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BERGMAN SPACES FOR THE SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

by

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§1. Introduction. The operator Re (taking the real part of a complex function in one complex variable) transforms the class of analytic functions into the class of harmonic functions. Since harmonic functions are the solutions of Laplace's equation, it is natural to ask whether similar possibilities exist for relating solutions of more general partial differential equations to holomorphic functions. Bergman [1] established various integral operators which associate the solutions of linear elliptic partial differential equations in two real variables with holomorphic functions of one complex variable.

Let L(U) be a second linear partial differential equation given by

$$L(U) = U_{xx}+U_{yy}+a(x,y)U_x+b(x,y)U_y+c(x,y)U = 0 \qquad (1.1)$$

where a,b,c are real analytic functions in a domain $D \subset \mathbb{R}^2$. Now, continue a,b,c into entire functions in $\mathbb{C}^2$. Consider the following transformations of $\mathbb{C}^2$ into itself:

$$\begin{aligned} z &= x+iy \\ z^* &= x-iy \end{aligned} \qquad (x,y) \in \mathbb{C}^2$$

(z,z^* are complex conjugates if and only if x,y are real). Then (1.1) becomes

$$L(U) = U_{zz^*} + AU_z + BU_{z^*} + CU = 0, \tag{1.2}$$

where $B = \bar{A}$, $U(z,z^*) = U(x,y)$ and A,B,C are entire functions in two complex variables [1,p.6]. It is known that

$$U(z,z^*) = c_2(z,z^*,g)$$

is a solution of (1.2), where c_2 is the *Bergman* c_2*-integral operator* and g is a given holomorphic function in one complex variable z [1,p.13]. Solutions of equation (1.2) can be written as an infinite series:

$$U(z,z^*) = c_2(g)$$
$$= \exp\Big[-\int_0^{z^*} A(z,\zeta^*)d\zeta^*\Big]\Big[g(z) + \sum_{n=1}^{\infty} \frac{Q^{(n)}(z,z^*)}{2^{2n}B(n,n+1)} \int_0^z (z-\zeta)^{n-1} g(\zeta)d\zeta\Big]. \tag{1.3}$$

Here

$$Q^{(n)}(z,z^*) = \int_0^{z^*} P^{(2n)}(z,z^*)dz^*,$$

where $P^{(2n)}$ are holomorphic functions satisfying certan recursion relations [1,p.15, Equation 5]. Also, it is assumed that $g(0) = 0$, and $B(n,n+1)$ is a beta function.

In the rest of the paper D, E and F are positive constants which depend on the indicated parameters only and not on any function which may appear in a particular formula. The constants are not necessarily the same on any two occurrences.

§2. Bergman spaces for solution of linear partial differential equations.

Let Δ^N denote the unit polydisc with center at the origin, and $T^N = \{w \in \mathbb{C}^N : |w_j| = 1, 1 \leq j \leq N\}$ be the *Bergman-Silov boundary*. For $z \in \mathbb{C}^N$, let $dz = dz_1 \ldots dz_n$ and for $w \in T^N$, $w_j = \exp(i\theta_j)$, let $dm_N(w) = (2\pi)^{-N} d\theta_1 \ldots d\theta_N$. Given $r = (r_1, \ldots, r_N)$ and $z \in \mathbb{C}^N$, denote by rz the N-tuple $(r_1 z_1, \ldots, r_N z_N)$. A function f, holomorphic on Δ^N, is said to be in $H^p(\Delta^N)$, $0 < p < \infty$, if

$$M_p(r,f) = \left(\int_{T^N} |f(rw)|^p dm_N(w)\right)^{1/p}$$

is bounded independently of r for all $r \in I^N = [0,1) \times \ldots \times [0,1)$. The funtion f is said to be in the *Bergman space* $A^p(\Delta^N)$ $(0 < p < \infty)$, if

$$\|f\|_{A^p(\Delta^N)} = \left(\int_0^1 r_N dr_N \int_0^1 \ldots \int_0^1 M_p^p(r,f) r_1 dr_1\right)^{1/p} < \infty.$$

Yang [4, Theorem 5.1, p.96] proved that the Bergman c_2-integral operator preserves the H^p property, i.e. the c_2-operator maps $H^p(\Delta^1)$ into $H^p(\Delta^2)$ for $(1 \leq p < \infty)$.

We extend Tang's result and prove:

THEOREM. *For* $1 \leq p < \infty$ *if* $g \in A^p(\Delta^1)$, *then* $c_2(g) = U(z,z^*) \in A^p(\Delta^2)$, *where* c_2 *is the integral operator given by* (1.3).

Proof. First assume $g(0) = 0$. Then $U(z,z^*)$ is given by

(1.3). Since $A(z,\zeta^*)$ is an entire in $\mathbb{C}^2$, it is bounded in Δ^2, so that

$$\left|\exp\left(-\int_0^{z^*} A(z,\zeta^*)d\zeta^*\right)\right| \leqslant D. \tag{2.1}$$

Let $z = r_1e^{i\theta_1}$, $z^* = r_2e^{i\theta_2}$, $\zeta = se^{i\theta_1}$. Then by (2.1) and (3.1),

$$|U(r_1e^{i\theta_1},r_2e^{i\theta_2})| \leqslant D\Big[|g(r_1e^{i\theta_1})| + \\ + \sum_{n=1}^{\infty} H_n(z,z^*)\int_0^{|z|} |z-\zeta|^{n-1}|g(se^{i\theta_1})|ds\Big],$$

where

$$H_n(z,z^*) = \frac{Q^{(n)}(z,z^*)}{2^{2n}B(n,n+1)}.$$

Now $|z-\zeta| < r_1 < 1$, so that

$$|U(r_1e^{i\theta_1},r_2e^{i\theta_2})| \leqslant D\Big[|g(r_1e^{i\theta_1})| + \sum_{n=1}^{\infty} H_n(z,z^*)\int_0^{r_1} |g(se^{i\theta_1})|ds\Big].$$

It can be shown that $\sum_{n=1}^{\infty} H_n(z,z^*)$ is an entire function in $\mathbb{C}^2$, [4], therefore it is bounded in Δ^2 and

$$|U(r_1e^{i\theta_1},r_2e^{i\theta_2})| \leqslant D\Big[|g(r_1e^{i\theta_1})| + E\int_0^{r_1} |g(se^{i\theta_1})|ds\Big].$$

Since

$$(a+b)^p \leqslant 2^p(a^p+b^p), \quad a \geqslant 0,\ b \geqslant 0 \quad \text{for } 0<p<\infty, \tag{2.2}$$

[2], then

$$|U(r_1e^{i\theta_1},r_2e^{i\theta_2})|^p \leqslant D_p\Big[|g(r_1e^{i\theta_1})|^p + E^p\Big(\int_0^{r_1} |g(se^{i\theta_1})|ds\Big)^p\Big].$$

Integrate with respect to θ_1, θ_2 over $[0,2\pi]\times[0,2\pi]$, then take the p^{th} root of both sides and use (2.2). This gives

$$\left(\int_{T^2} |U(r_1e^{i\theta_1}, r_2e^{i\theta_2})|^p d\theta_1 d\theta_2\right)^{1/p}$$

$$\leqslant D_p\left[\left(\int_0^{2\pi} |g(r_1e^{i\theta_1})|^p d\theta_1\right)^{1/p} + E\left(\int_0^{2\pi} \left(\int_0^{r_1} |g(se^{i\theta_1})|ds\right)^p d\theta_1\right)^{1/p}\right].$$

Using Minkowski's inequality in continuous form on the second term of the right side, we obtain

$$\left(\int_{T^2} |U(r_1e^{i\theta_1}, r_2e^{i\theta_2})|^p d\theta_1 d\theta_2\right)^{1/p}$$

$$\leqslant D_p\left[\left(\int_0^{2\pi} |g(r_1e^{i\theta_1})|^p d\theta_1\right)^{1/p} + E\int_0^{r_1} \left(\int_0^{2\pi} |g(se^{i\theta_1})|^p d\theta_1\right)^{1/p} ds\right].$$

Since the mean of g is monotone nondecreasing, we have

$$\left(\int_{T^2} |U(r_1e^{i\theta_1}, r_2e^{i\theta_2})|^p d\theta_1 d\theta_2\right)^{1/p} \leqslant D_p\left(\int_0^{2\pi} |g(r_1e^{i\theta_1})|^p d\theta_1\right)^{1/p}$$

Now, raising both sides to the p^{th} power, multiplying by r_1, r_2 and integrating with respect to r_1, r_2 over $[0,1]\times[0,1]$, we obtain

$$\|U\|_{A^p(\Delta^2)} \leqslant D_p\|g\|_{A^p(\Delta^1)}. \tag{2.3}$$

Secondly, suppose that $g(0) = a \neq 0$. Set $G(z) = g(z) - a$. Then $c_2(g) = c_2(G)+c_2(a)$, since c_2 is a linear operator, where

$$c_2(a) = \exp\left[-\int_0^{z^*} A(z,\zeta^*)d\zeta^*\right]\left[a+a\sum_{n=1}^{\infty} h_n(z,z^*)\right]$$

and

$$h_n(z,z^*) = \frac{Q^{(n)}(z,z^*)z^n}{2^{2n}B(n,n+1)}.$$

But $h = \sum_{n=1}^{\infty} h_n$ converges uniformly in every compact subset of $\mathbb{C}^2$, [4, Lemma 5.1, p.94]; therefore, h is an entire function in $\mathbb{C}^2$. Thus, for $(z,z^*) \in \Delta^2$,

$$|h| \leqslant F.$$

Hence by using (2.1) we get

$$|c_2(a)| \leqslant D|a|[1+D]. \tag{2.4}$$

But for $g \in A^p(\Delta)$ $(p > 0)$, we have

$$|g(z)| \leqslant \frac{\|g\|_{A^p(\Delta)}}{(1-|z|^2)^{2/p}}, \quad |z| < 1$$

[3, p.476 Equation 17] and in particular for $z = 0$

$$|a| = |g(0)| \leqslant \|g\|_{A^p(\Delta)}, \tag{2.5}$$

so that (2.4) and (2.5) give

$$|c_2(a)| \leqslant D\|g\|_{A^p(\Delta)}. \tag{2.6}$$

Thus (2.3) and (2.6) give the theorem.

COROLLARY. *If* $g \in A^p(\Delta^1)$, $1 \leqslant p < \infty$, *then the* c_2*-operator is a bounded linear transformation from* $A^p(\Delta^1)$ *to* $A^p(\Delta^2)$.

Proof. From the proof of the theorem, there exists a constant D depending on p only such that

$$\|U\|_{A^p(\Delta^2)} \leqslant D_p\|g\|_{A^p(\Delta^1)}.$$

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