Revista Colombiana de Matemáticas
Vol. XIX (1985), pág. 9 - 24

REMARKS ON DA COSTA'S PARACONSISTENT SET THEORIES

Ayda Ignez Arruda*

ABSTRACT. In this paper we analyse da Costa's paraconsistent set theories, i.e., the set theories constructed over da Costa's paraconsistent logics $C_n^=$, $1 \leqslant n \leqslant \omega$. The main results presented here are the following. In any da Costa paraconsistent set theory of type **NF** the axiom schema of abstraction must be formulated exactly as in **NF**; for, in the contrary, some paradoxes are derivable that invalidate the theory. In any da Costa paraconsistent set theory with Russell's set $\mathbf{R} = \hat{x}\neg(x \in x)$, UUR is the universal set. In any da Costa paraconsistent set theory the existence of Russell's set is incompatible with a general (for all sets) formulation of the axiom schemata of separation and replacement.

1. INTRODUCTION.

A set theory is *paraconsistent* if it is inconsistent but non-trivial, i.e., at least one contradiction is derived but still there are formulas that are not theorems. Thus, the underlying logic of a paraconsistent set theory must be a *paraconsistent logic*, i.e., a logic in which there is a symbol of negation $\neg$, such that, from a formula A and its negation $\neg A$, it is not possible in general to obtain any formula B whatsoever.

Paraconsistent set theory appeared as an application of paraconsistent logic. The pioneering effort to construct a paraconsistent set theory was made by N.C.A. da Costa, in 1963, in [12], the same work in which he presented his hierarchy of paraconsistent logics (see also [14] and [15]). Further attempts can be found in Arruda and da Costa [6] and [8], Asenjo and Tamburino [9], Brady [10], Brady and Routley [11], and Goodman [17]. Except for da Costa's, and Asen-

* This paper is the text, now considerably expanded, of our lecture in the Fifth Latin American Symposium on Mathematical Logic (Bogotá, July of 1981). We thank IBM of Brazil for the financial help given to attend the meeting. This paper was written when the Author was Visiting Professor at the University of Warsaw (October of 1981), with a partial grant (81/0522-4) of the 'Fundação de Amparo à Pesquisa do Estado de São Paulo' (FAPESP), Brazil.

jo and Tamburino's set theories, the others are proved to be non-trivial. Those paraconsistent set theories already proven to be non-trivial may be called *weak* paraconsistent set theories for, because of their underlying logic, many of the basic results of *classical set theories* (the usual set theories based on classical logic) are not valid in them. The others, supposing that they are nontrivial, may be called *strong* paraconsistent set theories, for all or almost all results of classical set theories are valid in them. Of course, although useful in this paper, the above characterization of strong and weak paraconsistent set theories is not precise.

Let us call a set *non-classical* if it may exist in a paraconsistent set theory but not in a classical set theory, and *contradictory set* to a nonclassical set **X** such that $X \in X$ and $\neg(X \in X)$. Thus, Russell's set, $R = \hat{x}\neg(x \in x)$, is a contradictory set. In the above mentioned attempts to construct a paraconsistent set theory, Russell's set is, in general, introduced through a widening of the scope of validity of the axiom schema of abstraction, but its properties (as well as the properties of other nonclassical sets) have not been studied well. However, these properties may be very important, because they characterize the behavior of the non-classical sets in a paraconsistent set theory.

In this paper we restrict ourselves to da Costa's paraconsistent set theories analysing mainly two problems: first, the widening of the scope of validity of the axiom schemata of abstraction and separation, and second some properties of Russell's set and their consequences. The results presented here may be valid in other paraconsistent set theories, as will be shown in [5].

A *da Costa* paraconsistent set theory is a paraconsistent set theory whose underlying logic is one of da Costa's paraconsistent logics $C_n^=$, $1 \leqslant n \leqslant \omega$.

The postulates of $C_\omega^=$ are those of the positive intuitionistic first-order logic with equality, axiomatized as in Kleene [18], plus:

(1) $\neg\neg A \supset A$,

(2) $A \vee \neg A$.

The postulates of $C_n^=$, $1 \leqslant n \leqslant \omega$, are those of $C_\omega^=$, plus:

(3) $B^{(n)} \& (A \supset B) \& (A \supset \neg B) \supset \neg A$,

(4) $A^{(n)} \& B^{(n)} \supset (A \supset B)^{(n)} \& (A \& B)^{(n)} \& (A \vee B)^{(n)}$,

(5) $(x)(A(x))^{(n)} \supset ((x)A(x))^{(n)} \& ((Ex)A(x))^{(n)}$,

where $A^{(n)}$ is defined as follows: $A^1 = A^0 = \neg(A \& \neg A)$, $A^{n+1} = (A^n)^0$, $A^{(n)} = A^1 \& A^2 \& A^3 \& \ldots \& A^n$.

In each $C_n^=$, $1 \leqslant n \leqslant \omega$, $\neg^* A$ is defined as $\neg A \& A^{(n)}$, and it is proved that $\neg^*$ satisfies all the properties of the classical negation. Then classical logic can be obtained inside these systems; consequently, they are finitely trivializable. For, from any formula of the form $A \& \neg A \& A^{(n)}$ we can deduce any formula whatsoever. Nonetheless, $C_\omega^=$ is not finitely trivializable. Moreover, each systems in the hierarchy $C_1^=$, $C_2^=$, ..., $C_n^=$,... $C_\omega^=$ is strictly stronger than the following ones. Thus, we may construct a hierarchy of da Costa's paraconsistent set theories in which, at least intuitively, it seems that each system may admit more nonclassical sets than the preceding ones.

For a long time, in fact since 1964, we have been studying from time to time da Costa's paraconsistent set theories of type **NF** and $\mathbf{NF}_n$, $1 \leqslant n \leqslant \omega$. We have proved that da Costa's formulation of the axiom schema of abstraction for the systems $\mathbf{NF}_n$, $1 \leqslant n < \omega$, leadsto the trivialization of the systems (see [1], [3] and [4]). Thus, we have proposed to formulate the axiom schema of abstraction in these systems exactly as in **NF**.

The main objetive of this paper is to present some results (in fact some paradoxes) that we believe are very important in the construction of strong paraconsistent set theories, particulary, da Costa's paraconsistent set theories. In Section 2, we prove that tha Costa's formulation of the axiom schema of abstraction for $\mathbf{NF}_\omega$ leads to the *paradox of identity*, $(x,y).x = y$, and conclude with some argumentation showing that this axiom in any da Costa paraconsistent set theory of type **NF** should be formulated exactly as in **NF**. In Section 3, we summarize the syntactical development or our version of $\mathbf{NF}_\omega$ showing that this system may be considered as a strong paraconsistent set theory. In Section 4, we prove that if Russell's set is introduced in any da Costa set theory based on $C_n^=$, $1 \leqslant n < \omega$, then UUR is the universal set. In Section 5, we analyse some limitation in the construction of da Costa's set theories of type **ZF** where Russell's class is a set. Finally, Section 6 is a conclusion where we call attention to some open problems concerning paraconsistent set theory.

2. ON DA COSTA'S SET THEORIES OF TYPE NF.

In this section we analyse da Costa's formulation of the axiom schema of abstraction for the set theories $\mathbf{NF}_n$, $1 \leqslant n \leqslant \omega$. In [1] and [4], we proved that da Costa's formulation of the axiom schema of abstracion for his systems of type **NF** and $\mathbf{NF}_n^C$, $1 \leqslant n < \omega$, leads to their trivialization. Here we prove that da Costa's formulation of the axiom schema of abstraction for $\mathbf{NF}_\omega^C$ leads to the paradox of identity.

The systems $\mathbf{NF}_n^C$ are constructed, similarly to **NF**, over the respective calculus of descriptions $\mathbf{D}_n$. The calculi of descriptions $\mathbf{D}_n$, $1 \leqslant n \leqslant \omega$, are constructed as usual (see Rosser [20]) from the respective $C_n^=$.

The specific postulates of $\mathbf{NF}_\omega^C$ are the following:

EXTENSIONALITY. $(\alpha,\beta):.(x).x\in\alpha \equiv x\in\beta:\supset:\alpha=\beta$.

ABSTRACTION. $(E\alpha)(x):x\in\alpha\cdot\equiv\cdot F(x)$, *where* x *and* α *are different variables*

α does not occur free in F(x), F(x) *is stratified or it does not contain any subformula of the form* $A \supset B$.

In $\mathbf{NF}_\omega^C$, the restrictions regarding the use of non-stratified formulas obstruct a direct proof of the paradox of Curry. Russell's set **R**, defined as $\hat{x}\neg(x \in x)$, exists as well as many other non-classical sets. The paradox of Russell in the form $R \in R \& \neg(R \in R)$ is derivable but, apparently, it causes no harm to the system.

Due to its weakness, the primitive negation of $\mathbf{NF}_\omega^C$, $\neg$, is almost useless for set-theoretical purposes. Thus, let us define

$$\sim A \quad \text{for} \quad A \supset (x,y).x \in y \,\&\, x = y.$$

The universal set **V** is defined as $\hat{x}.x = x$, the empty set $\mathbf{\Lambda}$ as $\hat{x} \sim (x = x)$, and the complement of a set α, $\bar{\alpha}$, as $\hat{x} \sim (x \in \alpha)$. From now on the set theoretical notations and terminology in this and the next sections are mainly those of Rosser [20].

TEOREM 2.1. *In* $\mathbf{NF}_\omega^C$, $\sim$ *is a minimal intuitionistic negation.*

Proof. It is enough to prove $(A \supset B) \& (A \supset \sim B) \cdot\!\supset\!\cdot \sim A$. Let us suppose that $A \supset B$ and $A \supset \sim B$. Then, we obtain $A \supset (x,y).x \in y \,\&\, x = y$, i.e., $\sim A$.

COROLLARY 1. $\vdash A \supset (\sim A \supset \sim B)$.

$\vdash (A \supset B) \supset (\sim B \supset \sim A)$.

COROLLARY 2. *All the theorems of* **NF** *whose proofs depend only on the laws of the minimal intuitionistic first-order logic with equality and on the postulates of extensionality and abstraction of* **NF** *are valid in* $\mathbf{NF}_\omega^C$.

THEOREM 2.2. (Cantor's Theorem) $\vdash (\alpha) \cdot \sim (\alpha \text{ sm } SC(\alpha))$.

Proof. Let us suppose that α sm SC(α). Then, there exists a relation **S** such that $\mathbf{S} \in 1\text{-}1$, $\text{Arg}(\mathbf{S}) = \alpha$ and $\text{Val}(\mathbf{S}) = SC(\alpha)$. Now, let us take

$$\mathbf{X} = \hat{x}(x \in \alpha \cap \overline{S(x)}). \tag{1}$$

As the formula $x \in \alpha \cap \overline{S(x)}$ is non-stratified and does not contain any sub-formula of the form $A \supset B$, then **X** is a set. Moreover, as $\mathbf{X} \subseteq \alpha$, then there exists $y \in \alpha$ such that $S(y) = \mathbf{X}$. Then, by (1), we have:

$$y \in \mathbf{X} \cdot\!\equiv\!\cdot y \in \alpha \cap \overline{S(y)}.$$

However, $y \in \alpha \cap \overline{S(y)} \cdot\!\equiv\!\cdot y \in \alpha \,\&\, \sim (y \in \mathbf{S}(y))$. Then,

$$y \in \mathbf{X} \cdot\!\equiv\!\cdot y \in \alpha \,\&\, \sim (y \in \mathbf{X}). \tag{2}$$

From (2) we have $\sim(y \in \alpha)$, but, as $y \in \alpha$, then we obtain

$$(\alpha \text{ sm } SC(\alpha)) \supset y \in \alpha, \quad \text{and} \quad (\alpha \text{ sm } SC(\alpha)) \supset \sim(y \in \alpha).$$

Thus, by Theorem 2.1, the desired result follows.

COROLLARY. (Cantor's Paradox) $\vdash(\mathbf{V} \text{ sm } SC(\mathbf{V}))\,\&\sim(\mathbf{V} \text{ sm } SC(\mathbf{V}))$.

Proof. As $\mathbf{V} = SC(\mathbf{V})$, then we obtain $\mathbf{V}$ sm $SC(\mathbf{V})$. On the other hand, by the theorem, we have $\sim(\mathbf{V} \text{ sm } SC(\mathbf{V}))$. □

Apparently, Cantor's paradox does not trivialize $\mathbf{NF}_\omega^C$. For, from A and $\neg A$ we cannot obtain any formula B whatsoever. For instance, apparently, we cannot obtain any formula of the form $\neg B$, where B is a nonatomic formula.

THEOREM 2.3. I. $\vdash(\alpha,\beta).\alpha = \beta\,\&\sim(\alpha = \beta)$

II. $\vdash(\alpha,\beta).\alpha \in \beta\,\&\sim(\alpha \in \beta)$

III. $\vdash(\alpha).\alpha \in \alpha\,\&\sim(\alpha \in \alpha)$.

Proof. I. By the corollaries of theorems 2.1 and 2.2, we obtain $x = x \supset (\alpha,\beta).\alpha \in \beta\,\&\,\alpha = \beta$. Thus, as $x = x$, then $(\alpha,\beta).\alpha = \beta$. By the same corollaries we also obtain $(\alpha,\beta).\sim(\alpha = \beta)$. The proof of part II is similar to that of part I. Part III is an immediate consequence of part II. □

By Theorem 2.3, it could seem that $\mathbf{NF}_\omega^C$ is trivial. Nonetheless, apparently this is not the case. However, though it is nontrivial, $\mathbf{NF}_\omega^C$ is without interest, for not only are every two sets identical, but also every set belongs and "does not belong" to itself.

In order to avoid the results mentioned in Theorem 2.3, one could think of introducing more restrictions in da Costa's formulation of the axiom schema of abstraction when $\mathbf{F}(x)$ is non-stratified. Nonetheless, we belive that this is a worthless effort. For:

(i) The only non-stratified formula used in the proof of Cantor's Theorem (which is fundamental in the proof of Theorem 2.3) is a non-stratified formula of the form $\alpha \in \beta$. Then, the new restrictions must avoid those nonstratified atomic formulas of the form $\alpha \in \beta$ which determine a set.

(ii) A new proof of Theorem 2.3 may be obtained in the following way: in $\mathbf{NF}$ the formula $y = \{x\}$ cannot determine a relation because $\langle x,y\rangle = \langle x,y\rangle\,\&\,y = \{x\}$ is non-stratified. But, such a formula does not contain any subformula of the form $A \supset B$; then, in $\mathbf{NF}_\omega^C$ it determines a relation $\mathbf{S}$ such that $\mathbf{S} \in 1\text{-}1$. With such a relation we prove that $(\alpha).\alpha$ sm $USC(\alpha)$. In $\mathbf{NF}_\omega^C$ we also prove that $(\alpha).\sim(USC(\alpha) \text{ sm } SC(\alpha))$. Then, these new restrictions must also avoid that those non-stratified formulas whose atomic subformulas are of the form $\alpha = \beta$ determine a set.

From the above remarks we conclude that, in order to avoid the counterintuitive results mentioned in Theorem 2.3, the axiom schema of abstraction in $\mathbf{NF}_\omega$

should be formulated as in **NF**.

Due to the paradoxes obtained in $\mathbf{NF}^C_\omega$, $1 \leqslant n \leqslant \omega$, we conclude that in these systems the axiom schema of abstraction should be formulated as in **NF**. Thus, if we want these theories to be paraconsistent set theories, we need to postulate directly the existence of contradictory sets. Apparently, we may postulate the existence of Russell's set without any problem. Nonetheless, due to the two above considerations about the non-stratified formulas that lead to the proof of the paradox of identity, we believe that, besides Russell's set, very few other non-classical sets may exist in $\mathbf{NF}_n$, $1 \leqslant n \leqslant \omega$.

3. THE CLASSICAL PART OF NF_ω.

In this section we summarize the development of our version of $\mathbf{NF}_\omega$, in order to show that it can be considered a strong paraconsistent set theory, i.e., almost all the results of **NF** can be obtained in $\mathbf{NF}_\omega$. In our version of $\mathbf{NF}_\omega$, the axiom schema of abstraction is formulated as in **NF**, an axiom for the complement of a set is introduced, and no postulate is considered concerning the existence of non-classical sets.

The specific postulates of $\mathbf{NF}_\omega$ are extensionality and abstraction (both formulated as in **NF**), and the following one for the complement:

$$(\alpha,x).x \in \alpha \vee x \in \bar{\alpha}.$$

This axiom is fundamental if we want to preserve in $\mathbf{NF}_\omega$ at least the same properties of the algebra of classes of **NF**. Moreover, if we want to prove many other results of **NF** for instance, some of those in whose proof in **NF** it is necessary to use the principle of excluded middle or the law of double negation both for non-atomic formulas, this axiom is needed.

Observe that the universal set, the empty set, and the complement of a set are defined without using negation as follows, and the these definitions in **NF** are equivalent to the usual ones.

$\mathbf{\Lambda}$ for $\hat{x}(\alpha).x \in \alpha$

$\mathbf{V}$ for $\hat{x}(E\alpha).x \in \alpha$

$\bar{\alpha}$ for $\hat{x}(E\beta).x \in \beta \,\&\, \alpha \cup \beta = \mathbf{V} \,\&\, \alpha \cap \beta = \mathbf{\Lambda}$.

To express that α and β are different (or distinguishable) sets we use the symbol $\neq$ defined as follows:

$$\alpha \neq \beta \text{ for } (Ex).x \in \alpha \cap \bar{\beta} \vee x \in \bar{\alpha} \cap \beta.$$

LEMMA 3.1. I. $\vdash (Ex).x \in \mathbf{\Lambda} : \supset : (y).y \in \mathbf{\Lambda}$.

II. $\vdash (Ex).x \in \mathbf{\Lambda} : \supset : (y,z).y \in z \,\&\, y = z$.

LEMMA 3.2. I. $\vdash (\alpha,\beta).\alpha = \beta \vee \alpha \neq \beta$.

II. $\vdash (\alpha,\beta).\alpha = \beta \& \alpha \neq \beta : \supset : (Ex).x \in \Lambda$.

DEFINITION. $\sim A$ for $A \supset (Ex).x \in \Lambda$.

THEOREM 3.1. *In* $\mathbf{NF}_\omega$, $\sim$ *is a minimal intuitionistic negation.*

THEOREM 3.2. *For atomic formulas of* $\mathbf{NF}_\omega$, $\sim$ *is a classical negation.*

Proof. Let P and Q be variables for atomic formulas. Thus, by theorem 2.1, it suffices to prove that (i) $P \supset (\sim P \supset Q)$ and (ii) $P \vee \sim P$.

(i) From P and $\sim P$ we obtain $(Ex).x \in \Lambda$. Thus, from part II of Lemma 3.1, we obtain any atomic formula whatsoever.

(ii) P is of the form $\alpha \in \beta$. Supposing that $\alpha \in \beta$, we obtain $\alpha \in \beta \vee \sim(\alpha \in \beta)$. On the other hand, supposing that $\alpha \in \bar{\beta}$, we obtain $\alpha \in \beta \supset (Ex).x \in \Lambda$, i.e., $\sim(\alpha \in \beta)$; consequently, $\alpha \in \beta \vee \sim(\alpha \in \beta)$. Finally, using the axiom for the complement, we obtain the desired result.

P is of the form $\alpha = \beta$. Supposing that $\alpha = \beta$ we obtain $\alpha = \beta \vee \sim(\alpha = \beta)$. On the other hand, supposing that $\alpha \neq \beta$, then there exists an x such that $x \in \bar{\alpha} \cap \beta$ or $x \in \alpha \cap \bar{\beta}$. Taking the additional supposition that $\alpha = \beta$, in both cases we obtain $(Ex).x \in \Lambda$. Thus, from $\alpha \neq \beta$ we obtain $\alpha = \beta \supset (Ex).x \in \Lambda$, i.e., $\sim(\alpha = \beta)$, and so, by part I of Lemma 3.2, we have $\alpha = \beta \vee \sim(\alpha = \beta)$.

COROLLARY. I. $\vdash (x) : x \in \Lambda \cdot\equiv\cdot \sim(x = x)$.

II. $\vdash (x) : x \in V \cdot\equiv\cdot x = x$.

III. $\vdash (x) : x \in \bar{\alpha} \cdot\equiv\cdot \sim(x \in \alpha)$.

IV. $\vdash (x) : x \in \bar{\bar{\alpha}} \cdot\equiv\cdot \sim\sim(x \in \alpha)$.

V. $\vdash (\alpha,\beta) : \alpha \neq \beta \cdot\equiv\cdot \sim(\alpha = \beta)$.

THEOREM 3.3. *For positive formulas of* $\mathbf{NF}_\omega$ *(i.e., formulas in which no subformula is of the form* $\neg A$*)* $\sim$ *is an intuitionistic negation.*

Proof. Due to Theorem 3.1, we only have to prove that, if A^+ and B^+ denote positive formulas of $\mathbf{NF}_\omega$, then $A^+ \supset (\sim A^+ \supset B^+)$. In fact, let us suppose A^+ and $\sim A^+$; thus we obtain $(Ex).x \in \Lambda$. Then, using Lemma 3.1, part II, we have $(\alpha,\beta).\alpha = \beta \& \alpha \in \beta$. Now, by induction on the length of B^+ we conclude the proof. □

In the rest of this section, we follow Rosser [20], chapters IX to XIII. As almost all results of chapters IX and X that are valid in **NF** are proved in [19], the summary of these chapters is very short. The other chapters will be summarized section by section. From now on when we say that a theorem (or an exercise) of **NF** is valid in $\mathbf{NF}_\omega$ we are saying that the theorem (or exercise) is valid with the same formulation as in [20]; of course, negation is understood as the defined negation of $\mathbf{NF}_\omega$. When we say that the proofs are similar to Rosser's proofs, we are saying, in fact that either the proofs are exactly the same given in [20] or that small changes in Rosser's proofs are made in order to use the axiom for the complement and the above lemmas and theorems.

(CHAPTER IX) CLASS MEMBERSHIP. Except for part II of Theorem IX.4.11 and its corollaries, all the other theorems are valid in $\mathbf{NF}_\omega$, and the proofs are similar to Rosser's.

Let A^+ denote a positive formula of $\mathbf{NF}_\omega$, then part II of Theorem IX. 4.11 and its corollaries are proved in $\mathbf{NF}_\omega$ with the following formulations respectively:

$\vdash (x). \sim A^+ :\equiv: (x).x \in \Lambda \equiv A^+$

$\vdash (x). \sim A^+ :\supset: \exists \hat{x} A^+.$

$\vdash (x). \sim A^+ :\supset: \Lambda = \hat{x} A^+.$

(CHAPTER X) RELATION AND FUNCTION. All the theorems of this section are valid in $\mathbf{NF}_\omega$ and the proofs are similar to Rosser's proofs.

(CHAPTER XI) CARDINAL NUMBERS.

1. Cardinal similarity. All the theorems of this section are valid in $\mathbf{NF}_\omega$, and the proofs are similar to Rosser's proofs.

2. Elementary properties of cardinal numbers. All the theorems of this section are valid in $\mathbf{NF}_\omega$; concerning the exercises, only XI.2.12 apparently is not valid. This exercise guarantees in **NF** that if **S** is a relation not belonging to 1-1, then there exist x and y such that $x,y \in \mathrm{Arg}(\mathbf{S})$, $\mathbf{S}(x) = \mathbf{S}(y)$ and $x \neq y$. We call attention to the apparent nonvalidity of this exercise because it is used by Rosser to prove the 'Pigeonhole Principle'.

3. Finite classes and mathematical induction. Let us discuss first the theorems about mathematical induction. Weak Induction (Theorem XI.3.18) is proved in $\mathbf{NF}_\omega$ as in Rosser; Strong Induction is valid as in Theorem XI.3.19 and its Corollary, but apparently it is not valid in the form of Theorem XI.3.20. Nonetheless, a restricted form of Theorem XI.3.20 is valid in $\mathbf{NF}_\omega$; to wit, when the formula $\mathbf{F}(x)$ appearing in it is a positive formula. Apparently, Theorem XI.3.22 (Principle of Infinite Descent) is not valid in $\mathbf{NF}_\omega$. Rosser's proof does not obtain because he uses the principle of double negation ($\sim\sim A \equiv A$) for non-atomic formulas and we could not find another way to prove this theorem.

All the other theorems of this section are valid in $\mathbf{NF}_\omega$ and, except for the proof of Theorem XI.3.21; the proofs are similar to Rosser's ones. To prove Theorem XI.3.21 (every non-empty subset α of **Nn** has a minimum), it is enough to prove the lemma used in Rosser's proof. The proof of this lemma runs as follows:

Case 1. Supposing that $n+1 \leqslant m$, we obtain the desired result.

Case 2. Supposing that $m < n+1$, by the axiom for the complement, we obtain $m < n+1 \,\&\, (m \in \alpha \vee m \in \bar{\alpha})$, and from this formula the desired result follows.

Now, the lemma follows from cases 1 and 2, because in $\mathbf{NF}_\omega$ the following formula is provable: $(m,n).m \leqslant n \vee n < m$.

4. Denumerable classes. All the theorems of this section are valid in $\mathbf{NF}_\omega$ and the proofs are similar to Rosser's proofs, except for the proof of Theorem XI.4.4 ($(\alpha).\alpha \subseteq \mathbf{Nn} \supset \alpha \in \mathrm{Count}$). This proof in **NF** runs as follows:

Case 1. If $\alpha \subseteq \mathbf{Nn}$ and $\alpha \in \mathrm{Fin}$, then, obviously, $\alpha \in \mathrm{Count}$.

Case 2. Let us suppose that $\alpha \subseteq \mathbf{Nn}$ and $\alpha \in \mathrm{Infin}$, then it is easily proved that

$$(n): n \in \mathbf{Nn} \;.\supset.\; \hat{z}(z \in \alpha \,\&\, z > n) \neq \Lambda. \tag{1}$$

Consequently, we obtain

$$(n): n \in \mathbf{Nn} \;.\supset.\; (Ey).y \in \alpha \,\&\, y > n. \tag{2}$$

Now, by (2) and the corollary of Theorem XI.4.3, we obtain $\alpha \in \mathrm{Den}$, consequently $\alpha \in \mathrm{Count}$.

Finally, as it is easy to prove that $(\alpha).\alpha \in \mathrm{Fin} \vee \alpha \in \mathrm{Infin}$, the desired result follows from cases 1 and 2.

5. The cardinal number of the continuum. All the theorems of this section are valid in $\mathbf{NF}_\omega$, and the proofs are similar to Rosser's, except for the proof of Theorems XI.5.5 ($c = 2^{\mathrm{Den}}$). The proof of this theorem runs as follows. Let α be defined as in Rosser's proof.

LEMMA 1. $\vdash \alpha\ \mathrm{sm}(\mathrm{SC}(\mathbf{Nn}) \cap \mathrm{Fin})$.

Proof. Like in Rosser's Lemma 3.

LEMMA 2. $\vdash \mathrm{NTBX}\ \mathrm{sm}(\mathrm{SC}(\mathbf{Nn}) \cap \mathrm{Infin})$.

Proof. Similar to the proof of Lemma 1, but taking

$$\mathbf{W} = \hat{S}\hat{\beta}(\mathbf{S} \in \mathrm{NTBX} \,\&\, \beta = \hat{m}(m \in \mathbf{Nn} \,\&\, \mathbf{S}(m) = 1)).$$

LEMMA 3. $\vdash \alpha \cap \mathrm{NTBX} = \Lambda$.

LEMMA 4. $\vdash (\alpha \cup \mathrm{NTBX})\,\mathrm{sm}(\mathrm{PI} \upharpoonright \{0,1\})$.

The rest of the proof is similar to the rest of Rosser's proof.

(CHAPTER XII) ORDINAL NUMBERS.

1. Ordinal similarity. All the theorems of this section are valid in $\mathbf{NF}_\omega$, and the proofs are similar to Rosser's proofs.

2. Well-ordering relations. Except for Theorems XII.2.10 to XII.2.13 (all about definitions and proofs by transfinite induction), all the others are valid in $\mathbf{NF}_\omega$. Rosser's proofs of Theorems XII.2.10 to XII.2.13 do not obtain in $\mathbf{NF}_\omega$, and we could not find another way to prove them. It is worthwhile to mention that the proof of Theorem XII.2.14 (two well ordered sets either are similar or one is shorter than the other) is obtained without using transfinite induction (as mentioned in [20], p.462).

3. Elementary properties of ordinal numbers. All the theorems of this section are valid in $\mathbf{NF}_\omega$. The proof of theorem XII.3.4 ($\leqslant_0 \in$ Word) is a little different from Rosser's proof. To wit: suppose that $\beta \cap \mathbf{NO} \neq \Lambda$, then there exists ϕ such that $\phi \in \beta \cap \mathbf{NO}$. Case 1: $\beta \cap \mathbf{NO} = \{\phi\}$. Then there exists a minimal element in β. Case 2: $\beta \cap \mathbf{NO} \neq \{\phi\}$. Then, as in Rosser's proof, there exists a minimal element in β. Now, using Lemma 3.2, part I, and the fact that $\leqslant_0 \in$ Sord, we conclude the proof.

4. The cardinal number associated to an ordinal number. All the theorems of this section are valid in $\mathbf{NF}_\omega$, and the proofs are similar to Rosser's proofs.

(CHAPTHER XIII) COUNTING. The additional results about natural numbers given in Section 1 are valid in $\mathbf{NF}_\omega$ and, adding the *axiom of counting*, we also proved the Theorems of Section 2. Nonetheless, the main result of this chapter, the *pigeonhole principle*, apparently is not valid in $\mathbf{NF}_\omega$. Rosser's proof does not obtain because, as mentioned above, Exercise XI.2.12 apparently is not valid in $\mathbf{NF}_\omega$.

4. RUSSELL'S SET IN DA COSTA SET THEORIES.

In this section we show that in any da Costa set theory based on $C_n^=$, $1 \leqslant n < \omega$, UUR is the universal set; the same holds in a da Costa set theory based on $C_\omega^=$ when stregthened with some additional suppositions.

Let us denote by $\mathbf{DC}_n$ any da Costa set theory based on the respective $C_n^=$, where Russell's class is a set. Thus, in $\mathbf{DC}_n$, $1 \leqslant n < \omega$, the defined negation $\neg^*$ ($\neg^* A =_{df} \neg A \,\&\, A^{(n)}$) is a classical negation; and in $\mathbf{DC}_\omega$ the defined negation $\sim$ ($\sim A =_{df} A \supset (x,y).x \in y \,\&\, x = y$) is a minimal intuitionistic negation.

Let us denote by ϕ the empty set, defined in $\mathbf{DC}_n$, $1 \leqslant n < \omega$, as $\hat{x}\neg^*(x = x)$, and in $\mathbf{DC}_\omega$ as $\hat{x} \sim (x = x)$. Thus, in $\mathbf{DC}_n$, $1 \leqslant n < \omega$, we prove that $(x)\neg^*(x \in \phi)$, and in $\mathbf{DC}_\omega$ we prove that $(x).\sim(x \in \phi)$.

LEMMA 4.1. $\vdash \phi \in \mathbf{R}$.

Proof. In $\mathbf{DC}_n$, $1 \leqslant n < \omega$, if $\phi \in \phi$ then $\phi \in \phi \,\&\, \neg^*(\phi \in \phi)$. As this formula trivializes the system, then $\neg(\phi \in \phi)$. Consequently, $\phi \in \mathbf{R}$.

In $\mathbf{DC}_\omega$, if $\phi \in \phi$ then $\phi \in \phi \,\&\, \sim(\phi \in \phi)$; and so $(x,y).x \in y$. Thus, $\phi \in \mathbf{R}$. On the other hand, if $\neg(\phi \in \phi)$ then $\phi \in \mathbf{R}$.

LEMMA 4.2. $\vdash x \in \mathbf{R} \supset \{x\} \in \mathbf{R}$.

Proof. If $\neg(\{x\} \in \{x\})$, then $\{x\} \in \mathbf{R}$. On the other hand, if $\{x\} \in \{x\}$ then $\{x\} = x$; thus, by the hypothesis $x \in \mathbf{R}$, we obtain $\{x\} \in \mathbf{R}$.

LEMMA 4.3. I. $\vdash \mathbf{R} \subseteq \mathbf{UR}$
II. $\vdash \mathbf{R} \subseteq \mathbf{UUR}$.

Proof. I. If $x \in \mathbf{R}$ then, by Lemma 4.2, $\{x\} \in \mathbf{R}$. Now, as $x \in \{x\}$, then $x \in \mathbf{UR}$.
II. By part I, we have $\mathbf{R} \subseteq \mathbf{UR}$ and $\mathbf{UR} \subseteq \mathbf{UUR}$. Consequently, $\mathbf{R} \subseteq \mathbf{UUR}$.

LEMMA 4.4. I. $\vdash (x).\neg^*(x \in \mathbf{R}) \supset x \in \mathbf{UUR}$, *in* $\mathbf{DC}_n$ $(1 \leqslant n < \omega)$.
II. $\vdash (x).\sim(x \in \mathbf{R}) \supset x \in \mathbf{UUR}$, *in* $\mathbf{DC}_\omega$.

Proof. I. Let us suppose that $\neg^*(x \in \mathbf{R})$. Thus, if $x = \phi$, we obtain $\neg^*(\phi \in \mathbf{R})$, and, using Lemma 4.1, we have $\phi \in \mathbf{R} \,\&\, \neg^*(\phi \in \mathbf{R})$. Consequenlty

$$\neg^*(x = \phi). \tag{1}$$

Let us suppose that $\{\{x,\phi\}\} \in \{\{x,\phi\}\}$. Then, we obtain $\{\{x,\phi\}\} = \{x,\phi\}$. Consequently, $x = \phi$. Now, by (1), we have a contradiction that trivializes the system. Thus, $\neg(\{\{x,\phi\}\} \in \{\{x,\phi\}\})$, and so

$$\{\{x,\phi\}\} \in \mathbf{R}. \tag{2}$$

But, $\{x,\phi\} \in \{\{x,\phi\}\}$. Then, by (2), we have

$$\{x,\phi\} \in \mathbf{UR}. \tag{3}$$

However, $x \in \{x,\phi\}$. Then, by (3), we obtain $x \in \mathbf{UUR}$.

II. Let us suppose that $\sim(x \in \mathbf{R})$. Thus, if $x = \emptyset$, by Lemma 4.1, we obtain $\sim(\phi \in \mathbf{R})$ and $\phi \in \mathbf{R}$. Consequently, $(x.y).x \in y \,\&\, x = y$. Thus, $x = \phi \supset (x,y).x \in y \,\&\, x = y$. Then,

$$\sim(x = \phi). \tag{1}$$

Supposing that $\{\{x,\phi\}\} \in \{\{x,\phi\}\}$, as in part I, we obtain $x = \phi$. Thus, by (1), we have $(x,y).x \in y \,\&\, x = y$. Consequently, $(x,y).x \in y$, and so $\{\{x,\phi\}\} \in \mathbf{R}$. On the other hand, supposing that $\neg(\{\{x,\phi\}\} \in \{\{x,\phi\}\})$, we obtain $\{\{x,\phi\}\} \in \mathbf{R}$. consequently, $\{\{x,\phi\}\} \in \mathbf{R}$.

The rest of the proof follows as in part I.

THEOREM 4.1. *In* $\mathbf{DC}_n$ $(1 \leqslant n < \omega)$, $\mathbf{UUR}$ *is the universal set.*

Proof. It follows from Lemmas 4.3 and 4.4. □

The proof of Theorem 4.1 does not obtain in any $\mathbf{DC}_\omega$, but it obtain in any $\mathbf{DC}_\omega^u$, i.e., any $\mathbf{DC}_\omega$ with universal set $\mathbf{V}$, defined as $\hat{x}(x = x)$. To have a proof of Theorem 4.1 in any $\mathbf{DC}_\omega^u$ it is necessary to say what it means for a set to be different or distinguishable from the universal set. Thus, let us define

$$x \neq \mathbf{V} \text{ for } (Ey).\sim(y \in x).$$

Moereover, if there exists a universal set it is obvious that every set must be equal to or different from the universal set. If this is not a theorem, it must be introduced as postulate:

P1. $(x).x = \mathbf{V} \vee x \neq \mathbf{V}$.

THEOREM 4.2. *In* $\mathbf{DC}^{u}_{\omega}$ *plus* **P1**, *we prove that* $\mathbf{UUR} = \mathbf{V}$.

Proof. By P1 we have $\mathbf{UUR} = \mathbf{V}$ or $\mathbf{UUR} \neq \mathbf{V}$. If $\mathbf{UUR} = \mathbf{V}$, we have already the desired result. If $\mathbf{UUR} \neq \mathbf{V}$ then, by the above definition, we have $(Ey).\sim(y \in \mathbf{UUR})$. Thus, by Lemma 4.3, part II, it follows that $\sim(y \in \mathbf{R})$, and by Lemma 4.4, part II, $y \in \mathbf{UUR}$. Consequently, $(x,y).x \in y \,\&\, x = y$, and so $(x).x \in \mathbf{UUR}$. Thus, $\mathbf{UUR} = \mathbf{V}$.

REMARK. In the sistema $\mathbf{NF}_n$ $(1 \leqslant n \leqslant \omega)$ we prove that $\mathbf{UUR} = \mathbf{V}$. For $n < \omega$, the proof is the same as in Theorem 4.1; for $n = \omega$, the proof is the same as in Theorem 4.2, since **P1** is a theorem of $\mathbf{NF}_\omega$.

We have introduced some conditions in order to prove that **UUR** is the universal set in $\mathbf{DC}^{u}_{\omega}$. Thus, it could seem possible to construct a $\mathbf{DC}_\omega$ without universal set. In the next section we prove in such a system the paradox of identity is derivable.

5. ON DA COSTA'S SET THEORIES OF TYPE ZF.

In sections 2 and 3 we have analysed da Costa's set theories with universal set, constructed according to the pattern of **NF**. Now we analyse the possibility of constructing da Costa's set theories following the pattern of classical set theory without universal set. We choose to analyse da Costa's set theories of type **ZF**, denoted by $\mathbf{ZF}_n$, $1 \leqslant n \leqslant \omega$.

Firstly, we show that if **R** is a set in $\mathbf{ZF}_n$, $1 \leqslant n \leqslant \omega$, then the supposition of non-existence of a universal set leads to some paradoxes that invalidate these theories. Such a result may already be intuitively inferred from the results presented in Section 4. Secondly, we show that the axiom schema of separation, formulated for all sets, is incompatible with the existence of Russell's set. Consequenlty, the axiom schema of replacement is also incompatible with the existence of Russell's set.

Le us consider the set theories $\mathbf{ZF}_n$, $1 \leqslant n \leqslant \omega$, in which the axioms of pairing and union are postulated in general, and in which we also postulate the existence of the empty set and of Russell's set. Moreover, let us suppose that there is no universal set, i.e.,

Sn. $(x)(Ey).\neg^{*}(y \in x)$, in $\mathbf{ZF}_n$, $1 \leqslant n < \omega$;

Sω. $(x)(Ey).\sim(y \in x)$, in $\mathbf{ZF}_\omega$

Let us observe that the lemmas of the preceding section are provable in $\mathbf{ZF}_n$, $1 \leqslant n \leqslant \omega$.

THEOREM 5.1. *The set theories* $\mathbf{ZF}_n$ $(1 \leqslant n < \omega)$ *plus* **Sn** *are trivial.*

Proof. By Sn there exists y such that $\neg^*(y \in \mathrm{UU}\mathbf{R})$. By part II of Lemma 4.3, and part I of Lemma 4.4, we obtain $(x).x \subset \mathrm{UU}\mathbf{R}$. Consequently, $y \in \mathrm{UU}\mathbf{R} \& \neg^*(y \in \mathrm{UU}\mathbf{R})$, and this formula trivializes the system.

THEOREM 5.2. *The paradox of identity is derivable in* $\mathbf{ZF}_\omega$ *plus* $\mathbf{S}_\omega$.

Proof. By Sn, there exists a y such that $\sim(y \in \mathrm{UU}\mathbf{R})$. Using part II of Lemma 4.3, we obtain $y \in \mathrm{UU}\mathbf{R}$. Consequently, by the definition of $\sim$, $(x,y).x \in y \& x = y$. Thus, the paradox of identity, $(x,y).x = y$, follows. Moreover, we also obtain the other results mentioned in Theorem 2.3.

THEOREM 5.3. *The systems* $\mathbf{ZF}_n$ $(1 \leqslant n < \omega)$ *with Russell's set and the axiom schema of separation postulate for all sets are trivial.*

Proof. If the axiom schema of separation is postulated for all sets then there exists a subset α of **R** such that

$$(x) : x \in \alpha \;\cdot\!\equiv\!\cdot\; x \in R \,\&\, (x \in x)^{(n)}. \tag{1}$$

From (1) we obtain

$$\alpha \in \alpha \;\cdot\!\equiv\!\cdot\; \neg(\alpha \in \alpha) \,\&\, (\alpha \in \alpha)^{(n)}. \tag{2}$$

Consequently, we have $\alpha \in \alpha \,\&\, \neg^*(\alpha \in \alpha)$, and this formula trivializes the system.

THEOREM 5.4. *In* $\mathbf{ZF}_\omega$ *with Russell's set and the axiom schema of separation postulated for all sets, the paradox of identity is derivable.*

Proof. From the axiom schema of separation and Russell's set we obtain $(x) : x \in \alpha \;\cdot\!\equiv\!\cdot\; x \in R \,\&\, \sim(x \in x)$. Thus,

$$\alpha \in \alpha \;\cdot\!\equiv\!\cdot\; \neg(\alpha \in \alpha) \,\&\, \sim(\alpha \in \alpha). \tag{1}$$

Case 1. Let us suppose that $\alpha \in \alpha$. Then, by (1), we obtain $\sim(\alpha \in \alpha)$. Thus, $\alpha \in \alpha \,\&\, \sim(\alpha \in \alpha)$. Consequently, $(x,y).x = y$.

Case 2. Le us suppose that $\neg(\alpha \in \alpha)$. By (1) we obtain $\sim\neg(\alpha \in \alpha)$, i.e., $\neg(\alpha \in \alpha) \supset (x,y).x \in y \,\&\, x = y$. Thus, $(x,y).x = y$.

From cases 1 and 2, the paradox of identity follows. Moreover, the other results mentioned in Theorem 2.3 are also derivable. □

As a consequence of Theorems 5.3 and 5.4 we conclude that the existence of Russell's set is incompatible with a general (for all sets) formulation of the axiom schema of replacement. For, on the one hand, a general formulation of the axiom schema of replacement implies a general formulation of the axiom schema of separation. On the other hand, using the axiom schema of replacement we prove Cantor's Theorem. But, as **UUR** is the universal set, then Cantor's paradox is derivable. Consequently the $\mathbf{ZF}_n$, $1 \leqslant n < \omega$, are trivial and the paradox of identity is derivable in $\mathbf{ZF}_\omega$.

6. CONCLUDING REMARKS.

The main results presented in this paper are the following: (i) in any da Costa paraconsistent set theory with Russell's set the scope of validity of the classical formulations of the axiom schemata of abstraction, separation and replacement cannot be enlarged; (ii) it is not possible to construct da Costa's set theories with Russell's set and without universal set. These results may be obtained in many other strong paraconsistent set theories.

In a certain sense, these results may be considered as limitative ones. By (i), Russell's set as well as other non-classical sets have to be introduced by specific postulates. Thus, in each case, we must investigate if the non-classical set we want to introduce does not lead to a paradox that invalidates the theory. Still by (1), Russell's set is incompatible with a general formulation of the axiom schema of replacement. This fact makes it impossible to prove some interesting thing about some contradictory sets generated by **R**. For instance, let us define $SC^1(\mathbf{R})$ as $SC(\mathbf{R})$ and $SC^{n+1}(\mathbf{R})$ as $SC(SC^n(\mathbf{R}))$. If we could apply the axiom schema of replacement to these sets, we would prove that they are universes. This is the most interesting property of contradictory sets we have already devised. But, unfortunately, up to now we have not found any paraconsistent set theory in which this property is valid.

Set theories without universal set may be considered richer and more interesting than the ones with universal set. Moroeover, it is natural to guarantee the existence of Russell's set in paraconsistent set theories. But, by (ii) it seems that we cannot construct a strong paraconsistent set theory with Russell's and without universal set.

A natural question one may ask is if the above limitative results are valid in weak paraconsistent set theories. In [5] it is proved that Russell's set implies the existence of universal set in weak paraconsistent theories in whose underlying logic the law of excluded middle is valid. In [8] it is proved that Russell's set is not incompatible with a general formulation of the axiom schema of abstraction in some weak and non-trivial paraconsistent se theories. Nonetheless, it has not been investigated whether the paradox of identity is derivable or not in them. However, in [7] it is shown that the paradox of identity is derivable in some other weak and non-trivial paraconsistent set theories.

The weak paraconsistent set theories have the advantage of being non-trivial. But, even if they are free from the paradox of identity, they seem to be weak concerning the set-theoretical operations. Thus, it is interesting to know if they may be strengthened in a way similar to that used by Griss to construct his logic of species (see [2]). An idea of how to proceed in this direction is given in Section 3 above.

To finish, we mention some open problems whose solution we believe are important in the development of paraconsistent set theories. In da Costa paraconsistent set theories, is **R** different from the universal set? If the answer is affirmative, is **UR** different from the universal set? What is the meaning of the defined negation in **NF** (see Section 3, Theorems 3.1-3.3)? Is it possible to construct a paraconsistent set theory with Russell's set and without universal set? Apart from Russell's set what other non-classical sets may be introduced in paraconsistent set theory?

REFERENCES.

[1] A.I. Arruda, *Le schéma de la séparation dans les systèmes* $\mathbf{NF}_n$, C.R. Acad.Sc. Paris 280 (1975), 1341-1344.

[2] ————, *Some remarks on Griss' logic of negationless intuitionistic mathematics*, Mathematical logic: Proceddings of the first Brazilian Conference (Eds. A.I. Arruda, N.C.A. da Costa and R. Chuaqui), Marcel Dekker, 1978, pp.2-29.

[3] ————, *A survey of paraconsistent logic*, Mathematical logic in Latin America (Eds. A.I. Arruda, R. Chuaqui and N.C.A. da Costa), North-Holland. 1980, pp.1-41.

[4] ————, *The paradox of Russell in the systems* $\mathbf{NF}_n$, Proceedings of the third Brazilian Conference on mathematical Logic. (Eds. A.I. Arruda, N. C.A. da Costa and A.M. Sette), Sociedad Brasileira de Lógica, São Paulo 1980, pp.1-21.

[5] A.I. Arruda and D. Batens, *Russell's set versus universal set in paraconsistent set theories*. To appear.

[6] A.I. Arruda and N.C.A. da Costa, *Sur le schéma de la séparation*, Nagoya mathematical Journal 38 (1970), 71-84.

[7] ————, *Le schema de la séparation dans les systèmes* J_n, Mathematica Japonicae 19 (1974), 183-186.

[8] ————, *On the relevant systems* P *and* P^* *and some related systems*. To appear.

[9] F.G. Asenjo and H. Tamburino, *Logic of antinomies*, Notre Dame Journal of Formal Logic 16 (1965), 272-278.

[10] R. Brady, *The non-triviality of dialectical set theory*. To appear.

[11] R. Brady and R, Routley, *The non-triviality of extensional dialectical set theory*. To appear.

[12] N.C.A. da Costa, "Sistemas formais inconsistentes" (Thesis), Universidade Federal do Paraná, 1963.

[13] ——————, *Calculs de prédicats avec égalité pour les systèmes formels inconsistants*, C.R. Acad. Sc. Paris 258 (1964), 1111-1113.

[14] ——————, *Calculus de descriptions pour les systèmes formels inconsistants*, CR. Acad. Sc. Paris 258 (1964), 1366-1368.

[15] ——————, *Sur un système inconsistant de théorie des ensembles*, C.R. Acad. Sc. Paris 258 (1964), 3144-3147.

[16] ——————, *On the theory of inconsistent formal systems*, Notre Dame Journal of Formal Logic 15 (1975), 497-510.

[17] N. Goodman, *The Logic of contradiction*, Zeitschr.f.math.Logik und Grundlagen d. Math. 27 (1981), 119-126.

[18] S.C. Kleene, *Introduction to metamathematics*, Van Nostrand, 1952.

[19] C.R. Mendes dos Santos, "Sobre o sistema $\mathbf{NF}_\omega$" (Master's Dissertation), Departamento de Matemática, Universidade Estadual de Campinas, 1980.

[20] J.B. Rosser, *Logic for mathematicians*, Chelsea Publishing Company (2nd edition), 1980.

***

IMECC
Universidade Estadual de Campinas
Campinas, S.P., Brazil.