Revista Colombiana de Matemáticas
Vol. XVIII (1984), págs. 151-173

SEMIMETRIZABLE, SUBMETRIZABLE AND SIZABLE SPACES

by

Ali Ahmad FORA

ABSTRACT. In this paper we define sizable spaces and some related notions. We study their properties and their relations with other topological spaces. We discuss some metrization theorems and present several examples relevant ot our subject.

§1. Introduction. Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of all natural numbers and the set of all real numbers, respectively. Let $C\ell\, A$ denote the closure of the set A. Let $T(\mathcal{B})$ denote the topology generated by the base $\mathcal{B}$.

The familiar *Sorgenfrey line* S is defined to be the space of real numbers with the class of all half open intervals $[a,b)$, $a < b$, as a base. It is a well-known fact that S is hereditary Lindelöf, first countable, separable, submetrizable and nonmetrizable. It is also known that the Sorgenfrey plane S×S is not normal. (A space is called *submetrizable* if it is mapped onto some metrizable space by a continuous, one-to-one map).

Let $d: X\times X \to [0,\infty)$ be a function and let $p \in X$, $r > 0$. The d-*open sphere* in X with center p and radius r, denoted by $S_d(p,r)$, is defined by:

$$S_d(p,r) = \{x \in X : d(x,p) < r\}.$$

We shall write $S(p,r)$ for $S_d(p,r)$ when there will not be occasion for confusion. Let us define $\mathcal{B}^*$ to be the collection of all d-open spheres in X, i.e.,

$$\mathcal{B}^* = \{S(p,r) : p \in X,\ r > 0\}.$$

For a nonempty set $A \subseteq X$, if $\{d(x,y) : x,y \in A\}$ is a bounded set in $\mathbb{R}$, then we define the d-*diameter of* A, denoted by $\delta_d(A)$, by

$$\delta_d(A) = \text{d-diameter of } A = \sup\{d(x,y) : x,y \in A\}.$$

We shall write $\delta(A)$ for $\delta_d(A)$ if no confusion will arise. For $A \subseteq X$, we define the function $d(-,A): X \to \mathbb{R}$ by

$$d(-,A)(x) = d(x,A) = \inf\{d(x,a) : a \in A,\ x \in X\}.$$

A topological space X is said to be *symmetrizable* if there exists a function $d: X\times X \to [0,\infty)$ satisfying the following conditions:

(1) $d(x,y) = 0$ iff $x = y$,

(2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(3) $U \subseteq X$ is open if and only if whenever $x \in U$ there is a $t > 0$ such that $S(x,t) \subseteq U$.

A topological space X is called *semimetrizable* provided that there is a function $d: X\times X \to [0,\infty)$ such that

(1) $d(x,y) = 0$ iff $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(3) For any $A \subseteq X$ and for any $x \in X$ we have $x \in Cl\,A$ iff $d(x,A) = 0$.

Let (X,T) be a topological space and let $\mathcal{B}$ be a base for T. Let $L:(X\times X) \cup \mathcal{B} \to [0,\infty)$ be a function. Then L is called a *size function* for $(X,T(\mathcal{B}))$ if it satisfies the following conditions:

(L1) $L(x,y) = 0$ iff $x = y$;

(L2) For any $V,V' \in \mathcal{B}$ and for any $x,x' \in V$, $y,y' \in V'$, we have $L(x',y') \leq L(y,x)+L(V)+L(V')$;

(L3) For any $x \in X$, for any open set U_x containing x, and for any positive real number r, there exists a basic open set $V_{x,r} \in \mathcal{B}$ such that $x \in V_{x,r} \subseteq U_x$ and $L(V_{x,r}) < r$.

A space (X,T) is called *sizable* is the topology T on X has a base $\mathcal{B}$ for which the base has a size function L. This space will be denoted by $(X,T(\mathcal{B}),L)$. A space is called *subsizable* if it is mapped into some sizable space by a continuous, one-to-one map.

The author proved in [2] that a countable compact space in metrizable if and only if it is sizable. In this paper, we shall study the relation between the notion of sizable spaces and the other notions defined above (submetrizable, symmetrizable and semitrizable spaces). We shall also discuss the metrizability of certain spaces.

§2. Some properties of size functions.

Let us start with the following result.

2.1 THEOREM. *Let* L *be a size function for the space* $(X,T(\mathcal{B}))$. *Then* $L(x,y) = L(y,x)$ *for all* $x,y \in X$.

Proof. Let $x,t \in X$ and let r be any positive real number. Then by (L3) there exist basic open sets $U_x \in \mathcal{B}$, $U_y \in \mathcal{B}$ such that $x \in U_x$, $y \in U_y$, $L(U_x) < \frac{1}{2}r$ and $L(U_y) < \frac{1}{2}r$. Using (L2), we get

$$L(x,y) \leqslant L(y,x) + L(U_x) + L(U_y) < L(y,x)+r.$$

Since r is an arbitrary positive real number, $L(x,y) \leqslant L(y,x)$. Similarly, we have $L(y,x) \leqslant L(x,y)$. Thus $L(x,y) = L(y,x)$. ▲

From the definition of the size function L, we may observe that L actually consists of two function, $L_1 : X\times X \to [0,\infty)$ and $L_2 : \mathcal{B} \to [0,\infty)$. Theorem 2.2 shows that L_1 is well behaved. However, Example 6.1 shows that L_2 can behave strangely.

2.2 THEOREM. *If* L *is a size function for* $(X,T(\mathcal{B}))$, *then the restricted map* $L : X\times X \to \mathbb{R}$ *is continuous.*

Proof. Obvious from the inequality

$$|L(x',y')-L(x,y)| \leqslant L(V)+L(V'),\ x,x' \in V;\ y,y' \in V';\ V,V' \in \mathcal{B}.$$

The above inequality holds because of (L2) and Theorem 2.1. ▲

The following theorem studies the relation between $\delta(V)$ and $L(V)$ for $V \in \mathcal{B}$, and the relation between the closure operator $c\ell$ and the function $L(-,A)$.

2.3 THEOREM. *Let* $(X,T(\mathcal{B}),L)$ *be a sizable space. Then* *we have the following:*

(i) $\delta(V) \leqslant L(V)$ *for all* $V \in \mathcal{B}$.

(ii) *For any* $A \subseteq X$, *we have* $C\ell\, A \subseteq \{x \in X : L(x,A) = 0\}$.

(iii) *The equality in* (ii) *need not hold.*

(iv) $L(V) = 0$ *if and only if* V *is a singleton (i.e., a set consisting of only one element), for* $V \in \mathcal{B}$.

Proof. (i) Let $V \in \mathcal{B}$ and let $x,y \in V$. For each $n \in \mathbb{N}$ there exists $V_n \in \mathcal{B}$ such that $y \in V_n$ and $L(V_n) < 1/n$. Using (L2), we get $L(x,y) \leqslant L(y,y)+L(V)+L(V_n) < L(V)+1/n$. Hence $L(x,y) \leqslant L(V)$. Consequently $\delta(V) \leqslant L(V)$.

(ii) can be easily proved by using (L3) and (i).

(iii) In the Sorgenfrey line S, define

$$L(x,y) = |x-y|, x,y \in S; \; L([a,b)) = b-a, \; a < b.$$

Then S is sizable. Take $A = (-1,0)$. Then $L(0,A) = 0$ and $0 \notin C\ell A$.

(iv) Let $L(V) = 0$, where $V \in \mathcal{B}$. Then by (i) and (L1) we have the required result . Conversely, let $V = \{p\} \in \mathcal{B}$. Suppose that $L(V) = r > 0$. Then by (L3) there exists $V' \in \mathcal{B}$ such that $p \in V' \subseteq V$ and $L(V') < \frac{1}{2} r$. Therefore $V' = V$ and $L(V) < \frac{1}{2}r$ which is absurd. Consequently, we must have $L(V) = 0$. ▲

Actually, it is easy to prove that if $x \in C\ell\, S(p,r)$, then $L(x,p) \leqslant r$. The following theorem shows that a size function must satisfy a certain inequality which resembles, but is weaker than, the triangle inequality for metric spaces.

2.4 THEOREM. *Let* $(X,T(\mathcal{B}),L)$ *be a sizable space. Then for any* $V \in \mathcal{B}$, *for any* $a,b \in V$ *and for any* $x \in X$, *we have* $L(x,b) \leqslant L(x,a)+L(V)$.

Proof. By using (L3), for each $n \in \mathbb{N}$ there exists a basic open set $V_n \in \mathcal{B}$ such that $x \in V_n$ and $L(V_n) < 1/n$. Applying (L2) and Theorem 2.1 we get

$$L(x,b) \leqslant L(x,a)+L(V)+L(V_n) < L(x,a)+L(V)+1/n,$$

which implies the required conclusion. ▲

§3. Some properties of sizable spaces.

3.1 THEOREM. *Every metrizable space is sizable.*

Proof. Let (X,d) be a metric space and let

$$\mathcal{B} = \{S_d(p,r) : p \in X, \ r > 0\}.$$

Define $L:(X\times X) \cup \mathcal{B} \to [0,\infty)$ by the following: $L(x,y) = d(x,y)$, $x,y \in X$ and $L(V) = \delta_d(V)$, $V \in \mathcal{B}$. Then X is a sizable space with the size function L. ▲

The following result clarifies the relation between subsizable spaces and sizable spaces.

3.2 THEOREM. *Every subsizable space is sizable.*

Proof. Let (X,T) be a subsizable space. Then there exist a sizable space (Y,T_Y) and a one-to-one continuous map $f:X \to Y$. Since (Y,T_Y) is sizable, there exists a base $\mathcal{B}_Y$ for T_Y and a size function L_Y for $(Y,T(\mathcal{B}_Y))$. Define $\mathcal{B}$ as follows:

$$\mathcal{B} = \{G \cap f^{-1}(B) : G \in T \text{ and } B \in \mathcal{B}_Y\} - \{\emptyset\}.$$

Then $\mathcal{B}$ is a base for T because f is continuous and $\mathcal{B}_Y$ is a base for T_Y. Define $L:(X\times X) \cup \mathcal{B} \to [0,\infty)$ as follows:

$$L(x,x') = L_Y(f(x),f(x')), \ x,x' \in X,$$

$$L(U) = \inf\{L_Y(B) : U = G \cap f^{-1}(B), \ G \in T \text{ and } B \in \mathcal{B}_Y\}, \ U \in \mathcal{B}.$$

Then L is a size function for $(X,T(\mathcal{B}))$; (L1) is easy to prove because L_Y is a size function anf f is a one-to-one map. To prove (L2), let $V,V' \in \mathcal{B}$ and let $x,x' \in V$, $y,y' \in V'$. Let r be any positive real number. Then, from the definition of L, there exist $G,G' \in T$ and $B,B' \in \mathcal{B}_Y$ such that $V = G \cap f^{-1}(B)$, $V' = G' \cap f^{-1}(B')$ and $L(V) \leqslant L_Y(B) < L(V)+\frac{1}{2}r$, $L(V') \leqslant L_Y(B') < L(V') + \frac{1}{2}r$. Since $x,x' \in V$ and $y,y' \in V'$, therefore $f(x),f(x') \in B$ and $f(y),f(f') \in B'$. Applying (L2) for L_Y, we get

$$\begin{aligned} L(x',y') = L_Y(f(x'),f(y')) &\leqslant L_Y(f(y),f(x))+L_Y(B)+L_Y(B') \\ &< L(y,x)+L(V)+L(V')+r. \end{aligned}$$

Since r is an arbitrary small positive real number, $L(x',y') \leqslant L(y,x)+L(V)+L(V')$ and hence (L2) is proved. To prove (L3), let $x \in X$, let U_x be any open set in X containing x and let r be any positive real number. Let $B \in \mathcal{B}_Y$ be any basic open set such that $f(x) \in B \subseteq Y$ and $L_Y(B) < r$. Then $x \in V = U_x \cap f^{-1}(B) \in \mathcal{B}$ and $L(V) \leqslant L_Y(B) < r$. This completes the proof of the theorem. ▲

The following two results are corollaries to Theorem 3.2 and the fact that every submetrizable space is subsizable (use Theorem 3.1).

3.3 COROLLARY. *Being sizable is a topological property.*

3.4 COROLLARY. *Every submetrizable spaces is sizable.*

Since S is a submetrizable space, S^n $(n \in \mathbb{N})$ is submetrizable and hence S^n is sizable. This shows that the class of all sizable spaces contains properly the class of all metrizable spaces. It also shows that a sizable space need not even be normal.

Using Theorem 3.2 again we can prove the following two theorems:

3.5 THEOREM. *Any topology finer than a sizable one is sizable. In particular, any topology finer than a submetrizable (metrizable) one is sizable.*

Proof. Let (X,T) be a sizable space and let T' be any topology on X such that $T \subseteq T'$. Let $i:(X,T') \to (X,T)$ be the identity map; i is continupus because $T \subseteq T'$. Therefore (X,T') is a subsizable space and consequently it is sizable. ▲

3.6 THEOREM. *Any nonempty subspace of a sizable space is sizable.*

Proof. Let (X,T) be a sizable space and let A be a nonempty subspace of X. Then the inclusion map $j:A \to X$ is a continuous one-to-one map. Thus A is subsizable and hence sizable. ▲

The following theorem shows that sizable spaces must satisfy certain separation axioms and certain countability conditions.

3.7 THEOREM. *Let $(X,T(\mathcal{B}))$ be a sizable space with size function L. Then we have the following:*

(i) *X is a Hausdorff space.*

(ii) *Every point in X is a G_δ-set in X.*

(iii) *The diagonal $\Delta = \{(x,x) : x \in X\}$ is a zero-set in $X \times X$ and hence a G_δ-set in $X \times X$.*

Proof. (i) Let $x,y \in X$ such that $x \neq y$. Then $L(x,y) = r > 0$. Using (L3), there exist two basic open sets $U_x, U_y \in \mathcal{B}$ containing x,y, respectively, such that $L(U_x) < r/4$ and $L(U_y) < r/4$. Then $U_x \cap U_y = \emptyset$. For if $z \in U_x \cap U_y$, then by using (L2), we obtain

$L(x,y) \leq L(z,z)+L(U_x)+L(U_y) < \frac{1}{2}r$ which is absurd.

(ii) Let $p \in X$. For each $n \in \mathbb{N}$, there exists $V_n \in \mathcal{B}$ containing p such that $L(V_n) < \frac{1}{n}$. Then $\{p\} \subseteq \bigcap_{n=1}^{\infty} V_n$. If $t \in \bigcap_{n=1}^{\infty} V_n$, then $p,t \in V_n$ for all $n \in \mathbb{N}$. By Theorem 2.3 (i), we get that $L(p,t) \leq L(V_n) < \frac{1}{n}$ for all $n \in \mathbb{N}$. Therefore $L(p,t) = 0$ and consequently $p = t$. Hence $\{p\} = \bigcap_{n=1}^{\infty} V_n$.

(iii) According to Theorem 2.2, the restricted map $L: X \times X \to \mathbb{R}$ is continuous. Therefore $\Delta = L^{-1}(0)$ is a zero set and consequently it is a G_δ-set in the product space $X \times X$. ▲

In the modified Sorgenfrey line S_* (for the definition of S_* see Mrówka [3]), we know that the diagonal of $(S_*)^2$ is not a zero set in $(S_*)^2$ (see Tan [5]). Therefore S_* cannot be a sizable space. It is not difficult to prove that the product of two sizable spaces is again sizable. Combining this result together with Theorem 3.6 and Corollary 3.3, we obtain the fact that $(S_*)^n$ can not be embedded in S^m for any $m,n \in \mathbb{N}$.

§4. Sizable spaces and semimetrizable spaces.

Let us start this section by defining an operator $\wedge$ on subsets of a sizable space $(X,T(\mathcal{B}),L)$ as follows:

$$\wedge(A) = \hat{A} = \{x \in X : L(x,A) = 0\}, \quad A \subseteq X.$$

We also define a sizable space $(X,T(\mathcal{B}),L)$ to be *nice* if $\mathcal{B}^*$ (= the set of all L-open spheres in X) forms a base for some topology (this topology will be denoted by $T(\mathcal{B}^*)$) on X. We shall use $\mathcal{C}\ell^*A$ to denote the $T(\mathcal{B}^*)$-closure of A, where $A \subseteq X$.

A sizable space $(X,T(\mathcal{B}),L)$ is *locally nice at* $p \in X$ if for any L-open sphere $S(q,r)$ containing p, there exists $r' > 0$ such that $S(p,r') \subseteq S(q,r)$. A sizable space X is said to be *lo-*

cally nice if it is locally nice at each $p \in X$. It is clear that if a sizable space is locally nice then it is nice. However the converse need not be true (see Example 6.5).

We shall prove that $\mathcal{B}^*$ need not form a base for any topology on X (Example 6.2) and, if $\mathcal{B}^*$ forms a base for some topology on X then it is not necessarily Hausdorff (Example 6.3).

The following theorem shows that the operator $\wedge$ need not define a closure operator on X.

4.1 THEOREM. *Let* $(X,T(\mathcal{B}),L)$ *be a sizable space. Then we have the following:*

(i) *If* $A \subseteq B$ *then* $\hat{A} \subseteq \hat{B}$.

(ii) $(A \,\hat{\cup}\, B) = \hat{A} \cup \hat{B}$ *for any* $A,B \subseteq X$.

(iii) $\wedge$ *is not an idempotent closure operator.*

Proof. (i) Let $x \in \hat{A}$. Then $L(x,A) = 0$. Since $A \subseteq B$, therefore $L(x,B) \leq L(x,A)$. Hence $L(x,B) = 0$ and consequently $x \in \hat{B}$.

(ii) By (i), we have $\hat{A} \cup \hat{B} \subseteq (A \,\hat{\cup}\, B)$. To prove the other inclusion, let $x \in (A \,\hat{\cup}\, B)$. Then $L(x,A \cup B) = 0$. Consequently $L(x,A) = 0$ or $L(x,b) = 0$. This implies that $x \in \hat{A} \cup \hat{B}$.

(iii) Loooking at the example constructed in Example 6.2 let $A = \bigcup_{n=1}^{\infty} (2^{-2n}, 2^{-2n+1})$. Then $\hat{A} = \bigcup_{n=1}^{\infty} [2^{-2n}, 2^{-2n+1}] \cup \{0\}$ and $\hat{\hat{A}} = \hat{A} \cup \{1\}$. This shows that $\hat{\hat{A}} \neq \hat{A}$. Consequently $\wedge$ is not a closure operator on X. ▲

The following theorem gives us some relations between $T(\mathcal{B})$ and $T(\mathcal{B}^*)$ for a nice sizable space.

4.2 THEOREM. *Let* $(X,T(\mathcal{B}),L)$ *be a sizable space. Then we have the following:*

(i) $S(p,r)$ *is open in* $(X,T(\mathcal{B}))$ *for all* $p \in X$ *and* $r > 0$.

(ii) *If* X *is nice, then* $(X,T(\mathcal{B}^*) \subseteq T(\mathcal{B})$.

(iii) *If* X *is nice, then* $(X,T(\mathcal{B}^*))$ *is a* T_1*-space.*

Proof. (i) To prove that $S(p,r)$ is open in $(X,T(\mathcal{B}))$, let $y \in S(p,r)$. Then $L(p,y) < r$, i.e., $r' = r-L(p,y) > 0$. By (L3), there exists $V \in \mathcal{B}$ such that $y \in V$ and $L(V) < r'$. We claim that $V \subseteq S(p,r)$. To prove our claim, let $x \in V$. Then by Theorem 2.4, we get $L(p,x) \leqslant L(p,y)+L(V) < L(p,y)+r-L(p,y) = r$. Therefore $x \in S(p,r)$, i.e., our claim is proved and consequently $S(p,r)$ is open in $(X,T(\mathcal{B}))$.

(ii) It is clear by using (i).

(iii) Let $x,y \in X$ such that $x \neq y$. Then $L(x,y) = r > 0$. It is clear that if $0 < r' < \frac{1}{2}r$, then $x \in S(x,r')$, $y \notin S(x,r')$ and $y \in S(y,r')$, $x \notin S(y,r')$. ▲

The following theorem shows that a locally nice sizable space admits a weaker topology which is semimetrizable. It is also shown in this case that the operator ∧ is indeed a closure operator.

4.3 THEOREM. *Let* $(X,T(\mathcal{B}),L)$ *be a locally nice sizable space. Then* $A = C\ell^*A$ *for all* $A \subseteq X$, *and hence* $(X,T(\mathcal{B}^*))$ *is a semimetrizable space. Moreover, the operator* ∧ *is indeed a closure operator. Conversely, if* $(X,T(\mathcal{B}^*),L)$ *is semimetrizable (symmetrizable), then* $(X,T(\mathcal{B}),L)$ *is locally nice.*

Proof. Let $p \in C\ell^*A$, where $A \subseteq X$. Let r be any positive real number. Then $S(p,r) \cap A \neq \emptyset$, i.e., there exists $a_r \in A$ such that $L(p,a_r) < r$. Consequently, $L(p,A) = 0$, i.e., $p \in \hat{A}$. This proves that $C\ell^*A \subseteq \hat{A}$. To prove the other inclusion, let $p \in \hat{A}$, i.e., $L(p,A) = 0$. Let $S(q,r)$ be any open sphere containing p. Since $(X,T(\mathcal{B}),L)$ is locally nice at p, therefore there exists $r' > 0$ such that $S(p,r') \subseteq S(q,r)$. Since $L(p,A) = 0$, therefore there exists $a \in A$ such that $L(p,a) < r'$. Hence $S(p,r') \cap A \neq \emptyset$, i.e., $S(q,r) \cap A \neq \emptyset$. This implies that $p \in C\ell^* A$. Now, if $(X,T(\mathcal{B}^*),L)$ is semimetrizable then it is symmetrizable and consequently $(X,T(\mathcal{B}),L)$ is locally nice. ▲

Example 6.5, given later, illustrates that the assumption "locally nice" in Theorem 4.3 can not be replaced by the weaker condition "nice".

It is proved in Davis, Gruenhage and Nyikos ([1], Example 3.1) that there is a regular symmetrizable space X which is not subparacompact and contains a closed set which is not a G_δ-set. This will not occur if the symmetrizable space is sizable (Theorem 4.4). Davis, Gruenhage and Nyikos constructed (in [1], example 3.2) a symmetrizable space which is not semimetrizable. It is also known that all T_2 first countable symmetrizable spaces are semimetrizable. The following theorem shows that a symmetrizable sizable space is indeed semimetrizable. Note that a sizable space need not be first countable. Actually, the Radial Interval Topology constructed in Steen and Seebach [4] is a sizable space which is not first countable.

4.4 THEOREM. *Let* $(X,T(\mathcal{B}),L)$ *be a symmetrizable sizable space. Then we have the following:*

(i) $(X,T(\mathcal{B}),L)$ *is semimetrizable.*

(ii) *Every closed set in* X *is a* G_δ*-set in* X.

Proof. (i) Let $A \subseteq X$. Then $C\ell A \subseteq \hat{A}$ according to Theorem 2.3 (ii). To prove the other inclusion, let $x \in \hat{A}$, i.e., $L(x,A) = 0$. Then $L(x,C\ell A) = 0$. Since $(X,T(\mathcal{B}),L)$ is symmetrizable and $C\ell A$ is a closed set in X, $x \in C\ell A$. Consequently $C\ell A = \hat{A}$ and thus $(X,T(\mathcal{B}),L)$ is semimetrizable.

(ii) Let A be a closet set in X. For each $a \in A$, let

$V_n(a) \in \mathcal{B}$ be such that $a \in V_n(a)$ and $L(V_n(a)) < 1/n$. Let $V_n = \bigcup \{V_n(a) : a \in A\}$. Then V_n is open and $A \subseteq V_n$. To prove $\bigcap_{n=1}^{\infty} V_n \subseteq A$, let $p \in \bigcap_{n=1}^{\infty} V_n$. Then $p \in V_n$ for each $n \in \mathbb{N}$. Therefore there exists $a_n \in A$ such that $p \in V_n(a_n)$. Thus $L(p,a_n) \leqslant L(V_n(a_n)) < 1/n$. Hence $L(p,A) = 0$. Consequently $p \in C\ell\, A$. Hence $A = \bigcap_{n=1}^{\infty} V_n$, i.e., A is a G_δ-set in X. ▲

The following is our last result in this section.

4.5 THEOREM. *The Sorgenfrey line* S *is sizable and not symmetrizable (nor semimetrizable).*

Proof. We proved before that S is sizable (Corollary 3.4). To prove S is not semimetrizable, suppose on the contrary that it is so. Then there exists a semimetric d for S. For each point $x \in S$ there corresponds a positive real number r_x defined by $r_x = d(x,(-\infty,x))$ (note that $x \notin C\ell(-\infty,x)$ and d is a semimetric on S). Let $A_n = \{x \in \mathbb{R} : r_x > 1/n\}$, $n \in \mathbb{N}$. Then the collection $\{A_n : n \in \mathbb{N}\}$ forms a countable cover of the second catregory space $(\mathbb{R},T_u)$, where T_u is the Euclidean topology on $\mathbb{R}$. Thus some one of the sets A_n fails to be nowhere dense in $(\mathbb{R},T_u)$, so for some integer $m \in \mathbb{N}$, there is an interval (a,b) in which A_m is dense. Let $a_0 \in A_m \cap (a,b)$. Then $(a_o,a_o+1) \cap (a_o,b) \cap A_m \neq \emptyset$. Let $a_1 \in A_m \cap (a_o,a_o+1) \cap (a_o,b)$. Then we can inductively define the sequence $(a_n)_{n=1}$ in A_m such that $a_o < a_{n+1} \leqslant a_n$ and $|a_n - a_o| < 1/n$, $n \in \mathbb{N}$. Since $a_o \in C\ell\{a_1,a_2,..\}$, $d(a_o, a_1,a_2,\ldots) = 0$. Thus there exists $k \in \mathbb{N}$ such that $d(a_o,a_k) < 1/m$. Since $a_k \in A_m$, $d(a_k,(-\infty,a_k)) = r > 1/m$. Consequently $d(a_o,a_k) \geqslant r$, i.e., $d(a_o,a_k) > 1/m$ which is a contradiction. This completes the proof that S is not semimetrizable. But since S is a first countable Hausdorff space, S is not symmetrizable. ▲

§5. **Nice and locally nice sizable spaces**. This section is divided into three parts. The first part will discuss the metrizability of the semimetrizable spaces which is induced by a sizable space (Theorem 5.2). The second part will discuss certain conditions under which a sizable space will be locally nice (Theorem 5.3). Afterwards, we shall discuss certain conditions under which the space $(X,T(\mathcal{B}^*))$ will be a regular space (Theorem 5.4, Theorem 5.5 and Corollary 5.6). The third part will discuss certain condition under which $T(\mathcal{B}) = T(\mathcal{B}^*)$ (Theorem 5.7). Finally, we shall discuss the metrizability of a sizable space (Corollary 5.8 and Theorem 5.9).

5.1 THEOREM. *Let* $(X,T(\mathcal{B}),L)$ *be a separable locally nice sizable space. Then* $(X,T(\mathcal{B}^*))$ *is a second countable space.*

Proof. Let $D \subseteq X$ be a countable $T(\mathcal{B})$-dense set in X. Define $\mathcal{B}_1$ as follows:

$$\mathcal{B}_1 = \{S(p,r) : p \in D,\ r \text{ is a positive rational}\}.$$

Let $U \in T(\mathcal{B}^*)$ and let $y \in U$. Then there exists a positive rational number r such that $S(y,r) \subseteq U$. Since $S(y,r)$ is a $T(\mathcal{B})$-open set, there exists $V \in \mathcal{B}$ such that $y \in V \subseteq S(y,r/4)$ and $L(V) < r/4$. Since D is $T(\mathcal{B})$-dense, there exists $p \in V \cap D$. This implies that $p \in S(y,r/4)$, i.e., $L(p,y) < r/4$. Consequently, we have $y \in S(p,\frac{1}{2}r)$. To prove $S(p,\frac{1}{2}r) \subseteq S(y,r)$, let $z \in S(p,\frac{1}{2}r)$. Then $L(p,z) < \frac{1}{2}r$. Applying Theorem 2.4 we get $L(z,y) \leqslant L(z,p) + L(V) < r$. Hence $z \in S(y,r)$. Thus we have found that $y \in S(p,\frac{1}{2}r) \subseteq U$.

Consequently, $\mathcal{B}_1$ is a base for $(X,T(\mathcal{B}^*))$. Since $\mathcal{B}_1$ is obviously countable, $(X,T(\mathcal{B}^*))$ is second countable. ▲

The following theorem shows some conditions under which a sizable space will be submetrizable.

5.2 THEOREM. *Let* $(X,T(\mathcal{B}),L)$ *be a separable locally nice sizable space and assume that* $(X,T(\mathcal{B}^*))$ *is a regular space. Then* $(X,T(\mathcal{B}^*))$ *is metrizable and hence* $(X,T(\mathcal{B}))$ *is a submetrizable space.*

Proof. Using Theorem 5.1, we get that $(X,T(\mathcal{B}^*))$ is second countable. Since $(X,T(\mathcal{B}^*))$ is a regular T_1-space (Theorem 4.2), therefore by the classical theorem of Urysohn (see Willard [6], p.166), the space $(X,T(\mathcal{B}^*))$ is metrizable. Since $T(\mathcal{B}^*) \subseteq T(\mathcal{B})$ (Theorem 4.2), the identity map $i:(X,T(\mathcal{B})) \to (X,T(\mathcal{B}^*))$ defined by $i(x) = x$, $x \in X$, is continuous. Hence $(X,T(\mathcal{B}))$ is a submetrizable space. ▲

The following theorem shows that a sizable space can be locally nice without being metrizable.

5.3 THEOREM. *Let* $(X,T(\mathcal{B}),L)$ *be a sizable space satisfing the following condition: for any* $x \in X$ *there exists* $r_x > 0$ *such that for any* $r < r_x$ *and for any* $y \in S(x,r)$ *there exists* $V \in \mathcal{B}$ *such that* $\{x,y\} \subseteq V \subseteq S(x,r)$ *and* $L(V) < r$. *Then* $(X,T(\mathcal{B}),L)$ *is locally nice.*

Proof. Let $\mathcal{B}_1^* = \{S(x,r) : x \in X,\ 0 < r < r_x\}$. Let $S(x,r) \in \mathcal{B}_1^*$ and $y \in S(x,r)$. Then there exists a basic open set $V \in \mathcal{B}$ such that $\{x,y\} \subseteq V \subseteq S(x,r)$ and $L(V) < r$. If $t = r-L(V)$ then we claim that $S(y,t) \subseteq S(x,r)$. To prove our claim, let $z \in S(y,t)$, i.e., $L(z,y) < t$. Then using Theorem 2.4 we get $L(z,x) \leqslant L(z,y)+L(V) < t+L(V) = r$. This proves our claim. Now let $h = \frac{1}{2}\min\{t,r_y\}$. Then $S(y,h) \in \mathcal{B}_1^*$ and $y \in S(y,h) \subseteq S(x,r)$. This proves that $\{S(x,r) : 0 < r < r_x\}$ is a local base in

$(X,T(\mathcal{B}_1^*))$ at x. Consequently $(X,T(\mathcal{B}),L)$ is locally nice. ▲

The following three results show that a nice sizable space may satisfy under certain conditions a high separation axiom without being metrizable.

5.4 THEOREM. *Let* $(X,T(\mathcal{B}),L)$ *be a locally nice sizable space and let* $f_A:(X,T(\mathcal{B}^*)) \to (\mathbb{R},T_u)$ *with* T_u *= the Euclidean topology on* $\mathbb{R}$ *and* $f_A(x) = L(x,A)$, $x \in X$, *be continuous for all closed proper subsets* A *in* $(X,T(\mathcal{B}^*))$. *Then* $(X,T(\mathcal{B}^*))$ *is a* T_4-*space. Moreover, every closed set in* $(X,T(\mathcal{B}^*))$ *is a zero set in* $(X,T(\mathcal{B}^*))$.

Proof. Let A,B be any two closed proper disjoint sets in $(X,T(\mathcal{B}^*))$. Define the funcion $f:(X,T(\mathcal{B}^*)) \to (\mathbb{R},T_u)$ by $f(x) = L(x,A)/(L(x,A)+L(x,B))$, $x \in X$. Then f is well defined because of Theorem 4.3 and moreover f is a continuous function with $A = f^{-1}(0)$ and $B = f^{-1}(1)$. Consequently, $(X,T(\mathcal{B}^*))$ is a T_4-space and moveover, A,B are zero sets in $(X,T(\mathcal{B}^*))$. ▲

5.5 THEOREM. *Let* $(X,T(\mathcal{B}),L)$ *be a nice sizable space satisfying the following condition: if* $L(x,S(p,r)) = 0$ then $L(x,p) \leq r$ $(x,p \in X,\ r > 0)$. *Then* $(X,T(\mathcal{B}^*))$ *is a* T_3-*space.*

Proof. Since $(X,T(\mathcal{B}^*))$ is a T_1-space (Theorem 4.2), it suffices to prove that $(X,T(\mathcal{B}^*))$ is a regular space. To do this, let A be a $T(\mathcal{B}^*)$-closed set and let $p \in X-A$. Then there exists $q \in X, r > 0$ such that $p \in S(q,r) \subseteq X-A$. Since $L(p,q) < r$, $t = r-L(p,q) > 0$. Let $h = L(p,q)+\frac{1}{2}t$. Then $p \in S(q,h) \subseteq Cl^*S(q,h) \subseteq S(q,r) \subseteq X-A$ (we have used the condition stated in the Theorem for $Cl^*S(q,h) \subseteq S(q,r)$). This completes the proof that $(X,T(\mathcal{B}^*))$ is a regular space. ▲

The following result is an inmmediate corollary to the above theorem.

5.6 COROLLARY. *Let* $(X,T(\mathcal{B}),L)$ *be a nice sizable space. Let* $\{x \in X : L(x,p) \leqslant r\}$ *be closed in* $(X,T(\mathcal{B}^*))$ *for all* $p \in X$ *and all* $r > 0$. *Then* $(X,T(\mathcal{B}^*))$ *is a* T_3*-space.* ▲

To present our last result in this section, we need the following definition. Let (X,T) be a topological space and let $A \subseteq X$, then A is said to be T-*paracompact relative to* X if every open cover of A by members of T has a locally finite (in X) refinement by members of T.

5.7 THEOREM. *Let* $(X,T(\mathcal{B}),L)$ *be a nice sizable space and let* $(X,T(\mathcal{B}^*))$ *be Hausdorff. If each* $T(\mathcal{B})$*-closed proper subset of* X *is* $T(\mathcal{B}^*)$*-paracompact relative to* $(X,T(\mathcal{B}^*))$, *then* $T(\mathcal{B}) = T(\mathcal{B}^*)$. *Moreover* $(X,T(\mathcal{B}))$ *is a* T_4*-space.*

Proof. Since $T(\mathcal{B}^*) \subseteq T(\mathcal{B})$ (Theorem 4.2), it remains to prove that $T(\mathcal{B}) \subseteq T(\mathcal{B}^*)$. To do this, let F be a closed proper set in $(X,T(\mathcal{B}))$. Let $p \in X-F$. For each $x \in F$, there exists $y_x \in X$, $r_x > 0$ such that $x \in S(y_x,r_x) \subseteq C\ell^*S(y_x,r_x)$ and $p \notin C\ell^*S(y_x,r_x)$ (this is because $(X,T(\mathcal{B}^*))$ is a Hausdorff space). Let $\mathcal{C} = \{S(y_x,r_x) : x \in F\}$. Then $\mathcal{C}$ is a $T(\mathcal{B}^*)$-open cover of F. Thus there exists $\mathcal{C}' = \{U_\alpha : \alpha \in \Delta\}$ a locally finite (in $(X, T(\mathcal{B}^*))$) refinement which covers F and such that $U_\alpha \in T(\mathcal{B}^*)$ for all $\alpha \in \Delta$. Let $U = \bigcup\{U_\alpha : \alpha \in \Delta\}$, then $U \in T(\mathcal{B}^*)$ and $p \notin U$. Let $M = C\ell^*U$, then $F \subseteq U \subseteq M$ and $M = \bigcup\{C\ell^*U_\alpha : \alpha \in \Delta\}$, because $\mathcal{C}'$ is locally finite in $(X,T(\mathcal{B}^*))$. Since $\mathcal{C}'$ is a refinement of $\mathcal{C}$ and $p \notin C\ell^*S(x,r_x)$ for all $x \in F$, therefore $p \notin C\ell^*U_\alpha$ for all $\alpha \in \Delta$; hence $p \notin M$. This implies that $p \in X-M \subseteq X-F$, $F \subseteq U$ and $U \cap (X-M) = \emptyset$. Hence F is a $T(\mathcal{B}^*)$-closed set. Consequently $T(\mathcal{B}) = T(\mathcal{B}^*)$.

The above proof also shows that $(X,T(\mathcal{B}))$ is a regular space.

To prove that $(X,T(\mathcal{B}))$ is a normal space, let A,B be two closed proper disjoint sets in $(X,T(\mathcal{B}))$. For each $x \in A$, there exists $y_x \in X$, $r_x > 0$ such that $x \in S(y_x,r_x) \subseteq C\ell\, S(y_x,r_x)$ and $B \cap C\ell\, S(y_x,r_x) = \emptyset$ (this is because $(X,T(\mathcal{B}))$ is a regular space). Let $\mathcal{C}_1 = \{S(y_x,r_x) : x \in A\}$, then $\mathcal{C}_1$ is a $T(\mathcal{B}^*)$-open cover for A. Therefore there exists $\mathcal{C}_1' = \{V_\alpha : \alpha \in \Delta\}$ a $T(\mathcal{B}^*)$-open locally finite (in $(X,T(\mathcal{B}^*))$) refinement for $\mathcal{C}_1$ which covers A. Let $V = \bigcup\{V_\alpha : \alpha \in \Delta\}$, then V is open in $(X,T(\mathcal{B}^*))$ and $A \subseteq V$. Let $H = C\ell^* V = \bigcup\{C\ell^* V_\alpha : \alpha \in \Delta\}$, then H is a $T(\mathcal{B}^*)$-closed set and $A \subseteq V \subseteq H$. Now, it is clear that $A \subseteq V$, $B \subseteq X-H$ and $V \cap (X-H) = \emptyset$. Therefore $(X,T(\mathcal{B}))$ is a normal space. ▲

Actually we can strengthen the conclusion of Theorem 5.7 to deduce that $(X,T(\mathcal{B}))$ is a binormal space.

The following result in an immediate corollary of Theorem 5.2 and Theorem 5.7.

5.8 COROLLARY. *Let* $(X,T(\mathcal{B}),L)$ *be a locally nice sizable separable space and let* $(X,T(\mathcal{B}^*))$ *be a Hausdorff space. Let every* $T(\mathcal{B})$*-closed proper set in* X *be* $T(\mathcal{B}^*)$*-paracompact relative to* $(X,T(\mathcal{B}^*))$. *Then* $(X,T(\mathcal{B}))$ *is a metrizable space.* ▲

We observed that a sizable space can be locally nice by putting some condition on the size function. However, the author shows in [2] that a sizable space can be locally nice by putting certain conditions on the topological space itself. The following theorem is obtained in [2].

5.9 THEOREM. *Let* $(X,T(\mathcal{B}),L)$ *be a sizable countably compact space. Then it is locally nice and separable. Moreover,*

$T(\mathcal{B}) = T(\mathcal{B}^*)$ *and* $(X,T(\mathcal{B}))$ *is a metrizable space.* ▲

§6. Examples.

We start this section with the following example.

6.1 EXAMPLE. *There exists a size function* $L:(X\times X)\cup\mathcal{B} \to [0,\infty)$ *and there exist two basic open sets* $U,V \in \mathcal{B}$ *such that* $U \subset V$ *and* $L(U) \not\leq L(V)$.

Proof. Let $X = \mathbb{R}$, $\mathcal{B} = \{(a,b) : a < b\}$, $L(x,y) = |x-y|$, $x,y \in X$, $L((a,b)) = f(b-a)$, $a < b$, where $f:(0,\infty) \to (0,\infty)$ is any function with $f(t) \geq t$ and $\lim\limits_{t\to 0} \inf f(t) = 0$. Then $(X,T(\mathcal{B}),L(f))$ is a sizable space. For example, if we take $f(t) = \max\{t, 8t-t^2\}$, $t > 0$, then f will generate a size function on $(X,T(\mathcal{B}))$ with $L((0,8)) = 8$ and $L((0,4)) = 16$. Notice that $U = (0,4) \subseteq V = (0,8)$; $U,V \in \mathcal{B}$ and $L(U) \not\leq L(V)$. ▲

6.2 EXAMPLE. *Let* $(X,T(\mathcal{B}),L)$ *be a sizable space. Then* $\mathcal{B}^*$ *need not form a base for any topology on* X.

Proof. Let $X = [0,1]$. Define $\mathcal{B}$ as follows:

$$\mathcal{B} = \{\{0\},\{1\},\{2^{-n}\} : n\in\mathbb{N}\} \cup \{(a,b) : (a,b) \subseteq (2^{-n},2^{-n+1}) : n \in \mathbb{N}\}$$

Then $\mathcal{B}$ is a base for some topology $T(\mathcal{B})$ on X. Define L as follows:

$$L(1,x) = L(x,1) = x(1-x)+2 \text{ if } x \in \bigcup_{n=1}^{\infty} (2^{-2n},2^{-2n+1});$$

$$L(1,x) = L(x,1) = x(1-x) \text{ if } x \in \bigcup_{n=o}^{\infty} [2^{-2n-1},2^{-2n}];$$

$$L(x,y) = |x-y|, \text{ otherwise.}$$

Define also

$$L((a,b)) = b-a, \ (a,b) \in \mathcal{B},$$

$$L(\{0\}) = L(\{1\}) = L(\{2^{-n}\}) = 0, \quad n \in \mathbb{N}.$$

Then L is a size function for $(X,T(\mathcal{B}))$. Let r be any positive real number such that $r < 1/4$. By doing some calculations, one finds that

$$S(1,r) = ((0,\tfrac{1}{2}(1-t)) \cup (\tfrac{1}{2}(1+t),1]) \cap (\bigcup_{n=0}^{\infty} [2^{-2n-2},2^{2-n}]),$$

where $t = (1-4r)^{1/2}$. Let m be an even natural number such that $2^{-m} < r$. Then $S(2^{-m},r') = (2^{-m}-r',2^{-m}+r')$, $r' > 0$. Now if $v \in S(2^{-m},r') \cap (2^{-m},2^{-m+1})$, then $L(v,1) > 2$. Hence $S(2^{-m},r') \not\subseteq S(1,r)$ for all $r' > 0$. Consequently $\mathcal{B}^*$ can not be a base for any topology on X. Therefore $(X,T(\mathcal{B}),L)$ is sizable but not nice. ▲

6.3 EXAMPLE. *Let* $(X,T(\mathcal{B}),L)$ *be a locally nice sizable space. Then* $(X,T(\mathcal{B}^*))$ *is not necessarily a Hausdorff space.*

Proof. Let $X = [0,1]$, $\mathcal{B} = \{\{0\},\{1\}, (a,b) : 0 < a < b < 1\}$. Define $L(x,y) = |x-y|$ if $xy \neq 0$, $L(x,0) = L(0,x) = x(1-x)$ if $x \neq 1$, $L(0,1) = L(1,0) = 1$ for $x,y \in X$, and define $L((a,b)) = b-a$, $0 < a < b < 1$, $L(\{0\}) = L(\{1\}) = 0$. Then $\mathcal{B}$ is a base for some topology on X, and moreover L is a size function for $(X,T(\mathcal{B}))$, i.e., $(X,T(\mathcal{B}),L)$ is a sizable space. By doing some calculation we find that $S(0,r) = [0,\frac{1}{2}(1-t)) \cup (\frac{1}{2}(1+t),1)$, where $t = (1-4r)^{1/2}$ and $0 < r < 1/4$, and $S(1,r) = (1-r,1]$, $0 < r < 1$. Therefore $S(0,r) \cap S(1,r') \neq \emptyset$ for every $r,r' > 0$. Hence $(X,T(\mathcal{B}^*))$ is not Hausdorff. Notice that $(X,T(\mathcal{B}))$ is locally nice and hence nice. ▲

The following result is an immediate consequence of Example 6.3, Theorems 3.7, 4,2, 4,3 and 4.5, and the theorems of Section 5.

6.4 EXAMPLE. (i) *A refinement of a semimetrizable topo-*

logy need not be semimetrizable.

(ii) *A topology weaker then a sizable one need not be sizable.*

(iii) *Let* $(X,T(\mathcal{B}),L)$ *be a locally nice sizable space. Then we have the following:*

a) $(X,T(\mathcal{B}^*))$ *need not be regular.*

b) *The function* $L(-,A):(X,T(\mathcal{B}^*)) \to (\mathbb{R},T_u)$ *need not be continuous for all* A *closed in* $(X,T(\mathcal{B}^*))$.

c) *The set* $\{x \in X : L(x,p) \leqslant r\}$ *is not always closed in* $(X,T(\mathcal{B}^*))$ $(p \in X,\ r > 0)$.

The following example shows that the assumption of locally nice in Theorem 4.3 may not be weakened.

6.5 EXAMPLE. *There exists a nice sizable nonseparable space* $(X,T(\mathcal{B}),L)$ *in which* $\wedge$ *is not a closure operator and* $(X,T(\mathcal{B}^*))$ *is a separable space.*

Proof. Let $X = \mathbb{N} \cup [0,1]$, $\mathcal{B} = \{\{x\} : x \in X\}$. Define L as follows:

$L(x,y) = |x-y|$, if $x,y \in [0,1]$ or $x,y \in \mathbb{N}$;

$L(a,n) = L(n,a) = 1/n$, if a is irrational in $(0,1)$ and $n \geqslant 2$;

$L(b,n) = L(n,b) = 1$, if b is rational in $[0,1]$ and $n \geqslant 2$;

$L(\{x\}) = 0$ of $x \in X$.

Then L is a size function for the discrete space $(X,T(\mathcal{B}))$. If r is a small positive real number, then

$S(b,r) = (b-r,b+r) \cap X$, if b is rational in $[0,1]$;

$S(a,r) = (a-r,a+r) \cup \{n : n \in \mathbb{N}$ and $1/n < r\}$, if a is irrational in $(0,1)$, $0 < r < 1-a$;

$S(n,r) = \{n\}$ if $n \geqslant 2$, $r < 1/n$.

It is clear that $\mathcal{B}^*$ is a base for some topology on X (i.e.

$(X,T(\mathcal{B}),L)$ is a nice space), and it is also clear that $(X,T(\mathcal{B}),L)$ is not locally nice (in particular at irrational points in $(0,1)$). If $A = \mathbb{N}$ then $\hat{A} = \mathbb{N} \cup \{x : x \text{ is irrational in } (0,1)\}$ and $\hat{\hat{A}} = X$. This shows that the operator $\wedge$ is not a closure operator. Since the set of all rationals in X is dense in $(X,T(\mathcal{B}^*))$, then $(X,T(\mathcal{B}^*))$ is a separable space. ▲

We finish with two examples on sizable, semimetrizable and symmetrizable spaces.

6.6 EXAMPLE. *There exists a semimetrizable (symmetrizable) space which is no sizable.*

Proof. Let $(X,T(\mathcal{B}),L)$ be the sizable space constructed in Example 6.3. This space is locally nice. According to Theorem 4.3, the space $(X,T(\mathcal{B}^*))$ is semimetrizable with the semimetric L, and hence it is symmetrizable. Since $(X,T(\mathcal{B}^*))$ is not Hausdorff, it can not be sizable, by Theorem 3.7. ▲

6.7 EXAMPLE. *There exists a symmetrizable Hausdorff space* Y *which is not sizable.*

Proof. This example Y can be found in Davis, Gruenhage and Nyikos ([1], Example 3.2). Since Y contains a point which is not a G_δ-set, it can not be sizable. ▲

* *

REFERENCES

[1] Davis, S.W., Gruenhage, G. and Nyikos, P.J., *G_δ-Sets in symmetrizable and related spaces*. Gen. Top. and its Applications 9 (1978), 235-261.

[2] Fora, A.A., *Metrization of countably compact spaces*, Math. Japon., to appear.

[3] Mrówka, S., *Recent results on* E-*compact spaces*, Proc. Second International Conference in General Topology and its Applications (1972), 298-301.

[4] Steen, L.A. and Seebach, Jr.J.A., *Counterexamples in Topology*, Holt, Rinehart and Winston, Inc., New York, 1970.

[5] Tan, H.P., "N-Compactness and strong zero-dimensionality", PhD. Dissertation, SUNY at Buffalo, 1973.

[6] Willard, S., *General Topology*, Addison-Wesley, 1970.

*

Department of Mathematics
Yarmouk University
Irbid, JORDANIA.

(Recibido en marzo de 1984; versión revisada en Septiembre de 1984).