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NATURAL NUMBERS IN ILLATIVE COMBINATORY LOGIC

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1. INTRODUCTION. In this paper we attempt to develop natural numbers using the second order predicate calculus and the three axioms also used to obtain the set theory of [3]. Of these three axioms, two give us that the natural numbers, as we define them, are elements of the class of individuals A, the other says that all individuals are sets under the definition given in [3]. Besides the induction property and some elementary arithmetical properties of natural numbers, we can prove that the class of all natural numbers forms a set which is a subset of A.

It has been thought that this theory would be strong enough to contain all of first order arithmetic, and hence that it would be subject to Gödel's incompleteness results; this however seems not to be the case. In order to obtain the remaining Peano type axioms for arithmetic, we need to assume a restricted form of substitution of equality and a "type" for the paradoxical combinator *Y*.

2. THE SECOND ORDER LOGIC.

The primitive constants that we require, besides the combinators K and S (or λ-abstraction), are: A, H (the class of propositions) and Ξ (restricted generality). ΞUX (or $Uy \supset_y Xy$) expresses the fact that XY holds for all Y in U. ⊃ can be defined in terms of Ξ as in [1].

The rules of the logic are:

Eq If X = Y then X ⊢ Y.
P X ⊃ Y, X ⊢ Y.
Ξ ΞUX, UY ⊢ XY where U is A, H or FAH.(1)
DTP If Δ, X ⊢Y then Δ, HX ⊢X ⊃ Y .
DTΞ If Δ, UY ⊢XY then Δ ⊢ ΞUX where U is A, H or FAH.
H X ⊢HX.
PH HX, X ⊃ HY ⊢H(X ⊃ Y).

(1) F = λxλyλzΞx(Byz) and FXY can be interpreted as the class of functions from X to Y. FAH can therefore be interpreted as the class of first order predicates.

ΞH FUHX $\vdash H(\Xi UX)$ where U is A, H or FAH.

(The last five of these could be replaced by Axioms as in [1]).

Quantification over H is not used directly in this paper, but it is needed to define the connectives Γ, $\vee$ and $\wedge$, and the existential quantifier ΣA and to prove their appropriate properties (see [2]).

3. ARITHMETIC.

Natural numbers were defined in terms of λ-abstraction by Church in [4] and equivalently by Curry and Feys in [5] in terms of combinators. They have:

$$Z_0 = KI \qquad (= \lambda x\ \lambda y.y)$$
$$Z_{n+1} = SBZ_n \qquad (= \lambda x\ \lambda y.x(Z_n xy))$$

(Church starts with $Z_1 = \lambda x\ \lambda y.xy\ (=BI)$). These combinators, called *iterators*, have the property:

$$X^n = Z_n x$$

where

$$X^0 = I \qquad (= \lambda x.x)$$

and

$$X^{n+1} = BXX^n \quad (= \lambda u.X(X^n u)).$$

It is easy to prove that:

$$Z_{m+n} = \Phi BZ_m Z_n$$
$$Z_{mn} = BZ_m Z_n$$

and

$$Z_{n^m} = Z_m Z_n.$$

Thus the iterators themselves can be taken as numbers and addition, multi-cation and exponentiation can be defined so that the appropriate commutative, associative and distributive laws hold.

If however we want to represent numbers in a predicate calculus, a symbol for equality has to be introduced into that system; combinatory equality (=) is a primitive predicate.

Church, and also Kleene in his development of Church's system in [7], use

$$Q = \lambda x \lambda y(ux.\supset_u uy),$$

which is satisfactory when Church's strong deduction theorem for Ξ is used. Church's system is however inconsistent because of this deduction theorem, and our weaker theorem (or rule) requires a different equality. The one we use is an extensional equality given by:

$$Q_1 = \lambda x\ \lambda y.\quad FAHx \wedge FAHy \wedge Au \supset_u (xu \leftrightarrow yu)^{(2)}$$

In [3] sets and first order predicates were identified so that "FAHX" stands

(2) "$\leftrightarrow$" stands for "if and only if". We will often write $X =_1 Y$ for Q_1XY.

for "X is a set".

The main axiom of [3] asserts that all individuals are sets i.e.

(A) $$\vdash Au \supset_u FAHu.$$

The empty class, based on Q_1, given by:

$$0 = \lambda x.\Gamma(Q_1 xx)$$

could then be shown to be a set (i.e. $\vdash FAH0$), as could the class containing only 0 $(Q_1 0)$, the class containing only that $(Q_1(Q_1 0) = Q_1^2 0)$ etc.

Now 0, {0} (= $Q_1 0$), {{0}} (= $Q_1^2 0$)... have been used as definitions for 0,1,2,... in naive set theory and are very suitable here. Thus we define:

$$1 = Q_1 0,\ 2 = Q_1^2 0, \ldots, n = Q_1^n 0, \ldots$$

As $n = Q_1^n 0 = Z_n Q_1 0$, we can define the arithmetical operations in the following way:

$$n+m = \Phi B Z_m Z_n Q_1 0 = Q_1^m(Q_1^n 0)$$

$$n \cdot m = B Z_m Z_n Q_1 0 = (Q_1^n)^m 0$$

and

$$n^m = Z_m Z_n Q_1 0 = Q_1^{n^m} 0.$$

As before the appropriate commutative, associative and distributive laws hold over = ; this fact is independent of our definitions of 0 and Q_1 and of our axiom (A). Given (A) these laws also hold over Q_1 as we then have $\vdash FAH(Q_1^i 0)$ for $i \geqslant 0$.

To prove that for $i \neq j$, $Q_1^i 0$ and $Q_1^j 0$ are unequal in the Q_1 sense we need, as was shown in [3]:

$$\vdash A0 \tag{1}$$

and

$$\vdash FAAQ_1, \tag{2}$$

which guarantee that all our numbers are individuals.

These axioms, as was also shown in [3], allow us to prove the cancellation law for addition.

Thus we have all the simpler "Peano axioms" in the following form:

$$\vdash \Gamma(0 =_1 n + 1)$$

$$\vdash n + 1 =_1 m + 1 \supset n =_1 m, \quad \text{etc.,}$$

where n and m are natural numbers, but we do not have

$$\vdash (\forall x)(x \in N \supset \Gamma(0 =_1 x + 1) \quad \text{etc.}$$

Also to prove the mathematical induction axiom we need a class N of natural numbers. This we define as follows:

$$N = \cap(\lambda x(x0 \wedge Ay \supset_y (xy \supset x(Q_1 y))))$$

where

$$\cap Z = \lambda u.\ FAHv \supset_v Zv \supset vu.$$

We then prove that N is a set:

THEOREM 1. $\vdash$ FAHN.

Proof. By (1) $\quad FAHx \vdash H(x0)$

and $\quad FAHx, Ay \vdash H(xy)$

also by (2) $\quad FAHx, Ay \vdash H(x(Q_1y))$

so $\quad FAHx, Ay \vdash H(xy \supset x(Q_1y))$

and $\quad FAHx \vdash H(x0 \wedge Ay \supset_y (xy \supset x(Q_1y)))$

(this step uses rules PH and PΞ and results from [2]).

Let $W = \lambda x.x0 \wedge Ay \supset_y (xy \supset x(Q_1y))$

then, by DTΞ, $\quad \vdash F(FAH)HW$

Now, $\quad Au, FAHz \vdash H(zu)$

and $\quad FAHz \vdash H(Wz)$

so $\quad Au, FAHz \vdash H(Wz \supset zu)$

and so $\quad Au \vdash H(FAHz \supset_z Wz \supset zu)$

$\therefore \quad \vdash FAH(\cap W)$

i.e. $\quad \vdash FAHN$. □

We could note that (A) is not needed in this proof so that this theorem holds in alternative set theories where (1) and (2) hold but (A) does not.

We now show that N is, in a sense, a subclass of A; again this does not require (A).

THEOREM 2. $Nu \vdash Au$.

Proof. By (2) we have $\quad Ay \vdash A(Q_1y)$

and so $\quad Ay \vdash Ay \supset A(Q_1y)$

and by DTΞ $\quad \vdash Ay \supset_y Ay \supset A(Q_1y)$.

Then by (1), taking W as above we have:

$$\vdash WA$$

Now $Nu = \cap Wu = FAHv \supset_v Wv \supset vu$,

so as $\quad \vdash FAHA$

we have $\quad Nu \vdash Au$. □

Note that because DTΞ holds only for U = A, H and FAH we cannot conclude $\vdash Nu \supset_u Au$ from this. Using Theorem 1 we can get no more than $Au \vdash Nu \supset Au$, which is not very useful. If we use the stronger deduction theorem for Ξ given in [1]:

$$\text{If } \Delta, Xu \vdash Y \text{ then } \Delta, FAHX \vdash \Xi XY,$$

we could obtain $\vdash \Xi NA$, but this rule makes the system substantially stronger.

The theorem however means that $Au \supset_u (Nu \supset Xu)$ has all the properties that $Nu \supset_u Xu$ could have, namely:

$$Au \supset_u (Nu \supset X),\ Nu \vdash X$$

and if Δ, $Nu \vdash X$ then $\Delta \vdash Au \supset_u (Nu \supset X)$.

We now prove that mathematical induction holds for N.

THEOREM 3. *If* $\vdash XO \wedge Ay \supset_y (Xy \supset X(Q_1y))$ (a)

$\vdash Au \supset_u Xu \supset Nu$ (b)

$\vdash FAHX$, (c)

then $\vdash Q_1XN$,

Proof. With W as above we have:

$$Nu \vdash FAHv \supset_v Wv \supset vu$$

so by (c) $Nu \vdash WX \supset Xu$

and by (a), $Nu \vdash Xu$.

By Theorem 1, $Au \vdash H(Nu)$

so $Au \vdash Nu \supset Xu$.

Also by (b) $Au \vdash Xu \supset Nu$

so $Au \vdash Xu \backsim Nu$

$\therefore$ $\vdash Au \supset_u Xu \backsim Nu$

i.e. $\vdash Q_1XN$. □

The following restricted form of this theorem will be all we need in many cases:

COROLLARY. *If* (a) *and* (c) *hold then* $Nu \vdash Xu$.

We now look at some further basic properties of the natural numbers as we have defined them.

THEOREM 4. $Nx \vdash N(Q_1x)$.

Proof. Nx, $FAHv$, $[vo \wedge Ay \supset_y vy \supset v(Q_1y)] \vdash vx$.

By Theorem 2,

$$Nx \vdash Ax,$$

$\therefore$ Nx, $FAHv$, $[vo \wedge Ay \supset_y vy \supset v(Q_1y)] \vdash v(Q_1x)$.

Thus as $FAHv \vdash H[vo \wedge Ay \supset_y vy \supset y(Q_1y)]$

$Nx \vdash FAHv \supset_v [vo \wedge Ay \supset_y vy \supset v(Q_1y)] \supset v(Q_1x)$

i.e. $Nx \vdash N(Q_1x)$.

THEOREM 5. $Nx \vdash x =_1 0 \vee \Sigma A(\lambda z.x =_1 Q_1z \wedge Nz)$.

Proof. Let $X = \lambda x.x =_1 0 \vee \Sigma A(\lambda z.x =_1 Q_1z \wedge Nz)$

then $\vdash FAHX$ (3)

Now as $\vdash FAH0$,

$Az \vdash FAH(Q_1 z)$,

and $Az \vdash H(Nz)$

$\vdash H(\Sigma A(\lambda z \cdot 0 =_1 Q_1 z \wedge Nz))$.

Also $\vdash A0$

$\therefore$ $\vdash 0 =_1 0$

and so $\vdash X0$. (4)

Now $Ax \vdash x =_1 0 \supset Q_1 x =_1 Q_1 0 \wedge N0$,

so $Ax \vdash x =_1 0 \supset \Sigma A(\lambda z \cdot Q_1 x =_1 Q_1 z \wedge Nz)$. (5)

Also $Ax, Az, x =_1 Q_1 z \wedge Nz \vdash Q_1 x =_1 Q_1(Q_1 z)$

and by Theorem 4, $Ax, Az, x =_1 Q_1 z \wedge Nz \vdash N(Q_1 z)$

so also by Theorem 2, $Ax, x =_1 Q_1 z \wedge Nz \vdash \Sigma A(\lambda w \cdot Q_1 x =_1 Q_1 w \wedge Nw)$

$\therefore$ $Ax \vdash \Sigma A(\lambda z \cdot x =_1 Q_1 z \wedge Nz) \supset \Sigma A(\lambda z \cdot Q_1 x =_1 Q_1 z \wedge Nz)$ (6)

$\therefore$ by (5) and (6)

$$Ax \vdash Xx \supset X(Q_1 x)$$

and using (4) $\vdash X0 \wedge Ax \supset_x Xx \supset X(Q_1 x)$.

Thus by (3) and the Corollary to Theorem 3,

$$Nx \vdash x =_1 0 \vee \Sigma A(\lambda z \cdot x =_1 Q_1 z \wedge Nz).$$

THEOREM 6. $Nx \vdash Au \supset_u xu \supset \Sigma A(\lambda z \cdot Q_1 zu \wedge Nz)$.

Proof. $Nx, Au, xu \vdash \Gamma(x =_1 0)$

$\therefore$ $Nx, Au, xu \vdash \Sigma A(\lambda z \cdot x =_1 Q_1 z \wedge Nz)$

$Nx, Au, xu \vdash \Sigma A(\lambda z \cdot Q_1 zu \wedge Nz)$. □

If we have the following axiom of extent (mentioned in [3]):

(E2) $\vdash Au \supset_u Az \supset_z FAHt \supset_t Q_1 zu \supset tz \backsim tu$.

We also have as $\vdash FAHN$:

THEOREM 7. $Nx \vdash Au \supset_u xu \supset Nu$.

Note that this axiom for the weakest form of substitution for equality is suggested in [3]. The first rule:

$$Q_1 zu,\ tz \vdash tu$$

leads to anomalous (though not inconsistent results). The second, the axiom:

(E1) $\vdash FAHu \supset_u FAHz \supset_z FAHt \supset_t Q_1 zu \supset tz \backsim tu$

gives $\vdash RR \backsim RR$

where R is the Russell class (which is a *set* in this paper). We also have $RR = \Gamma(RR)$ and so by Eq, $\vdash RR \leftrightsquigarrow \Gamma(RR)$, but as we do not have $\vdash AR$ and hence not $\vdash H(RR)$ we cannot prove a contradiction from this anomalous result.

The third possibility, the axiom (E2) mentioned above seems to be free of anomalies and is sufficient for what we require above.

Note also that Theorem 7 says that N is a transitive set.

Going back now to arithmetical properties in the more general form we find that we can easily prove:

THEOREM 8. (i) $Nz, Nx, Ny \vdash x =_1 y \supset (x =_1 z \supset y =_1 z)$

(ii) $Nx, Ny \vdash x =_1 y \supset Q_1 x =_1 Q_1 y$

(iii) $Nx \vdash \Gamma(0 =_1 Q_1 x)$

(iv) $Nx, Ny \vdash Q_1 x = Q_1 y \supset x =_1 y$.

Proof. By Theorem 2, (A)

$$FAHx,\ FAHy \vdash H(x =_1 y)$$

and

$$FAHx \vdash FAH(Q_1 x)$$

it is easy to show that on the basis of the assumptions all the formulas to the right of $\vdash$ are propositions; (i) then follows by the definition of Q_1 and (ii) and (iv) follow directly from (i); (iii) follows by the definitions of Q_1 and 0.□

The four parts of Theorem 8 correspond to the first four Peano type axioms given by Mendelson [8]. To prove the next one:

$$Nx \vdash x + 0 =_1 x,$$

however seems to be impossible with + defined contextually as it is.

The alternative is to define addition (and also multiplication) by recursion. The recursion operator has been defined in [6] in terms of an ordered pair operator, which is also defined in [6] and a predecessor function which is also definable in terms of combinators. We can however define the predecessor relation in terms of terms definable using Ξ, A and H, in a much simpler fashion.

DEFINITION. $[\pi] = \lambda z\ \lambda x \cdot \Sigma A(\lambda y \cdot yx \wedge zy)$.

We can then prove:

THEOREM 9. (i) $FAHz \vdash FAH([\pi] z)$

(ii) $FAHt \vdash [\pi]\ (Q_1 t) =_1 t$

(iii) $\vdash [\pi] 0 =_1 0$.

Proof. (i) By (A), $FAHz,\ Ax,\ Ay \vdash H(yx \wedge zy)$

so $$FAHz,\ Ax, \vdash H(\Sigma A(\lambda y \cdot yx \wedge zy))$$

i.e. $$\mathrm{FAH}z \vdash \mathrm{FAH}([\pi]z).$$

(ii) $[\pi](Q_1t) = \lambda x\cdot\Sigma A(\lambda y\cdot yx \wedge Q_1ty)$, so by (A) and Theorem 2

$$Ay, Ax \vdash H(yx)$$

and $$\mathrm{FAH}t, Ay, Ax \vdash H(Q_1ty)$$

∴ $$\mathrm{FAH}t, Ax, [\pi](Q_1t)x \vdash tx$$

i.e. $$\mathrm{FAH}t, Ax \vdash [\pi](Q_1t)x \supset tx.$$

Also $$Ax, tx, \mathrm{FAH}t \vdash tx \wedge Q_1tt,$$

so $$Ax, tx, \mathrm{FAH}t \vdash \Sigma A(\lambda y\cdot yx \wedge Q_1ty),$$

and so $$Ax, \mathrm{FAH}t \vdash tx \supset [\pi](Q_1t)x.$$

∴ $$Ax, \mathrm{FAH}t \vdash [\pi](Q_1t)x \leftrightsquigarrow tx$$

so $$\mathrm{FAH}t \vdash Ax \supset_x [\pi](Q_1t)x \leftrightsquigarrow tx$$

i.e. $$\mathrm{FAH}t \vdash [\pi](Q_1t) =_1 t.$$

(iii) $[\pi]Ox = \Sigma A(\lambda y\cdot yx \wedge Oy)$.

But $$\vdash Ay \supset_y \Gamma(Oy)$$

so $$\vdash Ax \supset_x \Gamma([\pi]Ox)$$

and so $$\vdash Q_1([\pi]O)O. \quad \square$$

The ordered pair operator D can now be defined by:

DEFINITION D. $D = \lambda x\lambda y\lambda z\lambda u.(\Gamma(\Sigma Az) \supset xu) \wedge (\Sigma Az \supset y([\pi]x)u)$.

To prove all the expected results for this however, we need the following stronger form of (E2), which is however still weaker than (E1) and seems to avoid the anomalies mentioned earlier.

AXIOM (B). $\vdash Au \supset_u \mathrm{FAH}z \supset_z \mathrm{FAH}t \supset_t Q_1zu \supset tu \supset tz.$

This gives in particular:

THEOREM 10. (i) $Au, \mathrm{FAH}z, Q_1zu \vdash Az$
(ii) $Nu, \mathrm{FAH}z, Q_1zu \vdash Nz$,

and also the basic properties of D:

THEOREM 11. (i) $F_2(\mathrm{FAH})\mathrm{AH}y, \mathrm{FAH}x \vdash F_2(\mathrm{FAH})\mathrm{AH}(Oxy)$
(ii) $\mathrm{FAH}x, F_2(\mathrm{FAH})\mathrm{AH}y \vdash Dxy0 =_1 x$
(iii) $\mathrm{FAH}x, F_2(\mathrm{FAH})\mathrm{AH}y, At \vdash Dxy(Q_1t) =_1 yt.$

Proof. (i) By Theorem 9 (i) and (A),

$$F_2(\mathrm{FAH})\mathrm{AH}y, Au, \mathrm{FAH}x \vdash H(y([\pi]x)u)$$

also $$\mathrm{FAH}z \vdash H(\Sigma Az),$$

so $$F_2(\mathrm{FAH})\mathrm{AH}y, Au, \mathrm{FAH}x, \mathrm{FAH}z, \vdash H(Dxyzu)$$

∴ $$F_2(\mathrm{FAH})\mathrm{AH}y, \mathrm{FAH}x \vdash F_2(\mathrm{FAH})A(Dxy)$$

(ii) $Au, FAHx, F_2(FAH)AHy,\ DxyOu \vdash xu$

and by (i)

$$FAHx,\ F_2(FAH)AHy,\ Au \vdash DxyOu \supset xu.$$

Also $\quad FAHx,\ F_2(FAH)AHy,\ Au \vdash xu \supset DxyOu$

so $\quad FAHx,\ F_2(FAH)AHy \vdash Au \supset_u xu \backsim DxyOu$

i.e. $\quad FAHx,\ F_2(FAH)AHy \vdash Dxy0 =_1 x.$

(iii) As $\quad At \vdash Q_1tt$

$$At \vdash \Sigma A(Q_1t) \tag{7}$$

$$\therefore \quad Au, FAHx,\ F_2(FAH)AHy, At, Dxy(Q_1t)u \vdash y([\pi](Q_1t))u. \tag{8}$$

Now as $\quad Ar \vdash FAHr$

we have $\quad F_2(FAH)AHy, Ar, Au \vdash H(yru)$

i.e. $\quad F_2(FAH)AHy, Ar, Au \vdash H(Cyur)$

and $\therefore \quad F_2(FAH)AHy,\ Au \vdash FAH(Cyu).$

By (A) and Theorem 9 (i) $\quad At \vdash FAH([\pi](Q_1t))$

and by Theorem 9 (ii) $\quad At \vdash [\pi](Q_1t) =_1 t$

$\therefore$ by (B) and (S):

$$Au, FAHx, F_2(FAH)AHy, At, Dxy(Q_1t)u \vdash ytu.$$

Similarly using (B)

$$Au, FAHx,\ F_2(FAH)AHy, At, ytu \vdash y([\pi](Q_1t))u$$

so by (7),

$$Au, FAHx,\ F_2(FAH)AHy,\ At, ytu \vdash Dxy(Q_1t)u,$$

and the result can be proved. □

Now we define the recursion operator:

DEFINITION R. $R = \lambda x\lambda y.\mathbf{Y}(B(Dx)(Sy))$.

A result such as that in parts (i) of Theorems 9 and 11, about the funcionality (or type) of R seems to be impossible. We have

$$Rxy = Dx(Sy(Rxy)),$$

so to determine, for given x and y, the type of Rxy from that of D we need first the type of Rxy.

Alternatively, we need to know a type for $\mathbf{Y}$; however, this is known to be not derivable from the types for K and S. (These "types" in fact constitute the two basic axioms of the kind of system that we are dealing with - they allow the proof of the deduction theorem for Ξ - see [1], and for a discussion of the relation between axioms and types see [6]).

We can however postulate a type for $\mathbf{Y}$ that does not conflict with those for K and S and which will lead to a type for R.

($\mathbf{Y}$) $\quad \vdash F\{F[F(FAH)(FAH)][F(FAH)(FAH)]\}[F(FAH)(FAH)]\mathbf{Y}.$

($\vdash F(FTT)TY$ is reasonable for any T as then $FTTZ \vdash T(YZ)$ and $FTTZ \vdash T(Z(YZ))$. Below we only need the above special case).

We can now prove:

THEOREM 12. (i) $F_2(FAH)(FAH)(FAH)y, FAHt, FAHx \vdash FAH(Rxyt)$

(ii) $F_2(FAH)(FAH)(FAH)y, FAHx \vdash Rxy0 =_1 x$

(iii) $F_2(FAH)(FAH)(FAH)y, FAHx, At \vdash Rxy(Q_1t) =_1 yt(Rxyt)$.

Proof. $F(FAH)(FAH)u, FAHv \vdash FAH(uv)$

and $F_2(FAH)(FAH)(FAH)y, FAHv \vdash F(FAH)(FAH)(yv)$

$\therefore$ $F_2(FAH)(FAH)(FAH)y, F(FAH)(FAH)u, FAHv \vdash FAH(yv(uv))$

and so $F_2(FAH)(FAH)(FAH)y, F(FAH)(FAH)u \vdash F_2(FAH)AH(Syu)$

$\therefore$ by Theorem 11 (i):

$F_2(FAH)(FAH)(FAH)y, F(FAH)(FAH)u, FAHx \vdash F_2(FAH)AH(Dx(Syu))$

and so $F_2(FAH)(FAH)(FAH)y, FAHx \vdash F[F(FAH)(FAH)][F_2(FAH)AH](BDx(Sy))$.

Now $F_2(FAH)AH = F(FAH)(FAH)$, so by (Y) and Definition R:

$F_2(FAH)(FAH)(FAH)y, FAHx \vdash F(FAH)(FAH)(Rxy)$

so the result follows.

(ii) As $\vdash FAH0$, we have by (i) that

$F_2(FAH)(FAH)(FAH)y, FAHx \vdash FAH(Rxy0)$.

Now
$$\begin{aligned} Rxy0 &= Y(B(Dx)(Sy))0 \\ &= B(Dx)(Sy)(Rxy)0 \\ &= Dx(Sy(Rxy))0. \end{aligned}$$

Now $F_2(FAH)(FAH)(FAH)y, FAHu \vdash F(FAH)(FAH)(yu)$

so by (i) $F_2(FAH)(FAH)(FAH)y, FAHx, FAHu \vdash FAH(yu(Rxyu))$

$\therefore$ $F_2(FAH)(FAH)(FAH)y, FAHx \vdash F_2(FAH)A(Sy(Rxy))$. (9)

So by Theorem 11 (ii)

$F_2(FAH)(FAH)(FAH)y, FAHx \vdash Dx(Sy(Rxy)0) =_1 x$,

and the result holds.

(iii) This holds by (9) and Theorem 11 (iii). □

Now we can define addition.

DEFINITION +. $+xy = x + y = Rx(KQ_1)y$.

THEOREM 13. (i) $FAHx \vdash x + 0 =_1 x$

(ii) $FAHx, Ay \vdash x + Q_1y =_1 Q_1(x+y)$

(iii) $Nx, Ny \vdash N(x+y)$.

Proof. (i) By (A), $FAHv, Aw \vdash H(Q_1vw)$

i.e. $FAHu, FAHv \vdash FAH(KQ_1uv)$

and so $\vdash F_2(FAH)(FAH)(FAH)(KQ_1)$ (10)

$\therefore$ By (A) and Theorem 12 (ii)

$$FAHx \vdash Rx(KQ_1)0 =_1 x$$

i.e. $$FAHx \vdash x+0 =_1 x.$$

(ii) By (10) and Theorem 12 (iii),

$$FAHx, Ay \vdash x+Q_1y =_1 KQ_1y(x+y)$$

so $$FAHx, Ay \vdash x+Q_1y =_1 Q_1(x+y).$$

(iii) Theorem 12 (i) $\vdash$ FAH(x+0), so by (i) and Theorem 10 (ii)

$$Nx \vdash N(x+0)$$

$\therefore$ $$\vdash Ax \supset_x Nx \supset N(x+0).$$

Now $Ax \supset_x Nx \supset N(x+y), Ax, Nx \vdash N(x+y)$

so by Theorem 4,

$$Ax \supset_x Nx \supset N(x+y), Ax, Nx \vdash N(Q_1(x+y)),$$

also then $$Ax \supset_x Nx \supset N(x+y), Ax, Nx \vdash A(Q_1(x+y)).$$

Now $$Ay \vdash FAH(Q_1y)$$

so by Theorem 12 (i) $\quad Ay, FAHx \vdash FAH(x+Q_1y)$

$\therefore$ by (ii) and Theorem 10 (ii)

$$Ax \supset_x Nx \supset N(x+y), Ax, Ay, Nx \vdash N(x+Q_1y)$$

so $$\vdash Ay \supset_y [Ax \supset_x Nx \supset N(x+y)] \supset [Ax \supset_x Nx \supset N(x+Q_1y)].$$

So by the corollary to Theorem 3

$$Ny \vdash Ax \supset_x Nx \supset N(x+y)$$

so $\quad Nx, Ny \vdash N(x+y).$ □

Clearly (i) and (ii) of this theorem have as special cases:

$$Nx \vdash x+0 =_1 x$$

$$Nx, Ny \vdash x+Q_1y =_1 Q(x+y).$$

We now define the multiplication:

DEFINITION X. $\quad Xxy = x\cdot y = R0(K(+x))y.$

THEOREM 14. (i) $FAHx \vdash x\cdot 0 =_1 0$

(ii) $FAHx, Ay \vdash x\cdot Q_1y =_1 x + x\cdot y$

(iii) $Nx, Ny \vdash N(x\cdot y)$

Proof. (i) By Theorem 12 (i), $\quad FAHx, FAHy \vdash FAH(x+y)$

so $$FAHu, FAHx, FAHy \vdash FAH(K(+x)uy) \tag{11}$$

$\therefore$ $$FAHx \vdash F_2(FAH)(FAH)(FAH)(K(+x))$$

then by Theorem 12 (ii) $\quad FAHx \vdash R0(K(+x))0 =_1 0$

so (i) follows.

(ii) By (11) and Theorem 12 (iii)

$$FAHx, Ay \vdash x\cdot(Q_1y) =_1 K(+x)y(x\cdot y)$$

so $$FAHx, Ay \vdash x\cdot(Q_1y) =_1 x+x\cdot y.$$

(iii) By (i) and Theorem 10 (ii)

$$Nx \vdash N(x\cdot 0)$$

so

$$\vdash Ax \supset_x Nx \supset N(x\cdot 0),$$

$$Ax \supset_x Nx \supset N(x\cdot y), Ax, Nx \vdash N(x\cdot y)$$

∴ by Theorem 13 (iii)

$$Ax \supset_x Nx \supset N(x\cdot y), Ax, Nx \vdash N(x\cdot y + x).$$

Now by Theorem 12 (i), (2) and Definition X

$$FAHx, FAHy \vdash FAH(x\cdot Q_1 y)$$

∴ by Theorem 10 (ii) and (ii)

$$Ax \supset_x Nx \supset N(x\cdot y), Ax, Nx, Ay \vdash N(x\cdot Q_1 y)$$

∴

$$\vdash Ay \supset_y [Ax \supset_x Nx \supset N(x\cdot y)] \supset [Ax \supset_x Nx \supset_x N(x\cdot Q_1 y)]$$

so by the corollary to Theorem 3

$$Ny \vdash Ax \supset_x Nx \supset N(x\cdot y)$$

∴

$$Nx, Ny \vdash N(x\cdot y).\ \square$$

Thus given the extra axioms (B) and (Y) which we have had to introduce, we can develop all the Peano type axioms of [8], and hence Mendelson's development of formal number theory can be carried out here.

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