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SORTS OF HUGE CARDINALS

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In this note we consider some large cardinal properties related to huge cardinals. We establish some connections between these notions and the concepts of multihuge cardinals and superhuge cardinals introduced in [B.DP.T].

A cardinal κ is *huge* if there is an elementary embedding $j:V \to M$ of the universe V into a transitive model containing all the ordinals, such that κ is the critical point of j and M is closed under sequences of size $j(\kappa)$. If κ is huge and j is an embedding as described above, we say that $\lambda = j(\kappa)$ is a *target* for κ and denote this by $\kappa \to (\lambda)$. We use the notation $\kappa \underset{j}{\to} (\lambda)$ to make explicit which is the embedding under consideration. A cardinal κ is α *times huge* if there are cardinals $\lambda_0 < \lambda_1 < \ldots < \lambda_\xi < \ldots (\xi < \alpha)$ such that for each $\xi < \alpha$, $\kappa \to (\lambda_\xi)$. A cardinal κ is *superhuge* if it is α times huge for every ordinal α.

In the paper cited above it is shown that if the existence of a 2-huge cardinal is consistent (see for example [S.R.K]), then so is the existence of a superhuge cardinal (moreover, it is consistent that a stationarily superhuge cardinal exists).

The large cardinal properties considered in the present work are in between 2-huge and superhuge (*or stationarily superhuge*) in consistency strength. We also prove that, in this sense, stationarily superhugeness is strictly stronger than mere superhugeness.

1. A TARGET LIMIT OF TARGETS.

The proof of the consistency of a superhuge cardinal from the consistency of 2-huge cardinal [B.DP.T] indicates that it is enough to have a multihuge cardinal with a target which is a limit of targets to obtain the consistency of superhugeness, in fact:

THEOREM 1.1. *Let κ and λ be cardinals such that $\kappa \to (\lambda)$ and $\{\alpha < \lambda \mid \kappa \to (\alpha)\}$ is unbounded in λ. Then $V_\lambda \models$" κ is superhuge and the limit of superhuge car-*

dinals", *and* V_κ $\models$" *there are unboundedly many superhuge cardinals*".

Before proceeding to prove this theorem we need a lemma. We will denote by $\kappa \to (<\lambda)$ the fact that the targets of κ are unbounded below λ, $\kappa \to (\leq\lambda)$ means that $\kappa \to (<\lambda)$ and $\kappa \to (\lambda)$.

LEMMA 1.2. *If* $\kappa \to (<\lambda)$ *and* $\lambda \to (\gamma)$ *then* $\kappa \to (<\gamma)$.

Proof. The proof of the lemma is just routine; suppose $\kappa \underset{j}{\to} (<\lambda)$ and $\lambda \underset{k}{\to} (\gamma)$. Then, if $k:V \to M$, by elementarity M satisfies "*there are unboundedly many targets for* κ *below* γ", and by the closure properties of M this is true in V.

Proof of Theorem 1.1. Let λ be as in the statement and let $j:V \to M$ be such that $j(\kappa) = \lambda$ and ${}^{\lambda}M \subset M$. $\alpha < \lambda$ is a target for κ, i.e. $\kappa \to (\alpha)$, if and only if there is a normal ultrafilter on $[\alpha]^{\kappa}$. The inaccessibility of λ guarantees that this normal ultrafilter belongs to M, thus M $\models$" κ *has unboundedly many targets below* λ". From here it follows that the set $A = \{\alpha < \kappa | \alpha$ has unboundedly many targets below $\kappa\}$ is in the ultrafilter on κ generated by j; therefore, V_κ $\models$"*There are unboundedly many superhuge cardinals*". By the lemma 1.2, the set of targets of each element of A is unbounded below λ, and thus V_λ $\models$ "κ *is superhuge and limit of superhuge cardinals*".

COROLLARY 1.3. Con("*There is a cardinal* κ *with a target which is limit of targets for* κ") *implies* Con("*There are unboundedly many superhuge cardinals*") *and* Con(" *There is a superhuge cardinal limit of superhuge cardinals*").

In fact we have:

COROLLARY 1.4. *If* κ *is superhuge and has a target which is limit of targets of* κ *then there is a normal ultrafilter on* κ *concentrating on superhuge cardinals.*

We have thus seen that having a huge cardinal with a target limit of targets is consistency-wise stronger than superhugeness. Nevertheless, the local character of the former property implies that the first cardinal with this property is not above the first superhuge (suppose κ_1, the first huge with a target limit of targets, is above κ, the first superhuge, then it is enough to consider an elementary embedding associated to $\kappa \to (\lambda)$ for a λ bigger than the target limit of targets of κ_1, and apply the usual kind of argument. Corollary 1.4. above indicates that $\kappa_1 \neq \kappa$, so we have that if both cardinals exist, $\kappa_1 < \kappa$).

Moreover, the existence of a huge cardinal with a target limit of targets does not imply the existence of a superhuge cardinal. Indeed, if κ_1 is the first such cardinal and λ is its first target limit of targets, then let γ be the first inaccessible cardinal above λ, then $V_\gamma \models$ "κ *has a target limit of targets and there is no superhuge cardinal*". Nevertheless, the existence of a cardinal κ with a target limit of targets, does imply multihugeness.

PROPOSITION 1.5. *κ has a target limit of targets strongly implies that κ is many times huge. More precisely, there is a normal ultrafilter on κ concentrating on cardinals with a huge collection of targets.*

Proof. Let $\kappa \to (\leqslant\lambda)$ and let $j:V \to M$ be the embedding associated to $\kappa \to (\lambda)$. Then $M \models$ "$\kappa \to (<\lambda)$". Thus $\{\alpha < \kappa | \alpha \to (<\kappa)\}$ belongs to the normal ultrafilter induced by j on κ.

2. A STATIONARY COLLECTION OF TARGETS.

In this section, we show that the concept of stationarily superhugeness is strictly stronger than that of superhugeness.

PROPOSITION 2.1. *Let κ be a cardinal with a set of targets stationary below a regular cardinal λ. Then κ has a target which is a limit of targets. The converse is not true.*

Proof. Let $A \subseteq \lambda$ be the set of targets of κ below λ. And let $B = (A)$ = the set of limits of elements of A. The set B is closed and unbounded (below λ) so there is $\gamma \in A \cap B$. The cardinal γ is a target limit of targets of κ.

The converse is not true: just take V_γ where γ is the first strongly inaccessible above the first target limit of targets. In this model there is a target limit of targets but no stationary set of targets. Moreover, the existence of a cardinal with a target limit of targets does not imply the consistency of the existence of a cardinal with a stationary set of targets, because the preceding argument shows that Con("$\exists\kappa$ *with a stationary set of targets*") implies Con("$\exists\kappa$ *with a target limit of targets*"). So if a cardinal with a target limit of targets implies the consistency of a cardinal with a stationary set of targets we would have that the theory ZFC + "$\exists\kappa$ *with a target limit of targets*" implies its own consistency.

COROLLARY 2.2. *If κ is stationarily superhuge then there is a normal ultrafilter on κ concentrating on superhuge cardinals; moreover, there is a normal ultrafilter on κ concentrating on superhuge cardinals with a target limit of targets.*

Proof. The proof of Proposition 2.1 shows that κ has a stationary class of targets which are limits of targets. From this and corollary 1.4 we obtain that that there is a normal ultrafilter on κ concentrating on superhuge cardinals.

For the second part, suppose $\lambda_1 < \lambda_2$ are targets of κ which are limits of targets and let $j: V \to M$ be the elementary embedding associated with $\kappa \to (\lambda_2)$. We observe that $M \models$ "$\kappa \to (\lambda_1)$, λ_1 *is a limit of targets of* κ, *and* κ *has unboundedly many targets below* λ_2". i.e. $M \models$ "κ *has a target limit of targets and* $\kappa \to (<\lambda_2)$". Therefore the set $\{\alpha < \kappa \mid \alpha$ has a target limit of targets and $\alpha \to (<\kappa)\}$ belongs to the normal ultrafilter induced on κ by j. But by lemma 1.2, all these cardinals α are superhuge cardinals.

3. MANIFOLD HUGE CARDINALS.

DEFINITION. A sequence $\{\kappa_\alpha\}_{<\gamma}$ of cardinals is a γ-*fold sequence* if $\kappa_\alpha \to (\kappa_{\alpha+1})$ for all $\alpha, \alpha+1 < \gamma$ and $\kappa_\alpha \to (\kappa_\lambda)$ for all $\alpha < \lambda < \gamma$, λ a limit ordinal.

For cardinals $\kappa < \gamma$, we say that κ is α-*fold huge* (resp. $<\alpha$-*fold huge*) and γ is its α-*fold target* ($<\alpha$-*fold target*) if there is an $\alpha+1$-fold (α-fold) sequence $\{\kappa_\xi\}_{\xi<\alpha+1}$ ($\{\kappa_\xi\}_{\xi<\alpha}$) with $\kappa_0 = \kappa$ and $\kappa_\alpha = \gamma$ ($\bigcup_{\xi<\alpha} \kappa_\xi = \gamma$). We denote this by $\kappa \overset{\alpha}{\to} (\gamma)$ ($\kappa \overset{<\alpha}{\to} (\gamma)$).

The various gradations of κ-fold hugeness form a hierarchy between 2-huge cardinals and those discussed above. The following chart summarizes this ordering. The symbol " $- \Rightarrow \ldots$" indicates that Con(ZFC+ $-$) implies Con(ZFC+...) but not the reverse. The numbers refer to the proofs that follow.

κ *is 2-huge* $\overset{1}{\Rightarrow}$ $\exists\gamma(\kappa \overset{\gamma}{\to} (\gamma))$ $\overset{2}{\Rightarrow}$ $\exists\gamma(\kappa \overset{<\gamma}{\to} (\gamma))$

$\overset{3}{\Rightarrow}$ κ *is* α-*fold huge for all* α

$\overset{4}{\Leftrightarrow}$ κ *is super* α-*fold huge for all* α

$\overset{10}{\Rightarrow}$ κ *is super* κ-*fold huge* $\overset{6}{\Rightarrow}$ κ *is* κ-*fold huge*

$\overset{5}{\Rightarrow}$ κ *is super* $<\kappa$-*fold huge* $\overset{6}{\Rightarrow}$ κ *is* $<\kappa$-*fold huge*

$\overset{5}{\Rightarrow}$ κ *is super* α-*fold huge for all* $\alpha < \kappa$

$\overset{6}{\Rightarrow}$ κ *is* α-*fold huge for all* $\alpha < \kappa$ $\overset{7}{\Rightarrow}$ ...

$\overset{7}{\Rightarrow}$ κ *is super* β-*fold huge* $\overset{6}{\Rightarrow}$ κ *is* β-*fold huge*... (for $\beta < \alpha$)

and for λ a limit ($\lambda < \beta$):

$\overset{7}{\Rightarrow}$ κ *is super* λ-*fold huge* $\overset{6}{\Rightarrow}$ κ *is* λ-*fold huge*

$\overset{5}{\Rightarrow}$ κ *is super* $<\lambda$-*fold huge* $\overset{6}{\Rightarrow}$ κ *is* $<\lambda$-*fold huge*

$\overset{5}{\Rightarrow}$ κ *is super* α-*fold huge for all* $\alpha < \lambda$

$\overset{6}{\Longrightarrow}$ κ *is* α*-fold huge for all* $\alpha < \lambda \Longrightarrow \ldots$

$\overset{7}{\Longrightarrow}$ κ *is 2-fold huge* $\overset{8}{\Longrightarrow}$ κ *is stationarily superhuge*

$\overset{9}{\Longrightarrow}$ κ *has a target the limit of targets*

$\overset{9}{\Longrightarrow}$ κ *is superhuge, the limit of superhuge cardinals*

$\overset{6}{\Longrightarrow}$ *there are unboundedly many superhuge cardinals*

$\overset{6}{\Longrightarrow}$ κ *is superhuge.*

In general, the prefix "*super*" indicates unboundedly many targets of the relevant sort, e.g. "κ *is super* $<\kappa$*-fold huge*" means that for unboundedly many γ, $\kappa \overset{<\kappa}{\to} (\gamma)$. First a simple lemma:

LEMMA 3.1. *If* $\{\kappa_\alpha\}_{\alpha<\gamma}$ *is a* γ*-fold sequence, then* $\kappa_\alpha \to (\kappa_\beta)$ *for all* $\alpha < \beta < \gamma$.

Proof. It is true by definition for $\beta = \lambda$ a limit ordinal and, by induction on n, it is true for $\lambda+n$ since $\kappa \to (\delta)$ and $\delta \to (\eta)$ implies $\kappa \to (\eta)$ (see [B.DP.T], theorem 2C).

Our proofs will now proceed in each case by showing that the first property implies the consistency of the second.

1. Let κ be 2-huge and $j:V \to M$ witness this fact. $M \models$ "$\kappa \to (j(\kappa))$", so if μ is the normal ultrafilter induced by j on κ, $X = \{\alpha < \kappa \mid \alpha \to (\kappa)\} \in \mu$. Define $G:[\kappa]^2 \to 2$ by $G(\{\alpha,\beta\}) = 0$ if and only if $\alpha \to (\beta)$. Let $Y \subseteq X$ be homogeneous for G with $Y \in \mu$. For $\alpha \in X$, since $\alpha \to (\kappa)$ is true in M, $\alpha \to (\beta)$ for a set of β's in the ultrafilter μ, so there is a $\beta \in Y$ such that $\alpha \to (\beta)$. Thus, $G''[Y]^2 = \{0\}$. So every $\alpha \in Y$ has the property $\alpha \overset{\kappa}{\to} (\kappa)$.

2. We use Lemma 3.1. Let $\kappa \overset{\gamma}{\to} (\gamma)$ and $j:V \to M$ be an embedding associated with the fact that $\kappa \to (\gamma)$. As $M \models$ "$\kappa \overset{<\gamma}{\to} (\gamma)$", there are unboundedly many $\alpha < \kappa$ such that $\alpha \overset{<\kappa}{\to} (\kappa)$.

3. Let $\kappa \overset{<\gamma}{\to} (\gamma)$ and $\{\kappa_\alpha\}_{\alpha<\gamma}$ be a γ-fold sequence with $\kappa_0 = \kappa$ and $\bigcup_{\alpha<\gamma} \kappa_\alpha = \gamma$. For $\alpha < \gamma$, let $\kappa_\alpha \underset{j_\alpha}{\to} (\kappa_{\alpha+1})$. Then $j_\alpha V \models$ "κ_α *is an* α*-fold target of* κ", so there are (in V) unboundedly many $\delta < \kappa_\alpha$ which are α-fold targets of κ. If $\kappa_1 \underset{k_\alpha}{\to} (\kappa_\alpha)$ (such a k_α exists by lemma 3.1), then $k_\alpha V \models$ "*there are unboundedly many* α*-fold targets of* κ *below* κ_α", and then (in V) there are unboundedly many $\delta < \kappa_1$ which are α-fold targets of κ. As this holds for all $\alpha < \gamma$, we have that $\forall\alpha < \kappa_1$ $\exists\lambda < \kappa_1(\kappa \overset{\alpha}{\to} (\lambda))$. So, by the closure properties of j_0V, this statement holds in j_0V. Hence the set $\{\beta < \kappa \mid \forall\alpha < \kappa \ \ \exists\lambda < \kappa(\beta \overset{\alpha}{\to} (\lambda))\}$ is in the ultrafilter induced by j_0 on κ. Therefore, V_κ is the required model of ZFC+"$\exists\beta$, β *is* α*-fold huge for all* α". (Notice that we only used $\kappa \to (\kappa_1)$ and $\kappa_1 \overset{\kappa_1}{\to} (\gamma)$).

4. Obviously the second property implies the first. Suppose that κ is α-fold huge for every α. Given ξ and η we want to show that there is an η-fold target of κ

above ξ. Indeed, if we take $\gamma > \max(\xi^+, \eta^+)$, as κ is γ-fold huge, the first γ-fold target of κ must be above ξ. Moreover, there is a $\gamma+1$-fold sequence for κ above ξ. But then, by Lemma 3.1, there is an $\eta+1$-fold sequence for κ above ξ.

5. Proofs of this sort follow a similar pattern. We prove, for example that κ is κ-fold huge $\Rightarrow$ κ is super $<\kappa$-fold huge. Let $\{\kappa_\alpha\}_{\alpha<\kappa+1}$ be a $\kappa+1$-fold sequence with $\kappa_0 = \kappa$. Claim: $V_{\kappa_1} \models$ "κ *is super* $<\kappa$*-fold huge*". If not, let $\beta < \kappa_1$ be the number of $<\kappa$-fold targets of κ below κ_1. Let $\kappa_1 \underset{j}{\to} (\kappa_\kappa)$; then $jV \models$ "κ *has at least* $\beta+1$ $<\kappa$*-fold targets below* κ_κ" hence (V satisfies) κ has at least $\beta+1$ $<\kappa$-fold targets below κ_1, a contradiction.

6. Routine. For example, if κ is super $<\kappa$-fold huge, let γ be an inaccessible above a $<\kappa$-fold target of κ. Then V_γ is our model.

7. In general, κ is β-fold huge $\Rightarrow$ κ is super α-fold huge for $\alpha < \beta$, by the method of 5 above.

8. We prove that κ is 2-fold huge strongly implies that κ has a stationary set of targets below a regular cardinal. By hypothesis there is a sequence $\{\kappa_0, \kappa_1, \kappa_2\}$ with $\kappa_0 = \kappa$, $\kappa \underset{j_1}{\vec{\to}} (\kappa_1)$ and $\kappa_1 \underset{j_2}{\vec{\to}} (\kappa_2)$. We have that $j_2V \models \kappa \to (\kappa_1)$, and therefore the set $S = \{\xi < \kappa_1 | \kappa \to (\xi)\}$ is a stationary subset of κ_1. Moreover, for each $\xi \in S$, $j_1V \models$ "$\kappa \to (\xi)$", and thus $j_1V \models$ "κ *has a stationary set of targets below* κ_1" (since all subsets of κ_1 belong to j_1V and if $j_1V \models$ "$A \subseteq \kappa_1$ *is closed and unbounded*", then A is really closed and unbounded below κ_1). From here we conclude that $\{\alpha < \kappa | \alpha$ has a stationary set of targets below $\kappa\}$ is in the ultrafilter induced by j_1 on κ.

9. Proved previously.

10. Trivial.

This ordering can be expanded further in several ways. First, for all properties not involving unboundedness, an additional property can be placed above by requiring superhugeness as well, e.g., between "κ *is super* $<\kappa$*-fold huge*" and "κ *is* $<\kappa$*-fold huge*" can be placed: "κ *is* $<\kappa$*-fold huge and superhuge*" (If $\{\kappa_\alpha\}_{\alpha<\kappa}$ is a $<\kappa$-fold sequence, let $\gamma, \lambda > \kappa$ be such that $\kappa \underset{j}{\to} (\gamma)$, $\gamma \to (\lambda)$, and $V_\gamma \models$ "κ *is* $<\kappa$*-fold huge*". Use the technique of 8, to show $jV \models$ "κ *has a stationary set of targets below* γ". Conclude that $\{\alpha < \kappa | V_\kappa \models$ "α *is superhuge and* α *is* $<\alpha$*-fold huge*"$\}$ is in the ultrafilter induced by j on κ). Second, for most properties, the existence of unboundedly many, or stationarily many cardinals are two ways of generating statements of greater consistency power. Similarly, one can consider the analogue of stationarily superhuge, for example, a cardinal κ with a stationary set of κ-fold targets.

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