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A THEORY OF VARIABLE TYPES

Solomon Feferman[1]

1. INTRODUCTION.

Several publications in recent years have presented various formal theories T in which considerable portions of mathematical practice (particularly analysis) can be more or less directly formalized and which are proof-theoretically weak; cf. Feferman 1977, Takeuti 1978 and Friedman 1980. Indeed, on the classical side we have such T which are conservative over PA (Peano's Arithmetic)[2].

The paper Feferman 1977 will be taken as the point of reference here (but the reader need not be familiar with it to follow the present paper). It used *functional finite type theories* as the basic framework. One of the theories, denoted Res-$\hat{Z}^{(\omega)}+(\mu)$ is shown there to be conservative over PA, but stronger theories for more substantial portions of mathematics were also dealt with.

For the past several years I have been engaged (off and on) in working up the material of my 1977 paper into a book. One of the first improvements in carrying on that project was to obtain a theory VT of *variable types* which provides a much more natural framework for the direct formalization of mathematics. In this paper the system VT is further improved and presented in print for the first time. VT and its extension and restrictions to be considered are described formally in Sec. 2. Some conservation results are stated in Sec. 3 and their proofs are outlined. The concluding Sec. 4 outlines how one goes about formalizing substantial portions of classical and modern analysis in the VT systems.

2. VARIABLE TYPE SYSTEMS.

In ordinary functional finite type theories one begins by specifying the

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(2) There are comparable results for constructive theories T, e.g., such T in which Bishop's constructive analysis can be formalized and which are conservative over HA (Heyting's Arithmetic); cf. Friedman 1977, Feferman 1979 (esp. pp. 217 ff.), and Beeson 1980.

type symbols $\sigma,\tau,\ldots$. For Z^ω these are generated from 0 by closure under $\sigma,\tau \mapsto \sigma\times\tau$, $(\sigma\to\tau)$ [3]. For each type (symbol) one then has variables $x^\tau, y^\tau, z^\tau, \ldots$ of type τ. The intended interpretation is that these range over M_τ where $M_0 = \mathbb{N}$ = the set of natural numbers, $M_{\sigma\times\tau} = M_\sigma \times M_\tau$ and $M_{(\sigma\to\tau)}$ is a (the) set of (all) functions from M_σ to M_τ. One advantage of such a setting is that functional existence axioms are simply provided by the *typed λ-calculus*. However, there is no natural way of forming *sub-types* $\{x^\sigma | \phi(x^\sigma,\ldots)\}$ in this framework and then iterating the operations of × and → applied to them, etc. Further, equations between individual terms $t_1 = t_2$ are permitted only between terms of the same type. If we were to regard members of $\{x^\sigma | \phi(x^\sigma,\ldots)\}$ as belonging to a new type $\sigma|\phi$, we could not say that an object of type $\sigma|\phi$ is (equal to) an object of type σ. The VT systems to be described here have the following advantages: (i) the types are *variable* , so that statements of generality can be expressed directly, yet (ii) every individual term t is still syntactically of a unique type, and hence (iii) the typed λ-calculus may be extended to this language; but also (iv) equations between terms of arbitrary type are admitted, and (v) we can apply separation to form sub-types from given types.

The basic system to be described is denoted VT_0. To specify its language we generate simultaneously the following syntactic classes, together with the relation, t *is of type* T:

1. **individual terms** s,t,...
   a) With each type term T is associated an infinite list of individual variables $x^T, y^T, z^T, \ldots$ (of type T).
   b) If s is of type S and t of type T, then (s,t) is of type S×T.
   c) If u is of type S×T, then $p_1(u)$ is of type S and $p_2(u)$ is of type T.
   d) If s is of type S and t is of type (S → T), then ts is of type T.
   e) If t is of type T, then $\lambda x^S.t$ is of type S → T.

2. **type terms** S,T,...
   a) Each type variable X,Y,Z,... is a type term.
   b) If S,T are type terms and ϕ is a formula, then S×T, S → T, and $\{x^S|\phi\}$ are type terms.

3. **formulas** $\phi,\psi,\ldots$.
   a) Each equation $t_1 = t_2$ (t_1,t_2 of arbitrary type) is a formula.
   b) If ϕ, ψ are formulas so also are $\neg\phi$ and $\phi\to\psi$
   c) If ϕ is a formula and S is a type term, then $\forall x^S\phi$ is a formula.

NOTE. (i) In extensions of the language we may specify some *individual*

(3) Actually the Z^ω described in the 1977 paper only built types by the operation $\sigma,\tau \mapsto (\sigma\to\tau)$. The dispensability of product types is familiar from the combinatory literature.

constants (of certain types) and *type constants,* which are then counted as individual terms and type terms, resp. Other means of constructing individual terms may also be supplied. (ii) Quantifiers are not applied to type variables. This simplifies the conservation arguments below. However, one can also extend those results to VT systems *with* quantified type variables.

Before stating the axioms of VT_o we make some abbreviations and conventions.
(1) The operators $\wedge, \vee, \leftrightarrow, \exists x^S(\ldots)$ are defined classically.
(2) $\forall x^S \phi(x,\ldots)$ is written for $\forall x^S \phi(x^S,\ldots)$, and similarly for $\exists x^S \phi(x,\ldots)$. That is, once the type of a variable is established, we suppress it in the following context.
(3) *Types* are also called *classes* and *type variables* are called *class variables,* etc. The former terminology figures in our syntactic description, the latter in our mathematical uses of the theory.
(4) $t \in T$ is defined as $\exists x^T(t = x)$, where 'x^T' does not occur in t. $\{x \in T | \phi\}$ is written for $\{x^T | \phi\}$, and $\forall x \in T(\phi)$ for $\forall x^T(\phi)$.
(5) $S \subseteq T$ is defined as $\forall x \in S\ (x \in T)$, i.e., as $\forall x^S\ \exists y^T(x = y)$.
(6) $S = T$ is defined as $S \subseteq T \wedge T \subseteq S$.
(7) We write $t(s,\ldots)$ for $t(s/x)$ when $t(x,\ldots)$ is written for t; similarly for $\phi(x,\ldots)$ and $\phi(s,\ldots) = \phi(s/x)$.

AXIOMS OF VT_o:

I. **Abstraction-Application.**

$\forall y \in X[\lambda x^X . t(x,\ldots)y = t(y,\ldots)]$.

II. **Pairing-Projections.**

i) $\forall x \in X\ \ \forall y \in Y[p_1(x,y) = x \wedge p_2(x,y) = y]$

ii) $\forall z \in X{\times}Y[z = (p_1(z),p_2(z))]$.

III. **Separation.**

$\{x \in X | \phi(x,\ldots)\} \subseteq X \wedge \forall y \in X[y \in \{x \in X | \phi(x,\ldots)\} \leftrightarrow \phi(y,\ldots)]$.

The logic of VT_o is that of the many-sorted classical predicate calculus. Since the type variables are treated as free, we use the rule of substitution for these: $\phi(X,\ldots)/\phi(T,\ldots)$.

The system VT is an extension of VT_o with axioms for the *natural numbers*. We adjoin a constant type symbol $\mathbb{N}$, individual constants 0 and sc, and individual terms r_T for each type term T, where 0 is of type $\mathbb{N}$, sc is of type $(\mathbb{N} \to \mathbb{N})$, and r_T is of type $(\mathbb{N}{\times}T \to T){\times}T \to (\mathbb{N} \to T)$. The variables 'n,' 'm,' 'p' with or without subscripts are reserved for variables of type $\mathbb{N}$. We write n' for sc(n). We shall tend to use letters 'f,' 'g,' etc. for members of function types $(S \to T)$.

AXIOMS OF VT (= VT_0 *plus*):

IV. 0, Successor.

i) $(n' \neq 0)$.

ii) $(n' = m' \rightarrow n = m)$.

V. Induction.

$0 \in X \wedge \forall n[n \in X \rightarrow n' \in X] \rightarrow \mathbb{N} \subseteq X$.

VI. Recursion.

$f \in (\mathbb{N}\times X \rightarrow X) \wedge a \in X \wedge r_X(f,a) = g \rightarrow g0 = a \wedge gn' = f(n,gn)$.

REMARK. Officially, Axiom VI would be written

$$\forall f^{(\mathbb{N}\times X\to X)}\forall a^X \forall g^{\mathbb{N}\to X}\forall x^{\mathbb{N}}[r_X(f,a) = g \rightarrow g0 = a \wedge gx' = f(x,gx)].$$

We put $1 = 0'$, $2 = 1'$, etc. Then $\{0,1\}$ is defined as $\{n|n = 0 \vee n = 1\}$. By a *characteristic function* on T we mean an element c of $T \rightarrow \{0,1\}$. Identify 0 with "true" and 1 with "false"; then write $x \in c$ for $cx = 0$. The elements of $T \rightarrow \{0,1\}$ are also called *sets, more precisely subsets of* T, and we also write $S(T)$ for the class of all such, i.e., for $T \rightarrow \{0,1\}$. *Set-induction* (on $\mathbb{N}$) or *restricted induction* is the principle

$$c \in S(\mathbb{N}) \wedge 0 \in c \wedge \forall n(n \in c \rightarrow n' \in c) \rightarrow \forall n(n \in c).$$

This is equivalent to the statement

$$f,g \in (\mathbb{N} \rightarrow \mathbb{N}) \wedge f0 = g0 \wedge \forall n(fn = gn \rightarrow fn' = gn') \rightarrow \forall n(fn = gn),$$

as well as the same with $g = \lambda n\cdot 0$. By *restricted recursion* we mean the principle VI taken only for $X = \mathbb{N}$; this means use only of primitive recursion with values in $\mathbb{N}$. By Res-VT is meant the system VT in which V is replaced by restricted induction and VI by restricted recursion.

Primitive recursive arithmetic and Kleene's extension of it to higher finite types are routinely developed in Res-VT. The following compares the present systems with those of Feferman 1977.

LEMMA. (i) VT *is an extension of* Z^ω.
(ii) Res-VT *is an extension of* Res-$\hat{Z}^\omega$.

Each type symbol τ of Z^ω corresponds to a closed type term T_τ, where $T_0 = \mathbb{N}$, $T_{(\sigma\times\tau)} = T_\sigma \times T_\tau$ and $T_{(\sigma\to\tau)} = (T_\sigma \rightarrow T_\tau)$. The T_τ are called *the finite types*.

For classical analysis we need to adjoin various non-constructive functions to VT. The first of these is the *unbounded minimum operator* μ of type $(\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$. When this is adjoined as a constant symbol, the associated axiom is taken to be:

(μ) $\qquad f \in (\mathbb{N} \to \mathbb{N}) \wedge fn = 0 \to f(\mu f) = 0 \wedge \mu f \leqslant n.$

This allows us to define *quantification over* $\mathbb{N}$ as a functional operator: $\exists^{\mathbb{N}} f = [0$ if $f(\mu f) = 0$, 1 otherwise$]$. Stronger systems are obtained by introducing functionals corresponding to the *Suslin quantifier, quantification over* $(\mathbb{N} \to \mathbb{N})$, etc. We shall not detail those here.

The final principle to be considered is the *Axiom of Choice* taken as a scheme:

$$(AC) \qquad \forall x^X \exists y^Y \phi(x,y) \to \exists z^{X\to Y} \forall x^X \phi(x, zx).$$

We denote by $(AC)_{S,T}$ the result of replacing X by S and Y by T in (AC). By *restricted* $(AC)_{S,T}$ is meant the statement:

$$\text{Res}(AC)_{S,T} \qquad c \in \mathcal{S}(S\times T) \wedge \forall x^S \exists y^T [(x,y) \in c] \to \exists z^{S\to T} \forall x^S [(x,zx) \in c].$$

In other words, this takes AC only for matrices ϕ which define sets. The scheme $(AC)_{\mathbb{N},\mathbb{N}}$ is already quite strong (stronger than full second-order analysis). Writing $\mathbb{N}_0$ for $\mathbb{N}$ and $\mathbb{N}_1$ for $\mathbb{N} \to \mathbb{N}$, we write $(AC)_{0,1}$ for $(AC)_{\mathbb{N}_0,\mathbb{N}_1}$. In Feferman 1977 the scheme $\text{Res}(AC)_{S,T}$ for all finite types was denoted (QF-AC) [QF = quantifier-free]. We shall use the same designation here. Then $(\text{QF-AC})_{0,1}$ is $\text{Res}(AC)_{\mathbb{N}_0,\mathbb{N}_1}$. By way of comparison with familar systems, $Z^\omega + (\mu) + (\text{QF-AC})_{0,1}$ contains the second-order system $(\Sigma^1_1\text{-AC})$. The same thus holds for $VT + (\mu) + (\text{QF-AC})_{0,1}$

3. CONSERVATION RESULTS.

The *type levels* lev(T) of finite type terms are defined by $\text{lev}(\mathbb{N}) = 0$, $\text{lev}(S\times T) = \max(\text{lev}(S),\text{lev}(T))$ and $\text{lev}(S\to T) = \max(\text{lev}(S)+1,\text{lev}(T))$. For $T = T_\tau$, we put $\text{lev}(\tau) = \text{lev}(T)$. By a *second-order sentence of the language of* Z^ω is meant one, all of whose variables are of type-level $\leqslant 1$.

MAIN THEOREM.

(i) $VT \pm (\mu) \pm (\text{QF-AC})_{0,1}$ *is a conservative extension of* $Z^\omega \pm (\mu) \pm (\text{QF-AC})_{0,1}$ *for second-order sentences.*

(ii) *The same holds with* Res-VT *in place of* VT *and* Res-$\hat{Z}^\omega$ *in place of* Z^ω.

COROLLARY.

(i) $\text{Res-VT} + (\mu) + (\text{QF-AC})_{0,1}$ *is a conservative extension of* PA.

(ii) $VT + (\mu) + (\text{QF-AC})_{0,1}$ *is a conservative extension of* $(\Sigma^1_1\text{-AC})$ [*and hence of* $(\Pi^0_1\text{-CA})_{<\varepsilon_0}$].

The corollary follows by Feferman 1977, 8.6-8.7 [and Friedman's theorem for Σ^1_1-AC also proved loc. cit.]. Similar conservation results may be obtained with adjunction of stronger functional constants.

The steps in the proof of the main theorem are now outlined. For simplicity we concentrate on the reduction of VT to Z^{ω}. Each of the other stated results follows by a parallel argument.

Step 1. *Reduction of* VT *to a theory* CT *of (semi-) constant types.* CT differs from VT in that it has no type variables, though it has type terms which may vary depending on individual parameters. (For this reason they are called *semi-constant*.) The terms and formulas of CT are generated as in 1.-3. of the preceding section, omitting 2.a) (type variables), but including $\mathbb{N}$, 0, sc, and r_T for each semi-constant type term T. The axioms of CT are obtained from those of VT by substituting semi-constant type terms throughout for the type variables. The logic of CT is the same as for VT except that one can dispense with the substitution rule for types. It is readily seen that VT is conservative over CT.

Step 2. *Reduction of* CT *to a theory* FT *of finite types.* The finite types were defined above. The language of FT is a part of CT with two essential restrictions: (i) there are no sub-type terms $\{x \in S \mid \phi\}$, and (ii) equations $t_1 = t_2$ are allowed only between terms of the *same* finite type. The axioms of FT consist of appropriate restrictions to its language of: I (Abstraction-Application), II (Pairing-Projections), IV (0, successor), V (Induction), and VI (Recursion), where now V consists of all instances of the *induction scheme* $\phi(0) \wedge \forall n[\phi(n) \rightarrow \phi(n')] \rightarrow \forall n\phi(n)$ for ϕ a formula of FT. (Note the Axiom III is dropped). The proof that CT is conservative over FT is by a model-theoretic argument. With each model $\boldsymbol{M}$ of FT is associated a model $\boldsymbol{M}^*$ of CT which satisfies the same sentences of FT.

Without loss of generality one can assume that the types of $\boldsymbol{M}$ are disjoint. Let L^*_M, L_M be the languages of CT, FT, resp. with constants for all the individuals in $\boldsymbol{M}$. With each term or formula of L^*_M is associated a corresponding term or formula of L_M which will be its interpretarion in $\boldsymbol{M}$, except that type terms S are interpreted as pairs $(A,\phi(x))$ or formal terms $S^* = \{x^A \mid \phi(x)\}$ with A a finite type (of FT) and ϕ a formula of L_M. Given also $T^* = \{x^B \mid \phi(x)\}$ of the same kind, we take

$$(S\times T)^* = \{z^{A\times B} \mid \phi(P_1(z)) \wedge \psi(P_2(z))\},$$

$$(S\rightarrow T)^* = \{z^{A\rightarrow B} \mid \forall x^A[\phi(x) \rightarrow \psi(z(x))]\},$$

$$\{x^S \mid \theta(x)\}^* = \{x^A \mid \phi(x) \wedge \theta^*(x)\}.$$

t^* is then defined in an obvious way for individual terms t. Next, for formulas, if s, t are terms of type S, T, resp. we take

$$(s = t)^* = \begin{cases} s^* = t^* & \text{if } A = B \\ 0 \neq 0 & \text{if } A \neq B. \end{cases}$$

This definition is appropriate since if $A \neq B$, then A and B are disjoint by hypothesis. $(\ )^*$ preserves $\neg$ and $\rightarrow$, while $(\forall x^S \theta(x))^* = \forall x^A[\phi(x) \rightarrow \theta^*(x)]$ for $S^* = \{x^A \mid \phi(x)\}$. It is then straightforward to prove that this interpretation of L_M^* in L_M serves to define a model M^* of CT [4]

Step 3. *Reduction of* FT *to* FT[0] *with type* 0 *equations.* The system FT[0] is obtained from FT by use only of those formulas built up from equations $t_1 = t_2$ between objects of type-level 0. Equality at higher types is introduced by definition. This is used in re-expressing Axioms I, II, VI of VT and CT. To verify the laws of equality at higher types we need the axiom Ext of Extensionality. It is then shown that FT[0] + (Ext) is interpretable in FT[0], by the following (formal) model of *hereditarily extensional objects*. With each finite type A is associated a pair of formulas $x \equiv_A y$ and $E_A(x)$ for objects x, y of type A by:

(i) $E_N(x) \leftrightarrow x = x, \quad x \equiv_N y \leftrightarrow x = y,$

(ii) $E_{A\times B}(x) \leftrightarrow E_A(P_1(z)) \wedge E_B(P_2(z)), \ z \equiv_{A\times B} w \leftrightarrow P_1(z) \equiv_A P_1(w) \wedge P_2(z) \equiv_B P_2(w),$

(iii) $E_{(A\rightarrow B)}(z) \leftrightarrow \forall x^A[E_A(x) \rightarrow E_B(zx)] \wedge \forall x^A \forall y^A[x \equiv_A y \rightarrow zx \equiv_B zy],$

$z \equiv_{(A\rightarrow B)} w \leftrightarrow \forall x^A[E_A(x) \rightarrow zx \equiv_B wx].$

Note that when $\text{lev}(A) \leqslant 1$ we have $\forall x^A . E_A(x)$. It follows that FT is conservative over FT[0] for second-order statements. (This is the point where restriction of conservation to second-order statements enters the Main Theorem).

Step 4. The system FT[0] is actually a form of Z^ω. As noted in ftn.3, the system Z^ω of Feferman 1977 is practically the same, but without product types. The latter are eliminable in the presence of extensionality, i.e., FT[0] + (Ext) is conservative over Z^ω + (Ext). Then Z^ω + (Ext) is reduced to Z^ω as in Step 3. The present step is unnecessary if the conservation results of Feferman 1977, 8.6-8.7 are established directly for FT[0] in place of Z^ω. That can be done by the same methods described loc. cit.

Step 5. The conservation results apply to extensions by the Axioms (μ) and/or $(\text{QF-AC})_{0,1}$ since these are second-order statements.

Finally it may be seen that each step can be carried out just as well to reduce Res-VT to Res-$\hat{Z}^\omega$, again with conservation for second-order statements,

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(4) Because of the interpretation of VT (via CT) in FT, I have also called VT a theory of *variable finite types* and denoted it VFT. For formalisms in which one can construct *transfinite types* cf. Feferman 1979.

and thence the same for extensiones by (μ) and/or $(QF\text{-}AC)_{0,1}$.

4. MATHEMATICS IN Res-VT+(μ)

The following is an outline of an informal development which can be formalized directly in Res-VT+(μ). This shows that a considerable portion of mathematical analysis is *predicative* and, indeed, is no stronger than PA. Cf. also Feferman 1977, §3.2, Takeuti 1978, and Friedman 1980[5]. In the approach taken here neither extensionality nor AC is needed, though both can be admitted to a certain extent by the formal results of Sec. 3.

A,B,C,...,X,Y,Z range over classes (which are treated formally as the type variables of VT). All constructions on classes are given explicitly, so all statements about classes are given in universal form $\forall X_1,\ldots,\forall X_n\phi(X_1,\ldots,X_n)$; this is justified in our framework when the formula $\phi(X_1,\ldots,X_n)$ is established in Res-VT+(μ). *Structures* are of the form $\mathbf{A} = \langle A,E,R_1,\ldots,R_m,f_1,\ldots,f_n,a_1,\ldots,a_p\rangle$ where $E \subseteq A^2$, $R_i \subseteq A^{k_i}$, $f_i \in A^{\ell_i} \to A$, and $a_i \in A$, and E is a congruence relation on A. E is called the *equality relation of* $\mathbf{A}$ and is often denoted $=_A$ even $=_A$. A homomorphism between structures $\mathbf{A} = \langle A,=_A,\ldots\rangle$, and $\mathbf{A}' = \langle A',=_{A'},\ldots\rangle$ of the same signature is a member h of $A \to A'$ such that $\forall x \in A\ \forall y \in A[x =_A y \to hx =_{A'} hy]$, and h preserves the operations and relations of $\mathbf{A}$. The appropriate notion of injective and surjective homomorphisms then leads to the notion of *isomorphism* for such structures.

We start with $\mathbf{N} = \langle\mathbb{N},=,<,+,\cdot,0,1\rangle$, where $=_{\mathbb{N}}$ is the identity relation. The *integers* $\mathbb{Z}$ are then defined to be $\mathbb{N}\times\mathbb{N}$ with $(x_1,y_1) =_{\mathbb{Z}} (x_2,y_2) \leftrightarrow x_1+y_2 = x_2+y_1$. An ordered integral domain structure $\mathbf{Z} = \langle\mathbb{Z},=_{\mathbb{Z}},<_{\mathbb{Z}},+_{\mathbb{Z}},\cdot_{\mathbb{Z}},0_{\mathbb{Z}},1_{\mathbb{Z}}\rangle$ is put on $\mathbb{Z}$ in the usual way, so that one has an injective homomorphism h of $\mathbf{N}$ into $\mathbf{Z}$, and $\mathbb{Z}$ is generated from the range of h. Similarly one passes from $\mathbf{Z}$ to the *rationals* $\mathbf{Q} = \langle\mathbb{Q},=_{\mathbb{Q}},<_{\mathbb{Q}},\ldots\rangle$, i.e., the quotient field of (an image of) $\mathbf{Z}$. Finally the *real number system* $\mathbf{R} = \langle\mathbb{R},=_{\mathbb{R}},<_{\mathbb{R}},\ldots\rangle$ is defined by taking $\mathbb{R}$ to consist of all *Cauchy sequences* of rationals, i.e.,

$$\mathbb{R} = \{z \in \mathbb{Q}^{\mathbb{N}} \mid \forall m\ \exists n\ \forall k_1,k_2[k_1 \geqslant n \wedge k_2 \geqslant n \to |zk_1 - zk_2| < \tfrac{1}{m+1}]\}$$

where the expresion '$|zk_1 - zk_2| < \frac{1}{m+1}$' is evaluated in $\mathbf{Q}$. The relations $=_{\mathbb{R}}, <_{\mathbb{R}}$ and the operations on $\mathbb{R}$ are then defined as usual. $\mathbf{R}$ forms an ordered field which is *Cauchy complete* in the sense that every Cauchy sequence of reals has a limit in $\mathbb{R}$. But $\mathbf{R}$ is not (provably) complete in the Dedekind sense that every Dedekind section in $\mathbb{Q}$ determines a real. The *complex number system* $\mathbf{C} = \langle\mathbb{C},=_{\mathbb{C}},\ldots\rangle$ is obtained in the standard way from $\mathbf{R}$.

From the reals we can move to *metric spaces*. All of the topological work

(5) For further original sources on predicative mathematics cf. Feferman 1964 and the references there to the work of Weyl, Lorenzen, and Kreisel.

is done with *separable metric spaces* $\mathbf{A} = (A,\ldots)$ wich carry as part of their structure a dense countable subset $\langle x_n \rangle_{n\in\mathbb{N}} \in A^{\mathbb{N}}$. Among the spaces wich are specially used are the real and complex finite-dimensional spaces $\mathbb{R}^n$ and $\mathbb{C}^n$, Cantor space $2^{\mathbb{N}}$, and Baire space $\mathbb{N}^{\mathbb{N}}$. All of these (and more) are show to be *locally sequentially compact,* i.e., every bounded sequence contains a convergent subsequence. The proof uses *König's Lemma,* which is here applied to trees t which are represented as members of $\mathbf{S}(\mathbb{N})$ (i.e., which have a characteristic function). Here the operator μ and the associated operator $\exists^{\mathbb{N}} \in (\mathbb{N}^{\mathbb{N}} \to \{0,1\})$ make an essential appearance. The definition of an infinite path through t is primitive recursive in $\exists^{\mathbb{N}}$ (and t). Only restricted induction is necessary to verify the required property of the path.

One cannot prove (local) *compactness* of these spaces in the usual sense of reduction of open covers to finite subcovers, but one can give a form of this for *countable open covers.* Some further general theorems which can be established in this setting for (Cauchy) complete separable metric spaces are the *Baire Category Theorem* and the *Contraction Mapping Theorem.*

Turning to classical analysis, the objects one deals with must usually be *presented* with additional information so as to be able to operate with them by the limited functional means provided in Res-VT+(μ). For example, an element of the class C(A,A') of *continuous functions* from A to A' (where A,A' are given metric spaces), is a pair (f,δ) for which f is a mapping in A → A' and δ is a *modulus-of-continuity function* δ(x,ε), i.e., such that

$$d_A(x,y) < \delta(x,\varepsilon) \to d_{A'}(f(x),f(y)) < \varepsilon.^{(6)}$$

Similarly, *uniformly continuous functions* are given as pairs (f,δ) where δ is a *modulus-of-uniform continuity* δ(ε) for f. It is shown for countably compact spaces that continuity implies uniform continuity and that maxima and minima are attained. Sequences and series of functions are studied in $C(A,\mathbb{R})$ when A is sequentially compact. This forms a metric space with respect to the sup-norm $\|f-g\| = \sup_{x\in A}|f(x)-g(x)|$; the *Stone-Weierstrass Theorem* can be proved, thus showing $C(A,\mathbb{R})$ to be separable.

Most classical topics in the *differential and integral calculus* (*Riemann integration*) go through quite readily. The extensions to *complex analysis* are

(6) In this respect we follow Bishop's lead in his development of constructive analysis (Bishop 1967); cf. also Feferman 1979, esp.pp.177 ff. The use of (μ) and thence of $\exists^{\mathbb{N}}$ is a way of incorporating mathematically what Bishop calls the *Limited Principle of Omniscience,* LPO. Bishop says that his results are constructive substitutes φ' for classical counterparts φ, such that φ'+LPO implies φ. Thus the formalization of Bishop's work in a system conservative over HA (cf.ftn.2 above) implies the formalization (in principle) of the corresponding body of classical mathematics in a system conservative over PA. The point of the approach here is to be able instead to step as directly as possible from current classical mathematics to its formalization in systemsof known limited strength.

also straightforward, as are establishment of the properties of the familiar stock of transcendental functions.

New considerations are required when one passes to more modern topics, viz. *measure theory* and *functional analysis*. Standard approaches which start Lebesgue measure theory in $\mathbb{R}^n$ with outer measure $\mu^*(X)$ make essential use of the g.l.b. operation on sets of reals, which in turn requires Dedekind completeness of $\mathbb{R}$; but that is not available in VT+(μ). Instead, one can define *measurable sets* X and their *measure* $\mu(X)$ directly, using sequences of covering approximations to each of X and the complement of X by countable unions of open intervals. Another elegant route is to obtain the theory of *Lebesgue integration* directly using Riesz's approach: every measurable function is represented as a difference of two monotone sequence of step functions which converge *a.e.*, and its integral is defined in terms of integrals of step functions. For this only the concept of set of *measure* 0 is needed. Then the theory of measurable sets is obtained from the integration theory. It turns out all of that can be carried out in Res-VT+(μ). However, when performing operations on measurable functions and sequences of such, one must consistently work with *presentations* of them in terms of sequences of step functions (as described).

Finally, one can obtain the main initial material from functional analysis for *linear operators* on *separable Banach spaces* and *Hilbert spaces*. Usable forms of the Riesz Representation Theorem, Hahn-Banach Theorem, Uniform Boundedness Theorem, and the Open Mapping Theorem are obtained (under heavy use of separability). Finally, I have verified that one can obtain the principal results of the *spectral theory* of compact self-adjoint operators on a Hilbert space. It seems then that all *applicable analysis* can be carried out in this conservative extension of PA.

A theme running throughout this development is that the *l.u.b.* (or *g.l.b.*) property of the reals, which is constantly used in classical analysis, but which is not derivable in VT+(μ), can be avoided by dealing systematically with *sequences of reals* rather than *sets of reals*. For bounded sequences we *do* have l.u.b., g.l.b. (and sequential compactness more generally).

There are of course many results of theoretical analysis which cannot be derived in this setting. Additionally, by the result of Paris-Harrington 1977, there are simple combinatorial Π_2^0 statements which are consequences of RT (infinite Ramsey's Theorem) but which are not provable in Res-VT+(μ). We leave the question of what can be done in various extensions of this theory to another occasion.

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Depts. of Mathematics and Philosophy.
Stanford University
Stanford, California, U.S.A.