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STUDIES IN PARACONSISTENT LOGIC II: QUANTIFIERS AND THE UNITY OF OPPOSITES

Newton C.A. da Costa and Robert G. Wolf

ABSTRACT. In this paper, the propositional logics introduced in a previous work (N.C.A. da Costa and R.G. Wolf, *Studies in paraconsistent logic I: the dialectical principle of the unity of opposites*, Philosophia 9(1980),pp.189-217) are extended to first-order predicate calculi. Our aim is to formalize certain aspects of dialectics, as they are interpreted by McGill and Parry (V.J. McGill and W.T. Parry, *The unity of opposites: a dialectical principle*, Science and Society 12(1948),pp.418-444).

In da Costa and Wolf 1980, we constructed a sentential calculus DL whose purpose was to formalize the dialectical principle of the unity of opposites, as that principle has been interpreted by McGill and Parry. As we insisted, such a sentencial logic is only a first step toward richer, more philosophically useful logics. Here we plan to extend DL (note, not the second system DL* also formulated in da Costa and Wolf 1980) to a first-order predicate logic DL^Q and show that motivations have not been sacrificed in the move to DL^Q. We shall also indicate how DL^Q can be extended, in a similar way as DL was previously, to DL^{Q*}.

We shall assume that our previous paper is available and will not repeat the motivating remarks we gave there nor some of the more easily adapted technical results. This paper will therefore be more straighforwardly technical, but such technicalities are, we feel, vital to the enterprise. Before moving on to such technical aspects, we would like however to remark that this (and the previous) paper is meant also to show the value of paraconsistent logics -those logics intended to formalize non-trivial inconsistent theories- in treating philosophical problems. Paraconsistent logics are as yet too little known or appreciated within the logical community. Hopefully, successful application of such logics will help change that situation.

1. THE FORMALIZATION OF DL^Q AND SOME METATHEOREMS.

DL^Q has the following primitive symbols. 1- The connectives: $\supset, \wedge, \vee$, and ^{o}. 2- The quantifiers: $\forall$ (for all) and $\exists$ (there exists). 3- Individual variables: an infinitely denumerable set of individual variables which we do not need to specify. 4- Three disjoint sets of individual constants, $\underset{\sim}{A}$, $\underset{\sim}{B}$ and $\underset{\sim}{C}$ such that $\underset{\sim}{A} \cup \underset{\sim}{B} \cup \underset{\sim}{C} = \underset{\sim}{D} \neq \emptyset$. 5- Three disjoint non-void sets, $\underset{\sim}{A}'$, $\underset{\sim}{B}'$ and $\underset{\sim}{C}'$, containing constant predicate symbols of any rank n, $0 < n < \omega$. 6- For every n, $0 < n < \omega$, an infinite denumerable set of predicate variables of rank n. 7- Parentheses. The individual variables and constants are called *terms*.

The common syntactical notions, for example those of formula, proof, the symbol $\vdash$, and deduction, are introduced as usual. The letters A, B and C, with or without subscripts, will be employed as metalinguistic variables for formulas; x, y and z, with or without subscripts, will denote individual variables; a, b and c are syntactical variables for individual constants; t will denote any term. The symbol of equivalence, $\equiv$, is introduced in the usual way. The metalinguistic abbreviations of implication and of equivalence are respectively $\Rightarrow$ and $\Leftrightarrow$.

DL^Q is an extension of DL, so we shall assume the axiom schemata given for DL in da Costa and Wolf 1980 (note that we are using schemata). To get DL^Q, we add the following schemata and rules, which are subjected to the standard restrictions:

A18. $C \supset A(x) / C \supset \forall x A(x)$

A19. $\forall x A(x) \supset A(t)$

A20. $A(t) \supset \exists x A(x)$

A21. $A(x) \supset C / \exists x A(x) \supset C$

A22. $\forall x (A(x))^{o} \supset (\forall x A(x))^{o}$

A23. $\forall x (A(x))^{o} \supset (\exists x A(x))^{o}$

A24. If A and B are congruent formulas in the sense of Kleene 1952, p.153, or one is obtained from the other by suppression of vacuous quantifiers, then $A \equiv B$ is an axiom.

As before A22 and A23 insure that the stability operador ^{o} makes well-behaved formulas obey the laws of classical logic.

Theorem 1 of da Costa and Wolf 1980 generalizes to this new context.

THEOREM 1. *All schemata and rules of classical positive predicate logic are valid in* DL^Q.

Proof. Consequence of the postulates of DL^Q.

In the next theorem, some notations of Kleene 1952 are employed:

THEOREM 2. *If* A, B, C_A *and* C_B *are formulas satisfying the conditions of theorem 14 of Kleene 1952, pp. 151-152, we have:* 1- *If the occurence of* A *in* C_A *is not within the scope of an occurrence of* ¬ *or of* ^{o}, *then:* $A \equiv B \vdash^{x_1,\ldots,x_n} C_A \equiv C_B$; 2- *If the prime componentes of* A, B, C *are* $A_1, A_2, \ldots A_k$, *then:* $A_1^o, A_2^o, \ldots, A_k^o$, $A \equiv B \vdash^{x_1,\ldots,x_n} C_A \equiv C_B$.

Proof. As in Kleene 1952: the postulate of DL^Q are selected partly so that this theorem would hold.

Theorem 3 of da Costa and Wolf 1980 also generalizes to this new context:

THEOREM 3. *Let* $\Gamma \cup \{A\}$ *be a set of formulas of* DL^Q, *in which* ^{o} *does not occur, and whose prime components are* $A_1, A_2, \ldots, A_n$. *Then* $\Gamma \vdash A$ *in the classical predicate calculus iff* $\Gamma, A_1^o, A_2^o, \ldots, A_n^o \vdash A$ *in* DL^Q.

It seems evident that theorem 3 can be generalized to cope with the case in which the formulas of $\Gamma \cup \{A\}$ belong to DL^Q, without any restrictions on the formulas.

THEOREM 4. DL^Q *is undecidable.*

Proof. Consequence of theorem 3 and of Church's result that the classical predicate calculus is undecidable.

We can in an obvious way introduce strong negation ~ into DL^Q just as was done with DL. Then the corollary to theorem 7 of da Costa and Wolf 1980 also generalizes to this new situation:

THEOREM 5. *In* DL^Q, *the symbols* $\supset, \wedge, \vee, \sim, \forall$ *and* $\exists$ *satisfy all the postulates of the classical predicate calculus. In particular, the following are provable:*

(i) $\vdash (A \supset B) \supset ((A \supset {\sim}B) \supset {\sim}A)$

(ii) $\vdash A \vee {\sim}A$

(iii) $\vdash {\sim}A \supset (A \supset B)$

(iv) $\vdash {\sim}{\sim}A \equiv A$

(v) $\vdash \forall x A(x) \equiv {\sim}\exists x {\sim}A(x)$

(vi) $\vdash \exists x A(x) \equiv {\sim}\forall x {\sim}A(x)$

(vii) $\vdash \forall x \forall y A \equiv \forall y \forall x A$

(viii) $\vdash \forall x \forall y A(x,y) \supset \forall z A(z,z)$

(ix) $\vdash \forall z A(z,z) \equiv \exists x \exists y A(x,y)$

(x) $\vdash {\sim}\forall x \forall y \forall z A \supset \exists x \exists y \exists z {\sim}A$.

Proof. Left to the reader. We note that the reasons for having the classical predicate calculus interpretable as a subsystem of DL^Q are the same as having the classical setential calculus interpretable as a subsystem of DL.

The concept of a k-*transform* of a formula (cf. Kleene 1952, p.178) is easily extended to the classical predicate calculus with individual constants and also to DL^Q. Definition of such a notion is useful to prove a motivationally crucial conservative extension result.

THEOREM 6. *If* $\Gamma \vdash A$ *in* DL^Q, *then any* k-*transform of* A *can be deduced in* DL *from the* k-*transforms of the formulas of* Γ.

Proof. Similar to the classical one, but taking into account the fact that the prime propositional components of the formulas of DL will not be propositional symbols (variables or constants), but predicate symbols of rank n, followed by n occurrences of the symbols (numerals) 1,2,3,...,k ($0 < n < \omega$).

The next theorem is crucial.

THEOREM 7. DL^Q *is a conservative extension of* DL, i.e. *schemata not valid in* DL *are not valid in* DL^Q *either*.

Proof. Apply theorem 6.

The import of theorem 7 is that adding quantifiers to DL does not disturb the intuitions underlying DL. If DL^Q did not conservatively extend DL, then either DL^Q would verify a formula scheme, in the vocabulary of DL, which on the intuitions that we are assuming is a false theorem; or DL would have been poorly formulated as it left out a theorem which it could have contained, since it uses only the connectives of DL, and which is on the same dialectical intuitions a true theorem. In one case DL^Q would be branded false, since it would lead from true assumptions to false conclusions; in the other, DL would be at best inadequate, precisely in the area where we have claimed adequacy.

The way in which we have added the quantifiers is not the only possible way; indeed it might be valuable to try other options. The value of the approach taken here is that it makes DL^Q as close as possible to the classical predicate logic (as indicated, the classical predicate calculus is close to DL^Q in another way; it can be interpreted as a subsystem of DL^Q). For our purposes, this is good for two reasons: 1- it facilitates proving metatheorems and obtaining technical information about DL^Q; and 2- it isolates the intuitions which separate dialectical logic from classical logic from the intuitions underlying other issues in the philosophy of logic. We do not need here to fight intuitionistic, modal or relevant battles, though we may opt to do so elsewhere.

To pinpoint some of the significance of theorem 7, we note that the following schemata are *not* valid in DL^Q:

(xi) $A \wedge \neg A. \supset B$

(xii) $A \wedge \neg A. \supset \neg B$

(xiii) $\neg A \supset (A \supset B)$

(xiv) $A \supset (\neg A \supset B)$

(xv) $\neg A \supset (A \supset \neg B)$

(xvi) $A \supset (\neg A \supset \neg B)$

(xvii) $(A \supset B) \supset (\neg B \supset \neg A)$

(xviii) $\neg\neg A \equiv A$

(xix) $\neg(A \wedge \neg A)$

(xx) $A \wedge \neg A$

However DL^Q also deviates from classical predicate logic on some properly quantificational theorems.

THEOREM 8. *In* DL^Q, *the following schemata are* not *valid:*

(xxi) $\neg \exists x \neg A(x) \equiv \forall x A(x)$

(xxii) $\neg \forall x \neg A(x) \equiv \exists x A(x)$.

Proof. Consider the k-transforms of the above schemata and apply theorem 6.

It is important to realize that we *don't* want either (xxi) or (xxii) to be valid. If we consider cases where neither $A(x)$ or $\neg A(x)$ is applicable, then (xxi) and (xxii) should fail.

We now move on to the semantics for DL^Q, which, as we shall see, is a generalization of that for DL. For similar semantics for predicate calculi, see Arruda and da Costa 1977 and Alves and Moura 1978.

2. A SEMANTICS FOR DL^Q.

A sentence is a formula without free individual variables. In what follows, Γ and A will always denote respectively a set of sentences and a sentence.

DEFINITION 1. Let D be a nonvoid set. An interpretation of DL^Q in D is a function i which associates to each individual constant of DL^Q an element of D. The diagram language of DL^Q relative to D is denoted by DL^QD. (See Schoenfield 1967.) A valuation of DL^Q in D, having i as its base, is a function v of the set of sentences DL^QD on {0,1}, such that:

(1) v satisfies the conditions of a valuation of DL;

(2) $v(\forall x A(x)) = 1 \Leftrightarrow$ For every individual constant c of DL^QD, $v(A(c)) = 1$;

(3) $v(\exists x A(x)) = 1 \Leftrightarrow$ For some individual constant c of DL^QD, $v(A(c)) = 1$;
(4) $v(\forall x(A(x))^{o}) = 1 \Rightarrow v((\forall x A(x))^{o}) = v((\exists x A(x))^{o}) = 1$;
(5) If A and B are sentences satisfying the conditions of postulate A24, then $v(A) = v(B)$;
(6) For any individual constants of DL^QD, a and b, if $i(a) = i(b)$, then $v(A(a)) = v(A(b))$.

The valuation v *satisfies a sentence* A of DL^QD (and of DL) if $v(A) = 1$.

DEFINITION 2. Suppose that $\Gamma \cup \{A\}$ is a set of sentences of DL^Q and that v is any valuation; v is said to be a *model of* Γ if, for every element B of Γ, $v(B) = 1$. A is called a *semantic consequence of* Γ, if every model v of Γ is such that $v(A) = 1$. In this case, we write $\Gamma \vDash A$. If $\Gamma = \emptyset$, we write $\vDash A$, and A is said to be *valid*.

As in da Costa and Wolf 1980, these definitions are meant to be as unstartling as possible. By paralleling previously studied cases, especially the classical one, our proofs also parallel, and can be presented very quickly.

We now move to prove DL^Q sound and complete relative to the above semantics. As typical, the soundness half is immediate.

THEOREM 9. *In* DL^Q: $\Gamma \vdash A \Rightarrow \Gamma \vDash A$.

Proof. By induction on the length of a deduction of A from Γ.

We now turn to the completeness half, which is very close to the classical case, given the following definitions.

DEFINITION 3. Γ is said to be *trivial* if for every sentence A, $\Gamma \vdash A$; otherwise Γ is called *nontrivial*. Γ is said to be *inconsistent* if there is a sentence A such that $\Gamma \vdash A$ and $\Gamma \vdash \neg A$; otherwise Γ is *consistent*. Γ is said to be $\neg$-*incomplete* if there is a sentence A such that $\Gamma \nvdash A$ and $\Gamma \nvdash \neg A$; otherwise Γ is called $\neg$-*complete*. Γ is *maximally nontrivial* if it is nontrivial and is not properly contained in any nontrivial set.

DEFINITION 4. Γ is called a *Henkin set* if, for every formula A(x) having x as its sole free individual variable, there exists an individual constant c (of DL^Q) such that $\Gamma \vdash \exists x A(x) \supset A(c)$.

We now prove some crucial preparatory theorems.

THEOREM 10. *If* Γ *is a nontrivial (Henkin) set, it is contained in a max-*

imal nontrivial (Henkin) set.

Proof. Similar to the classical proof.

THEOREM 11. *Let* Γ *denote a nontrivial Henkin set; then, one has:*

(1) Γ *has all the properties of a maximal nontrivial set of* DL;

(2) $\forall x A(x) \in \Gamma \Leftrightarrow$ *For every individual constant* c (of DL^Q), $A(c) \in \Gamma$;

(3) $\exists x A(x) \in \Gamma \Leftrightarrow$ *There is an individual constant* c (of DL^Q) *such that* $A(c) \in \Gamma$;

(4) *If* A *and* B *are sentences as in postulate* A24, *then* $A \equiv B \in \Gamma$.

Proof. The classical proof is immediately adaptable to the present situation.

THEOREM 12. *Every (consistent or inconsistent; ¬-complete or ¬-incomplete) nontrivial Henkin set has a model.*

Proof. Analogous to the proof of the corresponding theorem for DL.

COROLLARY. *Any nontrivial set of sentences of* DL^Q *has a model.*

We have now proven everything necessary for completeness.

THEOREM 13. *In* DL^Q: $\Gamma \vDash A \Rightarrow \Gamma \vdash A$.

Proof. Again, analogous to the classical case.

COROLLARY. $\Gamma \vdash A \Leftrightarrow \Gamma \vDash A$.

Proof. See theorems 9 and 13.

As the model theory here is so close to the classical case, some results of the usual model theory can be extended to DL^Q. For example, though we shall not prove it, a Löwenheim-Skolem theorem for any denumerable Γ in DL^Q is: if Γ has a model, then Γ has a infinite denumerable model. Such results are of course of more than technical interest, as they indicate that we have enough control over DL^Q to use it in (later) applications, without worrying overmuch as to DL^Q behaving in a pathological manner, fouling up attempted proofs of interesting results. It is worth while to observe that our semantics is a generalization of classical semantics; in particular, Tarski's scheme T remains valid.

We prove one last, motivationally important theorem:

THEOREM 14. DL^Q *is consistent and nontrivial.*

Proof. Consequence of theorem 6 and of the fact that DL is consistent and nontrivial.

3. THE UNITY OF OPPOSITES, AND CONCLUSION.

We can (as with DL*) formulate the dialectical principle of the unity of opposities explicitly using the resources of DL^Q (and also use some of the symbols introduced way back at the beginning of the paper).

We can formulate two important forms of the principle of the unity of opposities (see da Costa and Wolf 1980) as follows:

First form (#5 of McGill and Parry 1948): If $a \in \underset{\sim}{A}$ and P is a unary constant predicate symbol of $\underset{\sim}{A}'$, Then:

A25'. $\neg(P(a) \vee \neg P(a))$.

Second form (#6 of McGill and Parry 1948): Suppose that $b \in \underset{\sim}{B}$ and that Q is a unary constant predicate symbol belonging to $\underset{\sim}{B}'$; under these conditions, we have:

A26'. $Q(b) \wedge \neg Q(b)$.

Finally in order to insure the existence of well-behaved formulas, we assume:

A27'. $\forall x_1 x_2 \ldots \forall x_n (R(x_1, x_2, \ldots, x_n))^o$,

where R is a constant predicate symbol of rank n, $0 < n < \omega$, belonging to $\underset{\sim}{C}$.

A25' and A26' can obviously be generalized in various ways. DL^{Q*} (DL + A25' + A26' + A27') is both inconsistent and nontrivial and presumably can be interpreted as a logic of vagueness without any real difficulty.

We conclude that DL^Q (and DL^{Q*}) is a well-motivated and technically well-behaved logic which answers to (some of) the intuitions behind a dialectical logic. Its development and investigation promises to be of interest to philosephers interested in dialectical philosophy. Extension of DL^Q to get a tensed dialectical predicate logic seems to offer no real difficulty.

On to tenses!

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University of São Paulo
Institute of Mathematics
São Paulo, SP, Brazil.
and
Department of Philosophical Studies
Southern Illinois University
Edwardsville, Illinois, U.S.A.