THE COMPLETENESS AND COMPACTNESS OF A
THREE-VALUED FIRST-ORDER LOGIC

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ABSTRACT. The strong completeness and the compactness of a three-valued first order predicate calculus with two distinguished truth-values are obtained. The system was introduced in Sur un problème de Jaśkowski, I.M.L. D'Ottaviano and N.C. A. da Costa, C.R. Acad.Sc. Paris 270A (1970), pp. 1349-1353, and has several applications, especially in paraconsistent logics.

1. INTRODUCTION.

A theory T is said to be inconsistent if it has as theorems a formula and its negation; and it is said to be trivial if every formula of its language is a theorem.

A logic is paraconsistent if it can be used as the underlying logic for inconsistent but nontrivial theories.

Jaśkowski, motivated by some ideas of Łukasiewicz, was the first logician to construct a system of paraconsistent propositional logic (see [11], [12] and [13]). His principal motivations were the following: the problem of the systematization of theories which contain contradictions, as it occurs in dialectics; the study of theories in which there are contradictions caused by vagueness; the direct study of some empirical theories whose postulates or basic assumptions could be considered, under certain aspects, as contradictory ones (see [2] and [3]).

Jaśkowski proposed the problem of constructing a propositional calculus having the following properties:

i) an inconsistent system based on such a calculus should not be necessarily trivial;
ii) the calculus should be sufficiently rich as to make possible most of the usual reasonings;
iii) the calculus should have an intuitive meaning.

Jaśkowski himself introduced a propositional calculus which he named "Discussive logic" and which was a solution to the problem. However, he did recognize it was not the only solution (or even the best); in [11] he states:

"Obviously, these conditions do not univocally determine the solution, since they may be satisfied in varying degrees, the satisfaction of condition (iii) being rather difficult to appraise objectively".

In a previous paper (see [10]), we presented a propositional system, denoted by $J^*$, which is another solution to Jaśkowski's problem. A characteristic of $J^*$ is that it is a three-valued system with two distinguished truth-values. Furthermore, it reflects some aspects of certain types of modal logics.

In the same paper, we extended $J_3$ to the first-order predicate calculus with equality $J_{3**}$.

Some of these results about $J_3$ were improved by J. Kotas and N.C.A. da Costa (see [15]).

Our aim here is to develop further the calculus $J_3$.

In Sec. 2 we axiomatize $J_3$ and establish relations between this calculus and several known logical systems like, for example, intuitionism. We especially emphasize the close analogy between $J_3$ and Łukasiewicz' three-valued propositional calculus $L_3$.

Our solution to Jaśkowski's problem is discussed in the latter part of Sec. 2.

In Sec. 3 we introduce the $L_3$-Languages, among whose predicate symbols may appear in addition to identity other equalities. We axiomatize $J_3$-theories, which are three-valued extensions of $J_{3**}$, and we introduce a semantics for them.

In Sec. 4, after obtaining some theorems about first-order $J_3$-theories, we define a strong equivalence which is compatible with the fact that the matrices defining $J_3$ have more than one distinguished truth-value. This relation allows us to prove the Equivalence Theorems for $J_3$-theories and the Reduction Theorem for non-Trivialization.

Finally, in Sec. 5, after giving a suitable definition of canonical structure, we present a Henkin-type proof for the Completeness Theorem and the Compactness Theorem.

In this paper, definitions, theorems and proofs, when analogous to the corresponding classical ones, will be omitted.

The Model-theory we developed for $J_3$ allows us to obtain $J_3$-versions of the following classical results: Model Extension Theorem, Łoś-Tarski Theorem, Chang-Łoś Susko Theorem, Tarski Cardinality Theorem, Löwenheim-Skolem Theorem, Quantifier Elimination Theorem and many of the usual theorems on categoricity.

Some of the above results about $J_3$ were also extended to $J_n$-theories, $3 \leq n \leq \aleph_0$. 


The mentioned results about $J_n$-theories and Model-theory will appear elsewhere.

2. THE CALCULUS $J_3$.

The propositional calculus $J_3$ is given by the matrix $M = \langle\{0, h, 1\}, \{h, 1\}, \lor, \land, \neg\rangle$, where $\lor$, $\land$ and $\neg$ are defined as follows:

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The set of truth-values and the set of distinguished truth-values are denoted by $V$ and $V_d$ respectively.

The formulas of $J_3$ are constructed as usually from the propositional variables, by means of $\lor$, $\land$ and $\neg$ and parentheses. To write the formulas, schemas, etc. we use the conventions and notations of [14], with evident adaptations.

The concept of a truth-function is the usual one. The truth-functions defined by the tables above are denoted by $H_\lor$, $H_\land$, and $H_\neg$.

A truth-valuation $v$ for $J_3$ and the truth-value $v(A)$ for a formula $A$ are defined in the standard way; and we observe that $A$ is valid in $M$ if, for every evaluation $v$, $v(A)$ belongs to $V_d$ (see, for example, [22]).

The following abbreviations will be used:

$A \& B = \text{def} \neg(\neg A \lor \neg B)$
$\Delta A = \text{def} \neg \lor \neg A$
$\neg \neg A = \text{def} \neg \lor \neg A$
$A \Rightarrow B = \text{def} \lor \neg A \lor B$
$A \Rightarrow B = \text{def} (A \rightarrow B) \& (\neg B \rightarrow \neg A)$
$A \Rightarrow B = \text{def} \lor \lor B$
$A \equiv B = \text{def} (A \Rightarrow B) \& (B \Rightarrow A)$

$\neg$ is called weak negation or simply negation, $\neg \neg$ is called strong negation, and $\Rightarrow$ basic implication of $J_3$.

We present the tables of some of the non-primitive connectives:

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<tr>
<th>$A$</th>
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$A \Rightarrow B$

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In the following theorems, we mention only those results which are useful to the proofs of later theorems.

**THEOREM 2.1.** The following schemas of \( J_3 \) are valid in \( M \):

\[
\begin{align*}
\neg A &\equiv A \\
\neg (\neg A) &\Rightarrow \neg A \\
A &\lor (\neg A) \\
\neg (B \lor \neg B) &\equiv A \\
\neg (A \lor B) &\Rightarrow (\neg A \lor \neg B) \\
\neg (\neg A \lor \neg B) &\Rightarrow B \\
((A \Rightarrow B) \Rightarrow A) &\Rightarrow A \\
(A \Rightarrow B) &\Rightarrow (A \Rightarrow B) \\
(A \Rightarrow B) &\Rightarrow (\neg B \Rightarrow \neg A).
\end{align*}
\]

**THEOREM 2.2.** The following schemas are not valid in \( J_3 \):

\[
\begin{align*}
\neg A &\Rightarrow (A \Rightarrow B) \\
A &\Rightarrow (\neg A \Rightarrow B) \\
\neg A &\Rightarrow (\neg A \Rightarrow B) \\
\neg (\neg A \Rightarrow B) &\equiv A \\
A &\lor (\neg A \Rightarrow B) \\
A &\lor (\neg A \Rightarrow B) \\
A &\lor (A \Rightarrow B) \\
\neg (A \equiv \neg A) &\Rightarrow B \\
(A \equiv \neg A) &\Rightarrow B \\
(A \Rightarrow B) &\Rightarrow ((A \Rightarrow B) \Rightarrow \neg A).
\end{align*}
\]

It can be verified that, instead of \( \lor \), \( \Rightarrow \) and \( \neg \), it is possible to use only \( \neg \) and \( \Rightarrow \) as primitive connectives of \( J_3 \), considering \( A \Rightarrow B \) and \( \neg A \) as abbreviations respectively of \( (A \Rightarrow B) \Rightarrow B \) and \( \neg A \Rightarrow A \).
So, there is a close analogy between $J_3$ and Łukasiewicz' three-valued propositional calculus $L_3'$, defined by the matrix $M' = \langle \{0, \frac{1}{2}, 1\}, \{1, \wedge, \rightarrow\} \rangle$, in which the Łukasiewicz-Tarski operators $\wedge$ and $\rightarrow$ are given by the respective tables of $J_3$ (see [4]).

$J_3$ can be axiomatized by:

Axiom 1 : $\Delta(A \rightarrow (B \rightarrow A))$
Axiom 2 : $\Delta(((A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)))$
Axiom 3 : $\Delta((\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A))$
Axiom 4 : $\Delta(((A \rightarrow \neg A) \rightarrow A) \rightarrow A)$
Axiom 5 : $\Delta(\Delta(A \rightarrow B) \rightarrow \Delta(\Delta A \rightarrow \Delta B))$

Rule R1 : \[
\frac{A, \Delta(A \rightarrow B)}{B}
\]
Rule R2 : \[
\frac{\forall A}{A}
\]

The completeness theorem for $J_3$ is proved from the completeness of $L_3'$, due to Wajsberg (see [4] and [23]), using the following theorem.

THEOREM 2.3. If $A$ is a theorem of $L_3'$, then $\Delta A$ is a theorem of $J_3$.

Proof. As the axioms 1 to 4 are the axioms of $L_3'$ preceeded by $\Delta$, if $A$ is an axiom of $L_3'$, then $\Delta A$ is a theorem of $J_3$.

Let $A$ be obtained from $B$ and $B \rightarrow A$ by the rule $\frac{B, B \rightarrow A}{A}$ of $L_3'$. By induction hypothesis, $\Delta B$ and $\Delta(B \rightarrow A)$ are theorems of $J_3$. By axiom 5 and R1 we obtain $\Delta(\Delta B \rightarrow \Delta A)$. Applying R1, we have that $\Delta A$ is a theorem of $J_3$.

THEOREM 2.4. (Completeness theorem for $J_3$). A formula $A$ is a theorem of $J_3$ if and only if $A$ is valid in $M$.

Proof. A straightforward induction shows that if $A$ is a theorem of $J_3$, then $A$ is valid in $M$. On the other hand, if $A$ is valid in $M$, then $\forall v(\forall A) = 1$ for every truth-valuation $v$. By the axiomatization and completeness of $L_3'$, both $\forall A$ and $\Delta(\forall A \rightarrow \forall A)$ are theorems of $L_3'$. By the above theorem and R1, $\forall A$ is a theorem of $J_3$. By R2, $A$ is a theorem of $J_3$.

COROLLARY (Modus Ponens Rule). If both $A$ and $A \rightarrow B$ are theorems of $J_3$, then $B$ is a theorem of $J_3$.

However, contrary to $L_3'$, the Rule of Modus Ponens is not valid with respect to $\rightarrow$.

For some of the theorems that follow it will be convenient to assume that the language of $J_3$ contains, as primitive symbols, all the connectives introduced so far. In particular we shall often identify $J_3$ with the set of $M$-valid formulas in the expanded language.
The following theorems will be used in the proofs of many of the results about $J_3$.

THEOREM 2.5. $J_3$ is a non-conservative extension of the classical positive propositional calculus with connectives $\lor$, $\land$, $\supset$, $\equiv$.

THEOREM 2.6. $J_3$ is a conservative extension of the classical propositional calculus with connectives $\neg$, $\lor$, $\land$, $\supset$ and $\equiv$.

THEOREM 2.7. $J_3$ is a non-conservative extension of Łukasiewicz' three-valued logic $L_3$ with connectives $\neg$, $\rightarrow$.

THEOREM 2.8. $J_3$ is not functionally complete.

Proof. It is not possible to define a connective, from the primitive connectives of $J_3$, such that its truth-value is identically $\frac{3}{4}$.

On the other hand, if we add the Słupecki T operator to the primitive connectives of $J_3$, the calculus becomes functionally complete (see [21]).

By Theorem 2.4, the formulas $\forall A \supset (A \supset B)$, $A \supset (\forall A \supset B)$, $A \supset (\forall A \supset \forall B)$, $(A \land \forall A) \supset B$, $(A \supset B) \supset ((A \supset \forall B) \supset \forall A)$, $A \supset (B \land \forall B) \equiv A$, etc., are not theorems of $J_3$. So, in $J_3$, in general, it is not possible to deduce any formula whatsoever from a contradiction. Therefore, based on such a calculus we can construct nontrivial inconsistent deductive systems, in the sense of [11]. So, $J_3$ satisfies condition (i) of Jaśkowski's problem.

By Theorem 2.5 to 2.8, $J_3$ is quite a strong system, which evidently satisfies Jaśkowski's condition (ii).

$J_3$ admits intuitive interpretations. For instance, it can be used as the underlying logic of a theory whose preliminary formulation may involve certain contradictions, which should be eliminated in a later reformulation. This can be done as follows; among the truth-values of $J_3$, 0 can represent falsity, 1 truth, and $\frac{3}{4}$ can represent the provisional value of a proposition $A$, so that both $A$ and the negation of $A$ are theorems of the theory, in its initial formulation; in a later reformulation, the truth-value $\frac{3}{4}$ should be reduced, at least in principle, to 0 or to 1.

Therefore, $J_3$ is a solution to Jaśkowski's problem.

$J_3$ can also be used as a foundation for paraconsistent systems, in the sense of da Costa (see [5], [6], [7] and [8]). In this case, the value 0 represents falsity, 1 truth, and $\frac{3}{4}$ represents the logic value of a formula that is simultaneously true and false.
Finally, as the calculus $J_3$ was constructed from $L_3$, it is possible to obtain similar calculi $J_n$, from Łukasiewicz $n$-valued calculi $L_n$. $3 \leq n < \aleph_0$.

3. SEMANTICS FOR FIRST-ORDER $J_3$-THEORIES.

The symbols of a first-order $L_3$-language are the individual variables, the function symbols, the predicate symbols, the primitive connectives $\neg$, $\land$ and $\lor$, the quantifiers $\exists$ and $\forall$, and the parentheses.

The identity $=$ must be among the predicate symbols. Other equalities can be specified among the predicate symbols.

We use $x, y, z$ and $w$ as syntactical variables for individual variables; $f$ and $g$, for function symbols; $p$ and $q$, for predicate symbols, and $c$ for constants.

The definitions of term, atomic formula and formula are the usual ones; $a$, $b$, $c$, etc. are syntactical variables for terms and $A, B, C$, etc. for formulas.

By an $L_3$-language we understand a first-order language whose logical symbols include the ones mentioned above.

The symbols $\&$, $\lor$, $+$, $:>, =$, $1:1$ and $1^*$ are defined in the $L_3$-languages, as in $J_3$.

Free occurrence of a variable, open formula, closed formula, variable-free term and closure of a formula are used as in [22].

The definition of $a$ is substitutable for $x$ in $A$ is also the usual one.

We let $b[x_1,\ldots,x_n][a_1,\ldots,a_n]$ be the term obtained from $b$ by replacing all occurrences of $x_1,\ldots,x_n$ by $a_1,\ldots,a_n$ respectively; and we let $A[x_1,\ldots,x_n][a_1,\ldots,a_n]$ be the formula obtained from $A$ by replacing free occurrences of $x_1,\ldots,x_n$ by $a_1,\ldots,a_n$ respectively.

Whenever either of these is used, it will be implicitly assumed that $x_1,\ldots,x_n$ are distinct variables and that, in the case of $A[x_1,\ldots,x_n][a_1,\ldots,a_n]$, $a_i$ is substitutable for $x_i$, $i = 1,\ldots,n$.

In the following definitions, let $L$ be an $L_3$-language.

DEFINITION 3.1. A structure $\mathcal{A}$ for a first-order $L_3$-language $L$ consists of:

i) a nonempty set $|\mathcal{A}|$, called universe of $\mathcal{A}$;

ii) for each $n$-ary function symbol $f$ of $L$, a function $\mathcal{A}f$ from $|\mathcal{A}|^n$ to $|\mathcal{A}|$;

iii) for each $n$-ary predicate symbol $p$ of $L$, other than $=$, an $n$-ary predicate $p_{\mathcal{A}}$, such that $p_{\mathcal{A}}$ is a mapping from $|\mathcal{A}|^n$ to $\{0,1\}$.

As in [22], we construct the language $L(\mathcal{A})$; define $\mathcal{A}(a)$ for each variable free term of $L(\mathcal{A})$, and define $\mathcal{A}$-instance of a formula $A$.

We use $i$ and $j$ as syntactical variable for the names of individuals of $\mathcal{A}$.
DEFINITION 3.2. The truth-value $\alpha(A)$ for each closed formula $A$ in $L(\alpha)$ is given by:

i) if $A$ is $a = b$, then $\alpha(A) = 1$ iff $\alpha(a) = \alpha(b)$; otherwise, $\alpha(A) = 0$;

ii) if $A$ is $p(a_1, \ldots, a_n)$, where $p$ is not =, then $\alpha(A) = p_{\alpha}(\alpha(a_1), \ldots, \alpha(a_n))$;

iii) if $A$ is $\exists B$, then $\alpha(A) = H_{\exists}(\alpha(B))$;

iv) if $A$ is $\forall B$, then $\alpha(A)$ is $H_{\forall}(\alpha(B))$;

v) if $A$ is $B \lor C$, then $\alpha(A)$ is $H_{\lor}(\alpha(B), \alpha(C))$;

vi) if $A$ is a $\exists x B$, then $\alpha(A) = \max(\alpha(B_{x}[i])/i \in L(\alpha))$;

vii) if $A$ is a $\forall x B$, then $\alpha(A) = \min(\alpha(B_{x}[i])/i \in L(\alpha))$.

DEFINITION 3.3. (1) A formula $B$ of $L(\alpha)$ is true in $\alpha$ (or $\alpha$ is a model of $B$) iff $\alpha(B) \in V_d$.

(2) A formula $A$ of $L$ is valid in $\alpha$ iff for every $\alpha$-instance $A'$ of $A$, $A'$ is true in $\alpha$.

A first-order predicate calculus $J^*_3$ is the formal system whose language is an $L_3$ plus the following, with the usual restrictions (see [14]):

Axiom 6: $\forall x(x = x)$

Axiom 7: $x = y \Rightarrow (A[x] = A[y])$

Axiom 8: $A_x[a] \Rightarrow \exists xA$

Axiom 9: $\forall xA \Rightarrow A_x[a]$

Axiom 10: $\exists xA \equiv \forall x \exists xA$

Axiom 11: $\forall xA \equiv \exists x \forall xA$

Axiom 12: $\forall x \exists xA \equiv \forall xA$

Axiom 13: $\forall x \exists xA \equiv \exists x \forall xA$

Axiom 14: $\exists xA \equiv \forall x \forall xA$

Axiom 15: $\forall x \forall xA \equiv \forall xVA$

Rule R3: ($\exists$-introduction rule): $A \Rightarrow C$  \[\exists xA \Rightarrow C\]

Rule R4: ($\forall$-introduction rule): $C \Rightarrow A$  \[\forall xA \Rightarrow \forall xA\]

THEOREM 3.1. $J^*_3$ is a conservative extension of $J_3$.

Proof. We apply the Hilbert-Bernays theorem of $k$-transforms, that can be extended to this case.

THEOREM 3.2. $J^*_3$ is an extension of the classical predicate calculus, with connectives $\forall$, $\exists$, $\Rightarrow$, $\Leftrightarrow$ and $\forall$.

DEFINITION 3.4. A first-order $J^*_3$-theory is a formal system $T$ such that:

i) the language of $T$, $L(T)$, is an $L_3$-language;
ii) the axioms of T are the axioms of $J_3^a$, called the logical axioms of T, and certain further axioms, called the non-logical axioms;

iii) the rules of T are those of $J_3^a$.

A is a theorem of T, in symbols: $\vdash_T A$, and B is a semantical consequence of a set $\Gamma$ of formulas of $L(T)$ are defined in the standard way. If B is a semantical consequence of $\Gamma$, then we shall also say that "B is valid in $\Gamma$".

THEOREM 3.3. (Validity Theorem): Every theorem of a $J_3$-theory T is valid in T.

4. SOME THEOREMS IN FIRST-ORDER $J_3$-THEORIES AND THE EQUIVALENCE THEOREM.

DEFINITION 4.1. A $J_3$-theory T is finitely trivializable if there exists a fixed formula F such that, for any formula A, $F \rightarrow A$ is a theorem of T (see [2]).

THEOREM 4.1. The $J_3$-theories are finitely trivializable.

Proof. Any formula $\forall(A \lor A)$ trivializes a $J_3$-theory.

The following results hold in any $J_3$-theory T:

Generalization Rule: If $\vdash_T A$, then $\vdash_T \forall x A$.

Substitution Rule: Is $\vdash_T A$ and $A'$ is an instance of A, then $\vdash_T A'$.

Substitution Theorem: a) $\vdash_T A[x_1, \ldots, x_n[a_1, \ldots, a_n]] \Rightarrow 3x_1 \ldots x_n A$;

b) $\vdash_T \forall x_1 \ldots \forall x_n A \Rightarrow A[x_1, \ldots, x_n[a_1, \ldots, a_n]]$

Distribution Rule: If $\vdash_T A \rightarrow B$, then $\vdash_T 3x A \rightarrow 3x B$ and $\vdash_T \forall x A \rightarrow \forall x B$.

Closure Theorem: If $A'$ is the closure of A, then $\vdash_T A$ if and only if $\vdash_T A'$.

Theorem on Constants: If T' is a $J_3$-theory obtained from T by adding new constants (but no new nonlogical axioms), then for every formula A of T and every sequence $e_1, \ldots, e_n$ of new constants, $\vdash_T A$ if and only if $\vdash_{T'} A[x_1, \ldots, x_n[e_1, \ldots, e_n]]$.

In the case of classical logic, the equivalence $\equiv$ behaves as a congruence relation with respect to the other logical symbols. Unfortunately this is not the case in $J_3$-theories, for it is possible to have $\vdash_T A \equiv B$ and $\vdash_T \forall A \equiv \forall B$.

However we can introduce a stronger equivalence, $\equiv^*$, which is a $J_3^a$-congruence relation and thus allow us to prove a $J_3$-version of the equivalence theorem (see [22]).
DEFINITION 4.2. \( A \equiv^* B = \text{def}(A \equiv B) \& (\forall A \equiv \forall B) \).

THEOREM 4.2. If \( T \) is a \( J_3 \)-theory and \( \models T A \equiv^* B \), then \( \models T A \) if and only if \( \models T B \).

THEOREM 4.3. (Equivalence Theorem). Let \( T \) be a \( J_3 \)-theory and let \( A' \) be obtained from \( A \) by replacing some occurrences of \( B_1, \ldots, B_n \) by \( B'_1, \ldots, B'_n \) respectively. If \( \models T B_1 \equiv^* B'_1, \ldots, \models T B_n \equiv^* B'_n \), then \( \models T A \equiv^* A' \).

Proof. After considering the special case when there is only one such occurrence and it is all of \( A \), we use induction on the length of \( A \).

For \( A \) atomic, the result is obvious.

\( A \) is \( \forall C \) and \( A' \) is \( \forall C' \), where \( C' \) results from \( C \) by replacements of the type described in the theorem. By induction hypothesis, \( \models T C \equiv^* C' \), that is, \( \models T \forall C \equiv \forall C' \). As by Theorem 2.4, \( \models T \forall C \equiv \forall C' \), we have \( \forall C \equiv \forall C' \). So \( \forall C \equiv^* \forall C' \).

\( A \) is \( \exists C \) and \( A' \) is \( \exists C' \), with \( \models T C \equiv^* C' \). From \( \models T C \equiv C' \), it follows that \( \models T \exists C \equiv \exists C' \), by Theorem 2.6. Also from \( \models T C \equiv C' \) it follows that \( \models T \exists C \equiv \exists C' \), since \( \models T \forall C \equiv C \) by Theorem 2.4. Therefore, \( \models T \exists C \equiv \exists C' \).

\( A \) is \( C \lor D \) and \( A' \) is \( C' \lor D' \), with \( \models T C \equiv^* C \) and \( \models T D \equiv^* D' \). As by theorem 2.6,

\[ \models T ((C \equiv C') \& (D \equiv D')) \Rightarrow ((C \lor D) \equiv (C' \lor D')) \]

and

\[ \models T ((\forall C \equiv \forall C') \& (\forall D \equiv \forall D')) \Rightarrow ((\forall C \lor \forall D) \equiv (\forall C' \lor \forall D')) \]

we have that \( \models T C \lor D \equiv C' \lor D' \) and \( \models T \forall (C \lor D) \equiv \forall (C' \lor D') \).

\( A \) is \( \exists x C \) and \( A' \) is \( \exists x C' \), with \( C \equiv^* C' \). By Distribution Rule, \( \models T \exists x C \equiv \exists x C' \) and \( \models T \forall x \exists C \equiv \forall x \exists C' \). Using Axiom 12 we complete the proof.

If \( A \) is \( \forall x C \) and \( A' \) is \( \forall x C' \), with \( \models T C \equiv^* C' \), the proof is similar.

In the spirit of the equivalence theorem, we have the following corollaries and remark.

COROLLARY 1. In a \( J_3 \)-theory \( T \), it is possible to replace:

i) \( \exists \forall A \) by \( \forall \exists A \);

ii) \( \exists \exists \forall A \) by \( \forall \exists \forall A \);

iii) \( \exists (A \lor B) \) by \( \forall A \exists B \);

iv) \( \forall (A \lor B) \) by \( \exists \forall A \exists B \);

v) \( \forall \forall A \) by \( \forall \exists \forall A \);

vi) \( \forall \exists \forall A \) by \( \forall \exists \exists \forall A \);

vii) \( \forall \exists \exists A \) by \( \exists \exists \forall A \);

viii) \( \forall \exists \exists A \) by \( \exists \exists \forall A \);

ix) \( \forall \forall A \) by \( \forall \forall \forall A \).
Proof. It is enough to verify that $\vdash I \neg A \equiv^* A$, $\vdash I A \equiv^* \neg A \equiv^* \neg A$, etc.

COROLLARY 2. In a $J_3$-theory $T$, if $\vdash x = y$, then, for every formula $A$, $A(x)$ can be replaced by $A(y)$.

REMARK. Although $\vdash I A \equiv A$, it is not possible, in general, to replace $\neg^* A$ by $A$.

DEFINITION 4.3. A formula $A'$ is a variant of $A$ just in case $A'$ has been obtained from $A$ by renaming bound variables.

THEOREM 4.4. (Variant Theorem). If $A'$ is a variant of $A$, then $\vdash A \equiv^* A'$.

Proof. In view of Theorem 4.3 and Corollary 1, it is enough to observe that $\vdash 3x \equiv^* 3y [x]$. Let $T[\Gamma]$ be the $J_3$-theory whose non-logical axioms are those of $T$ plus the formulas of the set $\Gamma$.

THEOREM 4.5. (Reduction Theorem). Let $\Gamma$ be a set of formulas in the $J_3$-theory $T$ and let $A$ be a formula of $T$. $A$ is a theorem of $T[\Gamma]$ if, and only if, there is a theorem of $T$ of the form $B_1 \Rightarrow \ldots \Rightarrow B_n \Rightarrow A$, where each $B_1$ is the closure of a formula in $\Gamma$.

Given a non-empty set $\Gamma$ of formulas we let:

$\Gamma V \land \neg \neg = \{B \mid B$ is a disjunction of negations of closures of formulas of the type $\forall A$, with $A \in \Gamma\}$

$\Gamma V \land \neg \neg = \{C \mid C$ is a disjunction of negations of formulas of the type $\forall A'$, where $A'$ is the closure of a formula of $\Gamma\}$

THEOREM 4.6. (Reduction Theorem for non-trivialization). Let $\Gamma$ be a non-empty set of formulas in a $J_3$-theory $T$. Then the extension $T[\Gamma]$ is trivial, if and only if, there is a theorem of $T$ which belongs to $\Gamma V \land \neg \neg$.

Proof. The corollary to the replacement theorem gives us that every formula of $\Gamma V \land \neg \neg$ is strongly equivalent to a formula of $\Gamma V \land \neg \neg$. The proof of the theorem can be completed using the properties of strong negation.

COROLLARY. If $A'$ is the closure of $A$, then the formula $A$ is a theorem of $T$ if, and only if, $T[\neg^* A']$ is trivial.
5. THE COMPLETENESS AND THE COMPACTNESS THEOREMS FOR $J_3$-THEORIES

We study certain aspects of the $J_3$-theories and present a Henkin-type proof of the completeness theorem for this type of many-valued theories.

**DEFINITION 5.1.** If $T$ is a $J_3$-theory containing a constant, and if $a$ and $b$ are variable-free terms of $T$, then:

i) $a \equiv b \overset{\text{def}}{=} a = b$;

ii) $a^0 = \{b | a \equiv b\}$.

**DEFINITION 5.2.** A canonical structure for the $J_3$-theory $T$ is the structure $\alpha$:

i) whose universe $|\alpha|$ is the set of all equivalence classes under $\nu$;

ii) $f_{\alpha}(a^0_1, \ldots, a^0_n) = (f(a_1, \ldots, a_n))^0$;

iii) $\nu_{\alpha}(a^0_1, \ldots, a^0_n)$ is in $V_d$ iff $T \models p(a_1, \ldots, a_n)$.

Observe that (iii) could have been replaced by

$P_{\alpha}(a^0_1, \ldots, a^0_n) = 0$ iff $T \not\models p(a_1, \ldots, a_n)$.

**THEOREM 5.1.** If $\alpha$ is a canonical structure for $T$ and $p(a_1, \ldots, a_n)$ is a variable-free atomic formula in $L(T)$, then:

i) $\alpha(p(a_1, \ldots, a_n)) = 0$ iff $T \not\models p(a_1, \ldots, a_n)$;

ii) $\alpha(p(a_1, \ldots, a_n)) = \frac{1}{2}$ iff $T \models p(a_1, \ldots, a_n)$ and $T \not\models \neg p(a_1, \ldots, a_n)$;

iii) $\alpha(p(a_1, \ldots, a_n)) = 1$ iff $T \models p(a_1, \ldots, a_n)$ and $T \not\models \neg p(a_1, \ldots, a_n)$.

Proof.

ii) If $\alpha(p(a_1, \ldots, a_n)) = \frac{1}{2}$ then $\alpha(\neg p(a_1, \ldots, a_n)) = \frac{1}{2}$. By the last definition, $T \models p(a_1, \ldots, a_n)$ and $T \not\models \neg p(a_1, \ldots, a_n)$.

On the other hand, if $T \not\models p(a_1, \ldots, a_n)$ and $T \not\models \neg p(a_1, \ldots, a_n)$, also by Definition 5.2, $\alpha(p(a_1, \ldots, a_n))$ and $\alpha(\neg p(a_1, \ldots, a_n))$ belong to $V_d$. Then, $\alpha(p(a_1, \ldots, a_n)) = \frac{1}{2}$.

iii) If $\alpha(p(a_1, \ldots, a_n)) = 1$, then $\alpha(\neg p(a_1, \ldots, a_n)) = 0$; then, $T \not\models p(a_1, \ldots, a_n)$ and $T \not\models \neg p(a_1, \ldots, a_n)$.

On the other hand, if $T \not\models p(a_1, \ldots, a_n)$ and $T \not\models \neg p(a_1, \ldots, a_n)$, we have that $\alpha(p(a_1, \ldots, a_n))$ belongs to $V_d$ and $\alpha(\neg p(a_1, \ldots, a_n))$ does not belong to $V_d$; if $\alpha(p(a_1, \ldots, a_n)) = \frac{1}{2}$ then $\alpha(\neg p(a_1, \ldots, a_n)) = \frac{1}{2}$ and, so, $T \not\models \neg p(a_1, \ldots, a_n)$.

Then, $\alpha(p(a_1, \ldots, a_n)) = 1$.

Now, (i) is immediate.

As a consequence of the theorem we obtain that there is exactly one canonical structure for a $J_3$-theory. Furthermore, as in the classical case, in order for a canonical structure to characterize the theorems of a theory, the theory must be in some sense maximal, for there may be a closed formula $A$ such
DEFINITION 5.3. A formula $A$ of a $J_3$-theory $T$ is undecidable in $T$ if neither $A$ nor $\lnot^*A$ is a theorem of $T$. Otherwise, $A$ is decidable in $T$.

DEFINITION 5.4. A $J_3$-theory $T$ is complete if it is non-trivial and if every closed formula of $T$ is decidable in $T$.

THEOREM 5.2. A $J_3$-theory $T$ is complete if, and only if, $T$ maximal in the class of nontrivial theories.

DEFINITION 5.5. A $J_3$-theory $T$ is a Henkin $J_3$-theory if for every closed formula $\exists x A$ of $T$, there is a constant $e$ such that $\exists x A \Rightarrow A_x[e]$ is a theorem of $T$.

THEOREM 5.3. If $T$ is a Henkin $J_3$-theory, then for every closed formula $\forall x A$ in $T$ there is a constant $e$ such that $\exists x A \Rightarrow A_x[e]$ is a theorem of $T$.

Proof. As $T$ is a Henkin $J_3$-theory, there is $e$, such that $T \vdash \exists x A \Rightarrow A_x[e]$. We obtain the desired result, by successive applications of Theorem 2.6.

THEOREM 5.4. If $T$ is a complete Henkin $J_3$-theory and $\mathcal{A}$ is the canonical structure for $T$, then for all closed formulas $A$ of $L[T]$:

i) $\mathcal{A}(A) = 0$ iff $\not\models T A$

ii) $\mathcal{A}(A) = \frac{1}{2}$ iff $\models T A$ and $\not\models T \lnot A$

iii) $\mathcal{A}(A) = 1$ iff $\models T A$ and $\models T \lnot A$.

Proof. By induction on the height of $A$. For atomic $A$, the result follows from Theorem 5.1.

Case: $A$ is $\lnot B$. i) If $\mathcal{A}(A) = 0$, then $\mathcal{A}(B) = 1$. Thus $\not\models T \lnot B$, that is $\not\models T A$. On the other hand if $\models T A$, then since $T$ is complete $\models T \lnot^*A$, and then $\models T \lnot A$, $\models T \lnot B$, $\models T B$. Thus we have that $\models T B$ and $\not\models T \lnot B$, from which it follows that $\mathcal{A}(B) = 1$ and that $\mathcal{A}(A) = 0$.

ii) If $\mathcal{A}(A) = \frac{1}{2}$, then $\mathcal{A}(B) = \frac{1}{2}$. Thus $\models T B$ and $\not\models T \lnot B$, from which it follows that $\models T \lnot A$ and $\models T A$, the converse is analogous.

iii) If $\mathcal{A}(A) = 1$, then $\mathcal{A}(B) = 0$ and thus $\models T B$. Since $T$ is complete, $\models T \lnot^*B$ and thus $\models T \lnot B$. Since $\models T B$, we obtain that $\models T \lnot B$, in other words, we have that $\models T A$ and $\not\models T \lnot A$.

Assume next that $\not\models T A$ and $\models T \lnot A$, that is, $\not\models T \lnot B$ and $\models T \lnot B$. Then $\not\models T B$, and so by induction $\mathcal{A}(B) = 0$, from which it follows that $\mathcal{A}(A) = 1$.

Case: $A$ is $B \lor C$. i) If $\mathcal{A}(A) = 0$ then $\mathcal{A}(B) = 0$ and $\mathcal{A}(C) = 0$. Hence $\not\models T C$ and $\not\models T B$, from which it follows, since $T$ is complete, that $\models T B \lor C$. The converse is analogous.
ii) If $\alpha(A) = \frac{1}{2}$, then either: 
- $\alpha(B) = \frac{1}{2}$ and $\alpha(C) = \frac{1}{2}$,
- or $\alpha(B) = \frac{1}{2}$ and $\alpha(C) = 0$,
- or $\alpha(B) = 0$ and $\alpha(C) = \frac{1}{2}$.

Let us only consider the situation when $\alpha(B) = \frac{1}{2}$ and $\alpha(C) = 0$ (the others are analogous). The induction hypothesis gives us that

$$\models T B, ~ \models T \neg B, ~ \models T \neg C.$$ 

Since $T$ is complete we obtain that $\models T \neg C$ and $\models T \neg C$. From $\models T B$ we get $\models T B \lor C$, and from $\models T \neg B$ and $\models T \neg C$ we may conclude that $\models T (B \lor C)$.

Conversely, suppose that $\models T (B \lor C)$ and $\models T \neg (B \lor C)$. The latter gives us that $\models T B$ or $\models T C$. The induction hypothesis allows us then to conclude that $\alpha(B \lor C) = \frac{1}{2}$.

iii) If $\alpha(A) = 1$, then either:
- $\alpha(B) = 1$ and $\alpha(C) = 0$,
- or $\alpha(B) = 1$ and $\alpha(C) = \frac{1}{2}$,
- or $\alpha(B) = 0$ and $\alpha(C) = 1$,
- or $\alpha(B) = \frac{1}{2}$ and $\alpha(C) = 1$.

We will only consider the case when $\alpha(B) = 1$ and $\alpha(C) = \frac{1}{2}$. The induction hypothesis gives us that

$$\models T B, ~ \models T \neg B, ~ \models T C, ~ \models T \neg C.$$ 

From the first we obtain that $\models T (B \lor C)$. Suppose on the other hand that $\models T \neg (B \lor C)$. Then $\models T (\neg B \land \neg C)$, from which it would follow that $\models T \neg B$, contradicting that $\models T \neg B$. Thus $\models T (B \lor C)$.

On the other hand, suppose that $\models T (B \lor C)$ and $\models T \neg (B \lor C)$. Then from the completeness of $T$ we obtain that either

$$\models T B ~ or ~ \models T C.$$ 

From $\models T \neg (B \lor C)$, we obtain that

$$\models T \neg B ~ and ~ \models T \neg C.$$ 

The induction hypothesis then gives us that $\alpha(B \lor C) = 1$.

Case: $A$ is $\neg B$. i) If $\alpha(\neg B) = 0$. Then $\alpha(B) = 0$. Thus $\models T B$; from which it follows that $\models T \neg B$. Converse, analogous.

ii) $\alpha(\neg B)$ is never $\frac{1}{2}$.

iii) $\alpha(\neg B) = 1$ then either $\alpha(B) = \frac{1}{2}$ or $\alpha(B) = 1$.

Subcase: $\alpha(B) = \frac{1}{2}$. Then $\models T B$ and $\models T \neg B$, from which we obtain $\models T \neg B$ and $\models T \neg B$. Using that $T$ is complete we conclude $\models T \neg B$, and $\models T \neg B$.

Subcase: $\alpha(B) = 1$. Then $\models T B$ and $\models T \neg B$. Suppose that $\models T \neg B$. Then since $\models T B$, we should obtain that $T$ is trivial, which we are assuming it is not. Thus
Then $\vdash_B B$, and either $\vdash_B \neg B$ or $\vdash_B B$. In one case the induction hypothesis gives that $\alpha(B) = \frac{1}{2}$, and in the other that $\alpha(VB) = 1$. Thus $\alpha(VB) = 1$ in both. That is $\alpha(A) = 1$.

**Case: A is 3xB.**

i) If $\alpha(A) = 0$, then for every variable-free term $b$, $\alpha(B_x[b]) = 0$, and by induction hypothesis this is equivalent to $\vdash_T B_x[b]$. As $T$ is a Henkin theory this gives us that $\vdash_T 3xB$. The converse does not need to use that $T$ is a Henkin theory.

ii) If $\alpha(A) = 1$, then for all $b$ we have that $\alpha(B_x[b]) < \frac{1}{2}$. The induction hypothesis then tells us that

1. For those $b$ such that $\alpha(B_x[b]) = \frac{1}{2}$ (and there is at least one such):
   
   $\vdash_T B_x[b]$ and $\vdash_T \neg B_x[b]$.

2. For the remaining $b$'s: $\vdash_T B_x[b]$ and (because $T$ is complete) $\vdash_T \exists b x$. Thus we have that for all constants $b$: $\vdash_T \exists B_x[b]$; from which it follows that $\vdash_T \exists B$. From (1) we obtain $\vdash_T 3xB$.

   Conversely, suppose that $\vdash_T A$ and $\vdash_T \exists A$; that is $\vdash_T 3xB$ and $\vdash_T \exists x B$. Using that $T$ is a Henkin theory and induction, we obtain an $e$ such that $\vdash_T B_x[e]$, $\vdash_T \exists B_x[e]$, and thus $\alpha(B_x[e]) = \frac{1}{2}$. A proof by contradiction shows that there is no $b$ such that $\alpha(B_x[b]) = 1$. Hence $\alpha(3xB) = \frac{1}{2}$.

iii) If $\alpha(A) = 1$, then there is at least one $b$ such that $\alpha(B_x[b]) = 1$.

   From the induction hypothesis, we obtain that $\vdash_T B_x[b]$ and $\vdash_T \neg B_x[b]$. From the former, we obtain that $\vdash_T \exists x B$. Suppose next contrary to what we want to show, that $\vdash_T \exists x B$. Then $\vdash_T 3xB$ and thus $\vdash_T \exists B_x[b]$, a contradiction. Thus $\vdash_T \exists x B$.

**Corollary 1.** Let $T$ be a complete Henkin $J_3$-theory, $\alpha$ the canonical structure for $T$ and $A$ a closed formula of $T$; then, $\alpha(A)$ belongs to $\mathcal{V}_d$ if and only if $A$ is a theorem of $T$.

**Corollary 2.** If $T$ is a complete Henkin $J_3$-theory, then the canonical structure for $T$ is a model of $T$.

By the above corollary, to prove the completeness of a $J_3$-theory $T$, as in the classical case, it is enough to show that it is possible to extend $T$ to a complete Henkin $J_3$-theory.

Thus, given a nontrivial $J_3$-theory $T$, we will first extend it, conservative-ly, to a Henkin $J_3$-theory $T_c$, and then extend it to a complete Henkin $J_3$-theory $T_c'$. Given a $J_3$-theory $T$ with language $L$, we proceed as in [22] and define the
special constants of level \( n \), the language \( L_c \) with the special constants, and introduce the special axioms for the special constants.

**DEFINITION 5.6.** Let \( T \) be a \( J_3 \)-theory with language \( L \). Then \( T_c \) is the Henkin \( J_3 \)-theory whose language is \( L_c \) and whose nonlogical axioms are the nonlogical axioms of \( T \) plus the special axioms for the special constants of \( L_c \).

**THEOREM 5.5.** \( T_c \) is a conservative extension of \( T \).

**Proof.** By Theorem 4.4 and by Theorem 5.3, the proof is similar to the classical one.

**THEOREM 5.6.** (Lindenbaum's Theorem). If \( T \) is a nontrivial \( J_3 \)-theory, then \( T \) admits a complete simple extension.

Finally, we can obtain the completeness theorem for \( J_3 \)-theories.

**THEOREM 5.7.** (Completeness Theorem). A \( J_3 \)-theory \( T \) is nontrivial if, and only if, it has a model.

**Proof.** If \( \mathcal{O} \) is a model of \( T \) and \( A \) is a closed formula in \( T \), then \( \mathcal{O}(A \land \neg A) = 0 \). So, by the validity Theorem, \( A \land \neg A \) is not a theorem in \( T \). Then \( T \) is nontrivial.

If \( T \) is nontrivial, then we extend \( T \) to \( T_c \), which is a non-trivial Henkin \( J_3 \)-theory. By Lindenbaum's Theorem, we can extend \( T_c \) to a complete Henkin \( J_3 \)-theory \( T'_c \). By Corollary 2 to Theorem 5.4, \( T'_c \) has a model \( \mathcal{O} \). Therefore, \( \mathcal{O} \models L(T) \) is a model of \( T \).

**THEOREM 5.8.** (Gödel's Completeness Theorem). A formula \( A \) in the \( J_3 \)-theory \( T \) is a theorem in \( T \) if, and only if, it is valid in \( T \).

**Proof.** By supposing that the closed formula \( A \) is a theorem in \( T \) and using the above Completeness Theorem, we shall show that there is no model of \( T \) in which \( A \) is not valid.

Therefore, suppose that the closed formula \( A \) is a theorem in \( T \).

By the corollary to the Reduction Theorem for non-Trivialization, \( \models A \) if and only if \( T[\neg VA] \) is trivial; which, by Theorem 5.7, is equivalent to \( T[\neg VA] \) not having a model.

On the other hand, a model of \( T[\neg VA] \) is a model \( \mathcal{O} \) of \( T \) in which \( \neg VA \) is valid, that is, a structure \( \mathcal{O} \) such that \( \mathcal{O}(\neg VA) = 1 \). This is equivalent to \( \mathcal{O}(VA) = 0 \), and so \( \mathcal{O}(A) = 0 \).
Therefore, \( T \) \( A \) if and only if \( A \) is valid in \( T \).

**COROLLARY 3.** If \( T \) and \( T' \) are \( J_3 \)-theories with the same language, then \( T' \) is an extension of \( T \) if, and only if, every model of \( T' \) is a model of \( T \).

**THEOREM 5.9. (Compactness Theorem).** A formula \( A \) in a \( J_3 \)-theory is valid in \( T \) if, and only if, it is valid in some finitely axiomatized part of \( T \).

**COROLLARY 4.** A \( J_3 \)-theory \( T \) has a model if, and only if, every finitely axiomatized part of \( T \) has a model.

**REFERENCES.**


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