NATURAL NUMBERS IN ILLATIVE COMBINATORY LOGIC

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1. INTRODUCTION. In this paper we attempt to develop natural numbers using the second order predicate calculus and the three axioms also used to obtain the set theory of [3]. Of these three axioms, two give us that the natural numbers, as we define them, are elements of the class of individuals A, the other says that all individuals are sets under the definition given in [3]. Besides the induction property and some elementary arithmetical properties of natural numbers, we can prove that the class of all natural numbers forms a set which is a subset of A.

It has been thought that this theory would be strong enough to contain all of first order arithmetic, and hence that it would be subject to Gödel's incompleteness results; this however seems not to be the case. In order to obtain the remaining Peano type axioms for arithmetic, we need to assume a restricted form of substitution of equality and a "type" for the paradoxical combinator Y.

2. THE SECOND ORDER LOGIC.

The primitive constants that we require, besides the combinators K and S (or \( \lambda \)-abstraction), are: A, H (the class of propositions) and \( \exists \) (restricted generality). \( \exists U \)X (or \( \exists y \)XY) expresses the fact that XY holds for all Y in U. \( \Rightarrow \) can be defined in terms of \( \exists \) as in [1].

The rules of the logic are:

\[
\begin{align*}
\text{Eq} & \quad \text{If } X = Y \text{ then } X \vdash Y. \\
\text{P} & \quad X \in Y, \ X \vdash Y. \\
\exists & \quad \exists U, \ Y \vdash XY \text{ where } U \text{ is } A, H \text{ or } FAH. \\
\text{DTP} & \quad \text{If } \Delta, \ X \vdash Y \text{ then } \Delta, \ HX \vdash X \Rightarrow Y. \\
\text{DTE} & \quad \text{If } \Delta, \ UY \vdash XY \text{ then } \Delta \vdash \exists U X \text{ where } U \text{ is } A, H \text{ or } FAH. \\
\text{H} & \quad X \vdash HX. \\
\text{PH} & \quad HX, \ X \in HY \vdash H(X \Rightarrow Y).
\end{align*}
\]

(1) \( F = \lambda x \lambda y \lambda z \exists x (Byz) \) and \( FXY \) can be interpreted as the class of functions from X to Y. \( FAH \) can therefore be interpreted as the class of first order predicates.
3. ARITHMETIC.

Natural numbers were defined in terms of λ-abstraction by Church in [4] and equivalently by Curry and Feys in [5] in terms of combinators. They have:

\[ Z_0 = KI = \lambda x \lambda y.y \]
\[ Z_{n+1} = SBZ_n = \lambda x \lambda y.(Z_n xy) \]

(Church starts with \( Z_1 = \lambda x \lambda y.x(y) \)). These combinators, called *iterators*, have the property:

\[ x^n = Z_n x \]
\[ x^0 = I = \lambda x.x \]
and
\[ x^{n+1} = Bxx^n = \lambda u.x(x^n u) \].

It is easy to prove that:

\[ Z_{m+n} = \phi BZ_m Z_n \]
\[ Z_{mn} = BZ_m Z_n \]
and
\[ Z_{nm} = Z_{m+n} \].

Thus the iterators themselves can be taken as numbers and addition, multiplication and exponentiation can be defined so that the appropriate commutative, associative and distributive laws hold.

If however we want to represent numbers in a predicate calculus, a symbol for equality has to be introduced into that system; combinatory equality (=) is a primitive predicate.

Church, and also Kleene in his development of Church's system in [7], use

\[ Q = \lambda x \lambda y(ux \rightarrow u y), \]

which is satisfactory when Church's strong deduction theorem for \( \equiv \) is used. Church's system is however inconsistent because of this deduction theorem, and our weaker theorem (or rule) requires a different equality. The one we use is an extensional equality given by:

\[ Q_1 = \lambda x \lambda y. \ FAH x \land FAH y \land Au \rightarrow (ux \rightarrow yu), \]

In [3] sets and first order predicates were identified so that "FAHx" stands

(2) "\( \rightarrow \)" stands for "if and only if". We will often write \( X =_1 Y \) for \( Q_1 XY \).
for "X is a set".

The main axiom of [3] asserts that all individuals are sets i.e.

\[(A) \quad \forall u \exists u \quad FAHu.\]

The empty class, based on Q₁, given by:

\[0 = \lambda x. \Gamma(Q₁xx)\]

could then be shown to be a set (i.e. \(\forall FAH₀\)), as could the class containing only 0 (Q₁₀), the class containing only that \(Q₁(Q₁₀) = Q₁²₀\) etc.

Now 0, \{0\} (= Q₁₀), \{\{0\}\} (= Q₁²₀)... have been used as definitions for 0,1,2,... in naive set theory and are very suitable here. Thus we define:

\[1 = Q₁₀, \quad 2 = Q₁²₀, \ldots, n = Q₁^n₀, \ldots\]

As \(n = Q₁^n₀ = Z_n Q₁₀\), we can define the arithmetical operations in the following way:

\[n + m = \Phi_{Z_m Z_n Q₁₀} = Q₁^{m+n}₀\]

\[n \cdot m = Z_m Z_n Q₁₀ = (Q₁^n₀)\cdot m\]

and

\[n^m = Z_m Z_n Q₂₀ = Q₁^n₀\]

As before the appropriate commutative, associative and distributive laws hold over \(=\); this fact is independent of our definitions of 0 and Q₁ and of our axiom (A). Given (A) these laws also hold over Q₁ as we then have \(\forall FAH(Q₁^n₀)\) for \(i > 0\).

To prove that for \(i \neq j\), \(Q₁^i₀\) and \(Q₁^j₀\) are unequal in the Q₁ sense we need, as was shown in [3]:

\[\forall AO\] \hspace{1cm} (1)

and

\[\forall FAAQ₁,\] \hspace{1cm} (2)

which guarantee that all our numbers are individuals.

These axioms, as was also shown in [3], allow us to prove the cancellation law for addition.

Thus we have all the simpler "Peano axioms" in the following form:

\[\forall \Gamma(0 = ₁ n + 1)\]

\[\forall n \cdot (n + 1 = ₁ m + 1 \Rightarrow n = ₁ m), \text{ etc.,}\]

where \(n\) and \(m\) are natural numbers, but we do not have

\[\forall (\forall x)(x \in N \Rightarrow \Gamma(0 = ₁ x + 1)) \text{ etc.}\]

Also to prove the mathematical induction axiom we need a class N of natural numbers. This we define as follows:

\[N = \exists n (\lambda x(x₀ \land Ay \Rightarrow y(xy \Rightarrow x(Q₁y))))\]

where

\[n Z = \lambda u. \quad FA Hv \Rightarrow v Zv \Rightarrow vu.\]

We then prove that N is a set:
THEOREM 1. \( \vdash \) FAHx.

Proof. By (1) \( \vdash \) FAHx \( \vdash \) H(x0)
and \( \vdash \) FAHx,Ay \( \vdash \) H(xy)
also by (2) \( \vdash \) FAHx,Ay \( \vdash \) H(x(Q1y))
so \( \vdash \) FAHx,Ay \( \vdash \) H(xy \( \supset \) x(Q1y))
and \( \vdash \) FAHx \( \vdash \) H(x0 \( \wedge \) Ay \( \supset \) y(xy \( \supset \) x(Q1y)))
(this step uses rules PH and P" and results from [2]).

Let \( W = \lambda x.x0 \wedge Ay \supset y(xy \supset x(Q1y)) \)
then, by DTE, \( \vdash F(FAH)HW \)
Now, \( Au, FAHz \vdash H(zu) \)
and \( \vdash FAHz \vdash H(Wz) \)
so \( Au, FAHz \vdash H(Wz \supset zu) \)
and so \( Au \vdash H(FAHz \supset Wz \supset zu) \)

i.e. \( \vdash FAH(W) \)

\( \vdash \) FAHN. \( \Box \)

We could note that (A) is not needed in this proof so that this theorem holds in alternative set theories where (1) and (2) hold but (A) does not.

We now show that N is, in a sense, a subclass of A; again this does not require (A).

THEOREM 2. \( Nu \vdash Au. \)

Proof. By (2) we have \( Ay \vdash A(Q1y) \)
and so \( Ay \vdash Ay \supset A(Q1y) \)
and by DTE \( \vdash Ay \supset y Ay \supset A(Q1y). \)

Then by (1), taking W as above we have:

\( \vdash WA \)

Now \( Nu = N\nu = FAHv \supset vWv \supset \nuu, \)
so as \( \vdash FAHA \)
we have \( Nu \vdash Au. \) \( \Box \)

Note that because DTE holds only for \( U = A, H \) and FAH we cannot conclude \( \vdash Nu \supset Au \) from this. Using Theorem 1 we can get no more than \( Au \vdash Nu \supset Au, \)
which is not very useful. If we use the stronger deduction theorem for \( E \) given in [1]:

If \( \Delta, Xu \vdash Y \) then \( \Delta, FAHX \vdash EXY, \)
we could obtain \( \vdash ENA, \) but this rule makes the system substantially stronger.

The theorem however means that \( Au \supset (Nu \supset Xu) \) has all the properties that \( Nu \supset Xu \) could have, namely:
Au ⊨ u(Nu ⊨ X), Nu ⊨ X

and if Δ, Nu ⊨ X then Δ ⊨ Au ⊨ u(Nu ⊨ X).

We now prove that mathematical induction holds for N.

**Theorem 3.** If

\[ \Delta \vdash \forall x \phi(x) \wedge \forall y \phi(x(y)) \]

\[ \Delta \vdash \forall u \chi(u) \]

\[ \forall f - FAH \chi, \]

then

\[ \forall q \cdot N \chi \]

*Proof.* With \( W \) as above we have:

\[ \forall u - FAHv \vdash \forall v \cdot Wv \vdash \forall u \]

so by (c) \( \forall u - WX \vdash Xu \)

and by (a), \( \forall u - Xu \).

By Theorem 1, \( \forall u - H(Nu) \)

so \( \forall u - Nu \vdash Xu \).

Also by (b) \( \forall u - Xu \vdash Nu \)

so \( \forall u - Xu \vdash Nu \)

.: \( \forall u - Xu \vdash Nu \)

i.e. \( \forall q \cdot N \chi \).

The following restricted form of this theorem will be all we need in many cases:

**Corollary.** If (a) and (c) hold then \( \forall u - Xu \).

We now look at some further basic properties of the natural numbers as we have defined them.

**Theorem 4.** \( \forall x - N(Q,x) \).

*Proof.* \( \forall x, FAHv, [\forall v \wedge Ay \Rightarrow y \cdot v \Rightarrow v(Q_{1}y)] \vdash vx \).

By Theorem 2,

\[ \forall x - Ax, \]

.: \( \forall x, FAHv, [\forall v \wedge Ay \Rightarrow y \cdot v \Rightarrow v(Q_{1}y)] \vdash v(Q_{1}x) \).

Thus as \( FAHv \vdash H[\forall v \wedge Ay \Rightarrow y \cdot v \Rightarrow v(Q_{1}y)] \)

\[ \forall x - FAHv \Rightarrow [\forall v \wedge Ay \Rightarrow y \cdot v \Rightarrow v(Q_{1}y)] \Rightarrow v(Q_{1}x) \]

i.e. \( \forall x - N(Q_{1}x) \).

**Theorem 5.** \( \forall x - \exists z (\lambda z_x = 1 \cdot Q_{1}z \wedge Nz) \).

*Proof.* Let \( X = \lambda x \cdot x = 1 \cdot Q_{1}z \wedge Nz \)

then \( \vdash FAHx \) (3)
Now as

\[ \vdash FAH, \]

\[ \vdash FAH(Q_1z), \]

and

\[ \vdash H(Nz) \]

\[ \vdash H(\Sigma A(\lambda z \cdot Q_1z = 1 \land Nz)). \]

Also

\[ \vdash AO \]

\[ \vdash 0 = 1 \]

and so

\[ \vdash XO. \]

Now

\[ \vdash x = 1 \land Q_1x = 1 \land Q_10 \land NO, \]

so

\[ \vdash x = 1 \land Q_1x = 1 \land Q_10 \land Nz). \]

Also

\[ \vdash A, \Sigma, x = 1 \land Q_1z \land Nz \vdash Q_1x = 1 \land Q_1z) \]

and by Theorem 4, \[ \vdash A, \Sigma, x = 1 \land Q_1z \land Nz \vdash (Q_1z) \]

so also by Theorem 2, \[ \vdash A, x = 1 \land Q_1z \land Nz \vdash \Sigma A(\lambda z \cdot Q_1z = 1 \land Q_1w \land Nz) \]

\[ \vdash \Sigma A(\lambda z \cdot x = 1 \land Q_1z \land Nz) \]

\[ \vdash by (5) and (6) \]

\[ \vdash x = 1 \land Q_1x \]

and using (4)

\[ \vdash xo \land Ax \vdash x = 1 \land Q_1x. \]

Thus by (3) and the Corollary to Theorem 3,

\[ \vdash N x \vdash Au \vdash xu \vdash \Sigma A(\lambda z \cdot Q_1zu \land Nz). \]

THEOREM 6. \[ \vdash N x \vdash Au = xu \vdash \Sigma A(\lambda z \cdot Q_1zu \land Nz). \]

Proof. \[ \vdash N x, Au, xu \vdash \Gamma(x = 1) \]

\[ \vdash N x, Au, xu \vdash \Sigma A(\lambda z \cdot x = 1 \land Q_1z \land Nz) \]

\[ \vdash N x, Au, xu \vdash \Sigma A(\lambda z \cdot Q_1zu \land Nz). \]

If we have the following axiom of extent (mentioned in [3]):

(E2) \[ \vdash Au = xu \vdash \Sigma fz \vdash Q_1zu \vdash tz \land tu. \]

We also have as \[ \vdash FAH: \]

THEOREM 7. \[ \vdash N x \vdash Au = xu \vdash Nu. \]

Note that this axiom for the weakest form of substitution for equality is suggested in [3]. The first rule:

\[ Q_1zu, tz \vdash tu \]

leads to anomalous (though not inconsistent results). The second, the axiom:

(E1) \[ \vdash FAHu = _u FAHz = _z FAHu = _t Q_1zu \vdash tz \land tu \]

gives \[ \vdash RR \land RR \]
where R is the Russell class (which is a set in this paper). We also have \( RR = \Gamma(\text{RR}) \) and so by Eq, \( \vdash \text{RR} \not\equiv \Gamma(\text{RR}) \), but as we do not have \( \vdash \text{AR} \) and hence not \( \vdash \text{H(AR)} \) we cannot prove a contradiction from this anomalous result.

The third possibility, the axiom (E2) mentioned above seems to be free of anomalies and is sufficient for what we require above.

Note also that Theorem 7 says that N is a transitive set.

Going back now to arithmetical properties in the more general form we find that we can easily prove:

**Theorem 8.**

(i) \( N_z, N_x, N_y \vdash x =_1 y \supset (x =_1 z \supset y =_1 z) \)

(ii) \( N_x, N_y \vdash x =_1 y \supset Q_1 x =_1 Q_1 y \)

(iii) \( N_x \vdash Q_1(0 =_1 Q_1 x) \)

(iv) \( N_x, N_y \vdash Q_1 x =_1 Q_1 y \supset x =_1 y \).

**Proof.** By Theorem 2, (A)

\[
\text{FAH}_x, \text{FAH}_y \vdash \text{H}(x =_1 y)
\]

and

\[
\text{FAH}_x \vdash \text{FAH}(Q_1 x)
\]

it is easy to show that on the basis of the assumptions all the formulas to the right of \( \vdash \) are propositions; (i) then follows by the definition of \( Q_1 \) and (ii) and (iv) follow directly from (i); (iii) follows by the definitions of \( Q_1 \) and 0. □

The four parts of Theorem 8 correspond to the first four Peano type axioms given by Mendelson [8]. To prove the next one:

\[ N_x \vdash x + 0 =_1 x, \]

however seems to be impossible with + defined contextually as it is.

The alternative is to define addition (and also multiplication) by recursion. The recursion operator has been defined in [6] in terms of an ordered pair operator, which is also defined in [6] and a predecessor function which is also definable in terms of combinators. We can however define the predecessor relation in terms of terms definable using Ε, A and H, in a much simpler fashion.

**Definition.** \([\pi] = \lambda z \lambda x, \Sigma A(\lambda y, yx \land zy)\).

We can then prove:

**Theorem 9.**

(i) \( \text{FAH}_z \vdash \text{FAH}([\pi] z) \)

(ii) \( \text{FAH}_t \vdash [\pi] (Q_1 t) =_1 t \)

(iii) \( \vdash [\pi] 0 =_1 0. \)

**Proof.** (i) By (A), \( \text{FAH}_z, A_x, A_y \vdash \text{H}(yx \land zy) \)

\[
\text{FAH}_z, A_x, \vdash \text{H}(\Sigma A(\lambda y, yx \land zy))
\]
\[ \text{i.e. } FAHz \vdash FAH(\pi)z. \]

(ii) \[\pi](Q,t) = \lambda x \cdot \Sigma A(\lambda y \cdot yx \land Q,ty), \text{ so by (A) and Theorem 2} \]

\[ Ay, Ax \vdash H(yx) \]

and \[ FAHt, Ay, Ax \vdash H(Q,ty) \]

i.e. \[ FAHt, Ax, [\pi](Q,t)x \vdash tx \]

Also \[ Ax, tx, FAHt \vdash tx \land Q,tt, \]

do so \[ Ax, tx, FAHt \vdash \Sigma A(\lambda y \cdot yx \land Q,ty), \]

and so \[ Ax, FAHt \vdash tx = [\pi](Q,t)x. \]

\[ Ax, FAHt \vdash [\pi](Q,t)x \Rightarrow tx \]

i.e. \[ FAHt \vdash Ax \Rightarrow J(\pi)(Q,t)x \Rightarrow tx \]

so \[ FAHt \vdash [\pi](Q,t) = 1t. \]

(iii) \[\pi]Ox = \Sigma A(\lambda y \cdot yx \land Oy). \]

But \[ \vdash Ay \Rightarrow \gamma (Oy) \]

so \[ \vdash Ax \Rightarrow \gamma ([\pi]Ox) \]

and so \[ \vdash Q_1([\pi]O) = 0. \]

The ordered pair operator \( D \) can now be defined by:

**DEFINITION D.** \( D = \lambda x \lambda y \lambda z \lambda u. (\gamma(\Sigma Az) \supset xu) \land (\Sigma Az \supset y(\pi)xu). \)

To prove all the expected results for this however, we need the following stronger form of (E2), which is however still weaker than (E1) and seems to avoid the anomalies mentioned earlier.

**AXIOM (B).** \( \vdash Au \Rightarrow FAHz \Rightarrow FAHt \Rightarrow Q_1z \Rightarrow tu \Rightarrow tz. \)

This gives in particular:

**THEOREM 10.** (i) \( Au, FAHz, Q_1zu \vdash Az \)

(ii) \( Nu, FAHz, Q_1zu \vdash Nz, \)

and also the basic properties of \( D: \)

**THEOREM 11.** (i) \( F_2(FAH)Ah, FAHx \vdash F_2(FAH)Ah(Oxy) \)

(ii) \( FAHx, F_2(FAH)Ah \vdash Dxyo =_1 x \)

(iii) \( FAHx, F_2(FAH)Ah, At \vdash Dxy(Q,t) =_1 yt. \)

**Proof.** (i) By Theorem 9 (i) and (A),

\[ F_2(FAH)Ah, Au, FAHx \vdash H(y(\pi)xu) \]

also \[ FAHz \vdash H(\Sigma Az), \]

so \[ F_2(FAH)Ah, Au, FAHx, FAHz, \vdash H(Dxzu) \]

\[ \therefore F_2(FAH)Ah, FAHx \vdash F_2(FAH)A(Dxy) \]
(ii) \(Au, FAHx, F_2(FAH)Ah, DxyOu \vdash xu\)

and by (i)

\[FAHx, F_2(FAH)Ah, Au \vdash DxyOu \supset xu.\]

Also

\[FAHx, F_2(FAH)Ah, Au \vdash xu \supset DxyOu\]

so

\[FAHx, F_2(FAH)Ah \vdash Au \supset xu \supset DxyOu\]

i.e.

\[FAHx, F_2(FAH)Ah \vdash Dxy0 = 1 x.\]

(iii) As

\[\text{At } \vdash Q_1tt\]
\[\text{At } \vdash \Sigma A(Q_1t)\]

\[\therefore \quad \text{Au}, FAHx, F_2(FAH)Ah, At, Dxy(Q_1t)u \vdash y(\{\{\pi\}(Q_1t))u.\]  

(7)

(8)

Now as

\[Ar \vdash FAHr\]

we have

\[F_2(FAH)Ah, Ar, Au \vdash H(yru)\]

i.e.

\[F_2(FAH)Ah, Ar, Au \vdash H(Cyr)\]

and \(\therefore\)

\[F_2(FAH)Ah, Au \vdash FAH(Cyu).\]

By (A) and Theorem 9 (i) \(\text{At } \vdash FAH([\pi](Q_1t))\)

and by Theorem 9 (ii) \(\text{At } \vdash [\pi](Q_1t) = 1 t\)

\(\therefore\) by (B) and (S):

\[\text{Au}, FAHx, F_2(FAH)Ah, At, Dxy(Q_1t)u \vdash ytu.\]

Similarly using (B)

\[\text{Au}, FAHx, F_2(FAH)Ah, At, ytu \vdash y(\{\{\pi\}(Q_1t))u\]

so by (7),

\[\text{Au}, FAHx, F_2(FAH)Ah, At, ytu \vdash Dxy(Q_1t)u,\]

and the result can be proved. \(\square\)

Now we define the recursion operator:

DEFINITION R. \(R = \lambda x\lambda y. Y(B(Dx)(Sy))\).

A result such as that in parts (i) of Theorems 9 and 11, about the functionality (or type) of \(R\) seems to be impossible. We have

\(Rxy = Dx(Sy(Rxy))\),

so to determine, for given \(x\) and \(y\), the type of \(Rxy\) from that of \(D\) we need first the type of \(Rxy\).

Alternatively, we need to know a type for \(y\); however, this is known to be not derivable from the types for \(K\) and \(S\). (These "types" in fact constitute the two basic axioms of the kind of system that we are dealing with - they allow the proof of the deduction theorem for \(\exists\) - see [1], and for a discussion of the relation between axioms and types see [6]).

We can however postulate a type for \(y\) that does not conflict with those for \(K\) and \(S\) and which will lead to a type for \(R\).

(Y) \(\vdash F(F[FAH](FAH))[F(FAH)(FAH)](F(FAH)(FAH))Y.\)
(f - F(fTJ')TY is reasonable for any T as then FTT2 ⊢ T(YZ) and FTT2 ⊢ T(Z(YZ)). Below we only need the above special case).

We can now prove:

**THEOREM 12.** (i) $F_2(FAH)(FAH)(FAH)y, FAHx, FAHVx ⊢ FAH(Rxyt)$
(ii) $F_2(FAH)(FAH)(FAH)y, FAVx ⊢ Rxy0 = 1 x$
(iii) $F_2(FAH)(FAH)(FAH)y, FAHx, At ⊢ Rxy(Q_1t) = 1 yt(Rxyt)$.

Proof.

$F(FAH)(FAH)u, FAHVv ⊢ F(FAH)(uv)$

and

$F_2(FAH)(FAH)(FAH)y, FAVx, FAHVv ⊢ F(FAH)(FAH)(FAH)(uv)$

and so

$F_2(FAH)(FAH)(FAH)y, F(FAH)(FAH)u, FAHVv ⊢ F_2(FAH)(uv)$

by Theorem 11 (i):

$F_2(FAH)(FAH)(FAH)y, F(FAH)(FAH)u, FAHVv ⊢ F_2(FAH)(FAH)(uv)$

and so

$F_2(FAH)(FAH)(FAH)y, F(FAH)(FAH)u, FAHVv ⊢ F_2(FAH)(FAH)(uv)$

Now $F_2(FAV)(FAH)(FAH)$, so by (7) and Definition R:

$F_2(FAH)(FAH)(FAH)y, FAVx ⊢ F_2(FAH)(FAH)(Rxy)$

so the result follows.

(iii) As $\vdash \text{FAH}$, we have by (i) that

$F_2(FAH)(FAH)(FAH)y, FAVx, FAHVx ⊢ F_2(FAH)(FAH)(Rxy0)$.

Now

$Rxy0 = y(B(Dx)(Sy))0 = B(Dx)(Sy)(Rxy)0 = Dx(Sy)(Rxy))0$.

Now

$F_2(FAH)(FAH)(FAH)y, FAVx, FAHVx ⊢ F_2(FAH)(FAH)(yu)$

so by (i) $F_2(FAH)(FAH)(FAH)y, FAVx, FAHVx ⊢ F_2(FAH)(yu)(Rxyu))$

also by Theorem 11 (ii)

$F_2(FAH)(FAH)(FAH)y, FAVx ⊢ Dx(Sy)(Rxy))0 = 1 x$, and the result holds.

(iii) This holds by (9) and Theorem 11 (iii).

Next we can define addition.

**DEFINITION.** $x + y = x + y = Rx(KQ_1)y$.

**THEOREM 15.** (i) $FAHx ⊢ x + 0 = 1 x$
(ii) $FAHVx, ly ⊢ x + Q_1y = 1 Q_1(x+y)$
(iii) $Nx, Ny ⊢ N(x+y)$.

Proof. (i) By (A), $FAHV, Aw ⊢ H(Q_1vw)$

i.e. $FAHV, FAHV ⊢ FAH(KQ_1uv)$

and so

$\vdash F_2(FAH)(FAH)(FAH)(KQ_1)$. (16)
(. By (A) and Theorem 12 (ii)
\[ FAHx \vdash Rx(KQ_1)0 = 1x \]
i.e.
\[ FAHx \vdash x+0 = 1x. \]

(ii) By (10) and Theorem 12 (iii),
\[ FAHx,Ay \vdash x+Q_1y = 1KQ_1y(x+y) \]
so
\[ FAHx,Ay \vdash x+Q_1y = 1Q_1(x+y). \]

(iii) Theorem 12 (i) \( \vdash FAH(x+0) \), so by (i) and Theorem 10 (ii)
\[ Nx \vdash N(x+0) \]
so
\[ \vdash Ax \Rightarrow Nx \Rightarrow N(x+0). \]

Now \( Ax \Rightarrow Nx \Rightarrow N(x+y), Ax,Nx \vdash N(x+y) \)
so by Theorem 4,
\[ Ax \Rightarrow Nx \Rightarrow N(x+y), Ax,Nx \vdash N(Q_1(x+y)), \]
also then \( Ax \Rightarrow Nx \Rightarrow N(x+y), Ax,Nx \vdash A(Q_1(x+y)). \)

Now \( Ay \vdash FAH(Q_1y) \)
so by Theorem 12 (i) \( Ay,FAHx \vdash FAH(x+Q_1y) \)
so by (ii) and Theorem 10 (ii)
\[ Ax \Rightarrow Nx \Rightarrow N(x+y), Ax,Nx \vdash N(Q_1(x+y)), \]
\[ \vdash Ax \Rightarrow N(x+y), Ax,Nx \vdash N(x+y). \]

Clearly (i) and (ii) of this theorem have as special cases:
\[ Nx \vdash x+0 = 1x \]
\[ Nx,Ny \vdash x+Q_1y = 1Q(x+y). \]

We now define the multiplication:

DEFINITION X. \( Xxy = x\cdot y = RO(K(x))y. \)

THEOREM 14. (i) \( FAHx \vdash x+0 = 10 \)
(ii) \( FAHx,Ay \vdash x+Q_1y = 1x + x\cdot y \)
(iii) \( Nx,Ny \vdash N(x\cdot y) \)

Proof. (i) By Theorem 12 (i), \( FAHx,FAHy \vdash FAH(x+y) \)
so
\[ FAHu,FAHx,FAHy \vdash FAH(K(x+y)) \]
\[ \vdash FAHx \vdash F_2(FAH)(FAH)(K(x+y)). \]

then by Theorem 12 (ii) \( FAHx \vdash RO(K(x+y))0 = 10 \)
so (i) follows.

(ii) By (11) and Theorem 12 (iii)
\[ FAHx,Ay \vdash x\cdot (Q_1y) = 1K(x+y)(x\cdot y) \]
so
\[ FAHx,Ay \vdash x\cdot (Q_1y) = 1x+x\cdot y. \]
(iii) By (i) and Theorem 10 (ii) 
\[ Nx \vdash N(x'0) \]
so
\[ \vdash Ax \supset \forall x Nx \supset N(x'0), \]
\[ Ax \supset \forall x N(x'y), Ax, N x \vdash N(x'y)\]
\[ \therefore \text{by Theorem 13 (iii)} \]
\[ Ax \supset \forall x N(x'y), Ax, N x \vdash N(x'y + x) . \]
Now by Theorem 12 (i), (2) and Definition X
\[ F A x, F A h y \vdash F A H (x'Q1y) \]
\[ \therefore \text{by Theorem 10 (ii) and (ii)} \]
\[ Ax \supset \forall x N(x'y), Ax, N x, A y \vdash N(x'Q1y) \]
\[ \therefore \vdash A y \supset \forall y [Ax \supset \forall x N(x'y)] = [Ax \supset \forall x N(x'Q1y)] \]
so by the corollary to Theorem 3
\[ N y \vdash A x \supset \forall x N(x'y) \]
\[ \therefore \]
\[ N x, N y \vdash N(x'y) . \]

Thus given the extra axioms (B) and (Y) which we have had to introduce, we can develop all the Peano type axioms of [8], and hence Mendelson's development of formal number theory can be carried out here.

REFERENCES.


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