d(a, b)$ for every pair of elements $a, b \in X$. Denote by $\mathcal{M}$ the category of metric spaces and contractive maps.

2.2. Consider a directed system in $\mathcal{M}$, $(X_\alpha)_{\alpha \in A}$, $(\rho_{\beta\alpha})_{\beta \geq \alpha}$ where $A$ is a directed set. In particular, each $\rho_{\beta\alpha} : X_\alpha \to X_\beta$ is a contractive map such that
   
   i) $\rho_{\alpha\alpha} = \text{id}_{X_\alpha}$
   
   ii) if $\gamma \geq \beta \geq \alpha$, then $\rho_{\gamma\alpha} = \rho_{\gamma\beta}\rho_{\beta\alpha}$.

Let $X$ be the disjoint union of the family of metric spaces $(X_\alpha)_{\alpha \in A}$. Define on $X$ the relation $\sim$ by:

$u \sim v$ if and only if for every $\varepsilon > 0$, if $u \in X_\alpha$ and $v \in X_\beta$, there exists $\gamma \in A$ such that $\gamma \geq \alpha$, $\gamma \geq \beta$ and
It follows, by a straightforward verification that \( \sim \) is an equivalence relation [6].

2.3. Consider the quotient \( Z \) of \( X \) by the equivalence relation \( \sim \), \( Z = X/\sim \). Define on \( Z \times Z \),

\[
d(\bar{u}_\alpha, \bar{v}_\beta) = \inf d(\rho_{\gamma\alpha}(u_\alpha), \rho_{\gamma\beta}(v_\beta)),
\]

where the infimum is taken over all \( \gamma \in A \) such that \( \gamma \geq \alpha \) and \( \gamma \geq \beta \), and \( \bar{u}_\alpha, \bar{v}_\beta \) are the equivalence classes, module the equivalence relation \( \sim \), of \( u_\alpha, v_\beta \in X \). It can be shown that \( d \) is a well defined map and that it defines a metric on \( Z \) [6]. Moreover, the canonical map \( \tau_\alpha : X_\alpha \to Z \) is contractive.

2.4. The metric space \( Z \) and the maps \( (\tau_\alpha)_{\alpha \in A} \) define an inductive cone for the directed system of metric spaces. This cone turns out to be the directed colimit of the system. In fact, given another inductive cone, \( Y, \sigma_\alpha : X_\alpha \to Y \), where \( \alpha \) runs through \( A \), define \( \phi : Z \to Y \) by \( \phi(\bar{u}_\alpha) = \sigma_\alpha(u_\alpha) \). One can easily see that \( \phi \) is a well defined contractive map satisfying the universal property for \( Z \) [6]. Hence we have the following statement.

2.5. Every directed system of metric spaces and contractive maps has a directed colimit.

2.6. REMARK. The above construction remains valid even if the metric is allowed to take the value \( \infty \).

§3. Bundles of metric spaces.

3.1. DEFINITION. Let \( p : G \to T \) be a surjective function
A metric for $p$ is a map $d: G \times G \to [0, \infty]$ such that its restriction to each fiber $G_t = \{u \in G : p(u) = t\}$ is a metric in $G_t$ and $d(u, v) = \infty$ if $p(u) \neq p(v)$.

We refer to [4], [5] for each definition of selection, section and local section.

A set $M$ of selections is called bounded if $d(\alpha, \beta) = \sup \{d(\alpha(t), \beta(t)) : t \in \text{dom} \alpha \cap \text{dom} \beta\}$ is finite for every $\alpha, \beta \in M$, in this case $(\alpha, \beta) \to d(\alpha, \beta)$ is a metric on $M$. Nevertheless it is convenient for us to allow the value $\infty$ for $d$.

By definition, a bundle of metric spaces is a bundle of uniform spaces in the sense given in [5], such that the family of pseudometrics reduces to the metric $d$. In particular, the tubes around local sections are a basis for the topology of $G$ and the map $t \mapsto d(\alpha(t), \beta(t)): U \to \mathbb{R}$ is upper semicontinuous whenever $\alpha, \beta$ are local sections over $U$.

3.2. Let $(E, p, T)$ and $(F, q, T)$ be bundles of metric spaces, $\Sigma$ a presheaf of local sections in the field $(E, p, T)$ and $\Sigma'$ a presheaf of local sections in the field $(F, q, T)$, such that for every open set $U$ of $T$ the domain of each section in $\Sigma(U)$ or in $\Sigma'(U)$ is $U$.

Consider a morphism of presheaves $\phi: \Sigma \to \Sigma'$, then for every open subset $U$ of $T$ and $\alpha, \beta \in \Sigma(U)$, $d(\phi_U(\alpha), \phi_U(\beta)) \leq d(\alpha, \beta)$.

Assume that for every $t \in T$, every $x \in E_t = p^{-1}(t)$ and every open neighborhood $V$ of $t$, there exists $\alpha \in \Sigma(W)$ such $\alpha(t) = x$, with $W \subseteq V$. That is, assume that $\Sigma$ is full.

Define $f: E \to F$ by $f(x) = \phi_U(\alpha)(t)$ if $p(x) = t$, $\alpha(t) = x$ and $U = \text{dom} \alpha$ is an open neighborhood of $t$.

This is a well defined map: suppose $\beta \in \Sigma(V)$ is such that $\beta(t) = x$. Then there exists an open neighborhood $W \subseteq V \cap U$ of $t$.
such that
\[ d(\phi_W(\alpha_w)(t), \phi_W(\beta_w)(t)) \leq d(\phi_W(\alpha_w), \phi_W(\beta_w)) \leq d(\alpha_w, \beta_w) < \varepsilon \]
Thus \( \phi_U(\alpha)(t) = \phi_W(\alpha_w)(t) = \phi_W(\beta_w)(t)) = \phi_V(\beta)(t). \)

The map \( f \) is contractive fiberwise; in fact, take \( x, y \in E \) with \( p(x) = p(y) = t. \) Let \( \alpha, \beta \in \Sigma(U) \) be such that \( \alpha(t) = x \) and \( \beta(t) = y, \) where \( U \) is open in \( T \) and \( t \in U. \) Given \( \varepsilon > 0, \) there exists \( V \subset U \) open and containing \( t \) such that
\[ d(f(x), f(y)) = d(\phi_V(\alpha_V)(t), \phi_V(\beta_V)(t)) \leq d(\phi_V(\alpha_V), \phi_V(\beta_V)) \]
\[ \leq d(\alpha_V, \beta_V) < d(\alpha(t), \beta(t)) + \varepsilon = d(x, y) + \varepsilon. \]
Hence \( d(f(x), f(y)) \leq d(x, y). \)

3.3. LEMA. Let \( (E, p, T) \) and \( (F, q, T) \) be bundles of metric spaces, \( \Sigma \) a presheaf of local sections in \( (E, p, T) \) and \( \Sigma' \) a presheaf of local sections in \( (F, q, T). \) Assume that \( \Sigma \) is full. If \( \phi \) is a morphism between the presheaves \( \Sigma \) and \( \Sigma', \) let \( f : E \rightarrow F \) be defined as described above in terms of \( \phi, \) then

a) For every open subset \( U \) of \( T \) and every \( \alpha \in \Sigma(U), \)
\[ f \mathcal{T}_\varepsilon(\alpha_U) \subset \mathcal{T}_\varepsilon(f\alpha_U) = \mathcal{T}_\varepsilon(\phi_U(\alpha_U)). \]
b) \( f \) is continuous.
c) \( d(f\alpha, f\tau) \leq d(\alpha, \tau) \) for every pair of local sections \( \sigma, \tau \in \Sigma_p(U). \)

Proof. Parts a) and c) follow from the contractivity of \( f \) established above.

b) Let \( x \in E, t = p(x) \) and \( \sigma \) a local section in \( (F, q, T) \) such that \( f(x) \in \mathcal{T}_\varepsilon(\sigma). \) Take \( \alpha \in \Sigma(U) \) such that \( \alpha(t) = x, \) then
\[ W = \{ s \in U \cap \text{dom} \sigma : d(\phi_U(\alpha(s)), \sigma(s)) < \delta \}, \]with \( d(\alpha(t), \sigma(t)) < \delta < \varepsilon, \) is an open neighborhood of \( t = p(x) = q(f(x)). \) Now,
Let \( \sigma_w(\alpha_w) \) and \( \phi_w(\alpha_w(s)) \) be metrics. In fact, if \( y \in T_{\varepsilon-\delta}(\sigma_w(\alpha_w)) \), then 
\[
s = q(y) \in W \text{ and } d(y, \sigma(s)) \leq d(y, \phi_w(\alpha_w(s))) + d(\phi_w(\alpha_w(s)), \sigma(s)) < \varepsilon - \delta + \delta = \varepsilon.
\]
By part (a) of this lemma \( fT_{\varepsilon-\delta}(\alpha_w) \subseteq T_{\varepsilon-\delta}(\sigma_w(\alpha_w)) \). Thus \( f \) is continuous at \( x \).

3.4. Let \((E,p,T)\) and \((F,q,T)\) be bundles of metric spaces, a continuous map \( h:E \to F \) is called a morphism of bundles of metric spaces if \( h \) is fiber preserving (i.e. \( qh = p \)) and \( h \) is contractive.

To a bundle of metric spaces \((E,p,T)\) we can associate a sheaf of metric spaces \( \Sigma_{\hat{p}} \) such that for each open subset \( U \) of \( T \), \( \Sigma_{\hat{p}}(U) \) is the space of all local sections whose domain is \( U \), and to each morphism \( h:E \to F \) of bundles of metric spaces we can associate a sheaf morphism \( \Sigma_{\hat{p}h} = \theta \) such that if \( \alpha \in \Sigma_{\hat{p}}(U), \theta \alpha = h\alpha \in \Sigma_{\hat{q}}(U) \).

3.5. **THEOREM.** Let \( T \) be a topological space, \( A \) the set of all open subsets \( U \) of \( T \) and \((\Sigma(U)), U \in A, (\rho_{UU}), V \subseteq U \) a presheaf of metric spaces. Then there exists a bundle of metric spaces \((\hat{G},\hat{p},T)\) and maps \( \phi_u: \alpha + \hat{\alpha}, \Sigma(U) \to \Sigma_{\hat{p}}(U) \), where \( \Sigma_{\hat{p}}(U) \) are the local section for \( p \) over \( U \), compatible with restriction such that for every open subset \( U \) of \( T \) and every pair \( \alpha, \beta \in \Sigma(U) \), 
\[
d(\hat{\alpha}, \hat{\beta}) \leq d(\alpha, \beta).
\]

**Proof.** As in the second paragraph, \( \mathcal{M} \) denotes the category of metric spaces and contractive maps. For each \( t \in T \), denote by \( V(t) \) the directed set of all open neighborhoods of \( t \) in the space \( T \) and \((\Sigma(U)), U \in V(t), (\rho_{UU}), V \subseteq U \), the directed system determined by the given presheaf. Call \( \hat{G}_t \) its directed colim-
Define \( \hat{\mathcal{G}} \) be the disjoint union of the family \( \{ \hat{G}_t : t \in T \} \). Define \( p: \hat{\mathcal{G}} \to T \) by \( p(\hat{u}) = t \) if \( u \in \hat{G}_t \) and a metric \( \hat{d} \) for \( \hat{p} \) by
\[
\hat{d}(\hat{u}, \hat{v}) = \hat{d}_t(\hat{u}, \hat{v}) \text{ if } \hat{p}(\hat{u}) = \hat{p}(\hat{v}) = t \text{ and } \hat{d}(\hat{u}, \hat{v}) = \infty \text{ if } \hat{p}(\hat{u}) \neq \hat{p}(\hat{v}).
\]

Let \( \tau_{TU}: \Sigma(U) \to \hat{\mathcal{G}}_t \) be the colimit map. Given \( \alpha \in \Sigma(U) \), define \( \hat{\alpha}: U \to \hat{\mathcal{G}} \) by \( \hat{\alpha}(t) = \tau_{TU}(\alpha) \), with \( U \) and open neighborhood of \( t \). Clearly \( \hat{\alpha} = \text{id}_U \). Let \( \hat{\Sigma}(U) = \{ \hat{\alpha} : \alpha \in \Sigma(U) \} \) and \( \phi_U: \Sigma(U) \to \hat{\Sigma}(U) \) be defined by \( \phi_U(\alpha) = \hat{\alpha} \). It is apparent that \( \hat{G}_t = \{ \hat{\alpha}(t) \mid \alpha \in \hat{\Sigma}(U) \} \).

We show now that \( t \to \hat{d}(\hat{\alpha}(t), \hat{\beta}(t)): U \to \mathbb{R} \) is upper semicontinuous. Given \( \varepsilon > 0 \), let \( t \in T \) such that \( \hat{d}(\hat{\alpha}(t), \hat{\beta}(t)) < \varepsilon \).

By the definition of \( \hat{d}_t \) as an infimum, there exists an open neighborhood \( W \subset U \) of \( t \) in \( T \) such that
\[
\hat{d}_t(\hat{\alpha}(t), \hat{\beta}(t)) \leq d(\alpha_w, \beta_w) < \varepsilon,
\]
but \( W \) is also an open neighborhood of any \( s \in W \), hence
\[
\hat{d}_s(\hat{\alpha}(s), \hat{\beta}(s)) < \varepsilon. \text{ Then } W = \{ t \in U : d_t(\hat{\alpha}(t), \hat{\beta}(t)) < \varepsilon \}; \text{ that is } \{ t \in U : \hat{d}_t(\hat{\alpha}(t), \hat{\beta}(t)) < \varepsilon \} \text{ is an open subset of } T. \text{ This proves the asserted upper semicontinuity.}
\]

By Theorem 4 [5], we conclude that \((\hat{G}, \hat{p}, T)\) is a bundle of metric spaces and that each \( \hat{\alpha} \in \hat{\Sigma}(U) \) is a local section.

Let \( U \) be an open subset of \( T \), \( \alpha, \beta \in \Sigma(U) \) and \( \varepsilon > 0 \), then there exists \( t \in T \) such that \( d(\hat{\alpha}, \hat{\beta}) - \varepsilon < d(\hat{\alpha}(t), \hat{\beta}(t)) \leq d(\alpha, \beta) \). Thus \( d(\hat{\alpha}, \hat{\beta}) \leq d(\alpha, \beta) \).

3.6. THEOREM. Under the same hypothesis of the preceding theorem, assume that there is a bundle of metric spaces \((\tilde{\mathcal{G}}, \tilde{p}, T)\) and contractive maps \( \psi_U: \Sigma(U) \to \tilde{\Sigma}(U) \), \( \alpha \to \tilde{\alpha} \) compatible with restriction maps, where \( \tilde{\Sigma}(U) \) are the local section for \( \tilde{p} \) over \( U \). Then there exists a unique continuous map \( h: \hat{\mathcal{G}} \to \tilde{\mathcal{G}} \) such
a) $h$ is fiber preserving and contractive.

b) $h \tilde{\alpha} = \tilde{\gamma}$ for every $\alpha \in \Sigma(U)$.

Proof. Consider again the directed system $(\Sigma(U), \rho_U)$, where $V \subseteq U$ runs through the open neighborhoods of $t \in T$. Let $\tilde{G}_t$ the fiber above $t$ in the bundle $(\tilde{G}, \tilde{p}, T)$. For $V \subseteq V(t)$ define $\sigma_{tV} : \Sigma(V) \to \tilde{G}_t$ by $\sigma_{tV}(\alpha) = \tilde{\alpha}(t)$. Clearly $\sigma_{tW} \circ \rho_{WV} = \sigma_{tV}$ when $W \subseteq V$.

On the other hand, $d(\sigma_{tV} \alpha_V, \sigma_{tV} \beta_V) = d(\tilde{\alpha}(t), \tilde{\beta}(t)) \leq d(\alpha, \beta)$. Hence $\alpha_{tV}$ is contractive and consequently we have an inductive cone for the directed system.

By the universal property of $\tilde{G}_t$ there exists an unique contractive map $\theta_t : \tilde{G}_t \to \tilde{G}_t$ such that $\theta_t \circ \sigma_{tV} = \sigma_{tV}$, for every open neighborhood $V$ of $t$. Therefore, for every $t \in V$ and every $\alpha \in \Sigma(V)$, $\theta_t(\tilde{\alpha}(t)) = \tilde{\alpha}(t)$.

By means of the family $\{\theta_t : t \in T\}$ we can define $\theta_U : \Sigma(U) \to \Sigma(U)$ such that $\theta_U(\tilde{\alpha})(t) = \theta_t(\tilde{\alpha}(t)) = \tilde{\alpha}(t)$. Since $d(\tilde{\alpha}(t), \tilde{\beta}(t)) = d(\theta_t(\tilde{\alpha}(t)), \theta_t(\tilde{\beta}(t))) \leq d(\tilde{\alpha}(t), \tilde{\beta}(t))$ we have $d(\tilde{\alpha}, \tilde{\beta}) = d(\theta_U(\tilde{\alpha}), \theta_U(\tilde{\beta})) \leq d(\tilde{\alpha}, \tilde{\beta})$ for every $\alpha, \beta \in \Sigma(U)$. By Lemma 3.3 it follows that the map $h : \tilde{G} \to \tilde{G}$ defined by $h(\tilde{x}) = h(\tilde{\alpha}(t)) = \theta_t(\tilde{\alpha}(t)) = \tilde{\alpha}(t)$, is continuous.

Uniqueness of $h$ is obvious.

3.7. EXAMPLE. Let $T$ be a topological space and $\Sigma(U)$ be the (bounded) upper semicontinuous functions defined in $U$. As $U$ runs through the open sets of $T$, $\Sigma(U)$ defines a sheaf of metric spaces by taking $d(f, g) = \sup\{d(f(t), g(t)) : t \in T\}$ and the obvious restrictions maps.

By Theorem 3.5 there exists a bundle of metric spaces $(\tilde{\omega}, \tilde{p}, T)$ and contractive maps $\phi_U : f \to \tilde{f} : \Sigma(U) \to \tilde{\Sigma}(U)$ compatible with restrictions such that for every pair $f, g \in \Sigma(U)$, $d(\tilde{f}, \tilde{g}) = d(f, g)$. On the other hand if $(E, p, T)$ is any bundle of metric
spaces and \( \sigma, \tau \) are (bounded) local section for \( p \) over \( U \subseteq T \), then \( t \mapsto d(\sigma(t),\tau(t)) : U \to \mathbb{R} \) is upper semicontinuous and hence it determines an element of \( \hat{\mathcal{E}}(U) \), call it \( \hat{d}(\sigma,\tau) \). The bundle \( (\hat{\mathbb{R}},p,T) \) can thus be considered as the object "real numbers" in the category of bundles of metric spaces and contractive maps.

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