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ON INTUITIONISTIC SENTENTIAL CONNECTIVES I

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INTRODUCTION.

Recently there have appeared a series of articles on a non-classical logic, called the *Heyting-Brouwer Logic* (**H-B L**), see [2], [3], [5] and [6]. The **H-B** Logics are obtained by the addition of new sentential connectives to intuitionistic logic so that the resulting Lindenbaum algebras enjoy some duality properties.

In a pseudo-Boolean algebra $A = \langle A, \cap, \cup, \Rightarrow, 0 \rangle$, the element $a \Rightarrow b$ is the *pseudo-complement of a relative to* b and has the property that for every $x \in A$:

$$x \leqslant a \Rightarrow b \quad \text{iff} \quad a \cap x \leqslant b.$$

The dual notion to the pseudo-complement is the *pseudo-difference*. The pseudo-difference of b and a is denoted, when it exists, by "$b \dot{-} a$" and it has the property that for every $x \in A$:

$$x \geqslant b \dot{-} a \quad \text{iff} \quad a \cup x \geqslant b.$$

As is well known, a Boolean algebra always has both pseudo-complements and pseudo-differences. On the other hand, pseudo-Boolean algebras (also called Heyting algebras) have pseudo-complements but may fail to have pseudo-differences. The dual of Heyting algebras, the Brouwerian algebras, have pseudo-differences but may fail to have pseudo-complements.

The fusion of Heyting algebras and Brouwerian algebras are called *semi-Boolean algebras*; that is, $A = \langle A, \cap, \cup, \Rightarrow, \dot{-}, 0, 1\rangle$ is a semi-Boolean algebra iff $\langle A, \cap, \cup, \Rightarrow, 0\rangle$ is a Heyting algebra and $\langle A, \cap, \cup, \dot{-}, 1\rangle$ is a Brouwerian algebra.

The *Heyting-Brouwer Sentential Calculus*, **H-B SC**, is the extension of the Intuitionistic Sentential Calculus, ISC, obtained by adding a new sentential connective $\dot{-}$ to the language. The axioms and rules of inference were chosen so that the resulting Lindenbaum algebras are semi-Boolean algebras.

An interesting development of the H-**B SC**, obtain by C. Rauszer in [6], is that there is a complete and sound semantics for **H-B SC** in terms of Kripke models. The condition for a formula $(A \dot{-} B)$ to be forced (or satisfied) at a node N of Kripke model $K = \langle K, \leqslant, \ldots\rangle$ is given by:

$$K,N \Vdash (A \dot{-} B) \quad \text{iff} \quad \exists N'_{N' \leqslant N}[K,N' \Vdash A \ \& \ K,N' \nVdash B].$$

From the Kripke semantics for **H-B SC** it immediately follows that **H-B SC** is a conservative extension of the **ISC**. Thus it would appear that $\dot{-}$ might be considered as a new *intuitionistic* connective.

Unfortunately in a (complete) semi-Boolean algebra we have the following distributive law:

$$\bigcap_i (b \cup a_i) = b \cup \bigcap_i a_i$$

so that it is not surprising that in the **H-B** Predicate Calculus the schema:

$$\forall x(B \vee Ax) . \supset . B \vee \forall x Ax,$$

is provable, and thus **H-B PC** is not a conservative extension of the Intuitionistic Predicate Calculus, **IPC**. Thus we have second thoughts on whether $\dot{-}$ should be considered as an intuitionistic sentential connective.

This leads to the following problem:

PROBLEM. *Suppose that* S *is a schema, essentially involving quantifiers, such that* S *is provable in the Classical Predicate Calculus,* **CPC**, *but not in the* **IPC**. *Then, is there a sentential connective* $\oplus$ *(with associated rules) so that* ISC + $\oplus$ *is a conservative extension of* **ISC** *and* **IPC** + $\oplus$ $\vdash$ S?

Possible examples for S are: $[P \supset \exists x Qx . \supset . \exists x(P \supset Qx)]$ and $[\forall x \neg\neg Px . \supset . \neg\neg \forall x Px]$. A reason why we believe it may be possible, at least for some schemas is the following observation (suggested to us by the corresponding property for semi-Boolean algebras).

THEOREM. *If* **H** *is an extension of the* **IPC** *such that there is a binary operation* F *on the formulas of* **H** *such that for all formulas* A,B,C *of the first-order language:*

$$H \vdash F(A,B) \supset C \quad \text{iff} \quad H \vdash A \supset (B \vee C);$$

then the schema (restricted to first-order formulas):

$$\forall x(P \vee Qx) . \supset . P \vee \forall x Qx$$

is provable in **H**.

Proof. Let P,Qx be formulas of **IPC** and let a be an individual parameter which does not occur in $\forall x(P \vee Qx)$, nor in $F(\forall x(P \vee Qx),P)$. Then:

$$\mathbf{IPC} \vdash \forall x(P \vee Qx) . \supset . P \vee Qa$$

$$\mathbf{H} \vdash \forall x(P \vee Qx) . \supset . P \vee Qa$$

$$\mathbf{H} \vdash F(\forall x(P \vee Qx),P) \supset Qa$$

$$H \vdash \forall x[F(\forall x(P \vee Qx),P) \supset Qx]$$

$$H \vdash F(\forall x(P \vee Qx),P) \supset \forall x Qx$$

$H \vdash \forall x(P \vee Qx) . \supset . P \vee \forall xQx.$

1. SENTENTIAL CONNECTIVES IN INTUITIONISTIC LOGIC.

Before we can decide what it is meant by an intuitionistic sentential connective, we must have some agreement on what is understood by intuitionistic logic. And to define "intuitionistic logic" one must first define intituitionism and intuitionistic mathematics. Troelstra [8] suggests the following:

"*Intuitionistic mathematics*" is mathematics consistent with L. E. J. Brouwer's reconstruction of mathematics.

"*Intuitionism*" refers to the body of concepts used in the development of intuitionistic mathematics.

"*Intuitionistic logic*" is a formalization of (a part of) intuitionism.

It would thus appear that the place to look for intuitionistic connectives is in "Intuitionism" rather than in Intuitionistic Mathematics or Intuitionistic Logic. Since the principal activity in Intuitionistic Mathematics is obtaining constructions (that prove, or justify, mathematical assertions), we find that the concept "*the construction* c *proves* A" is one of the fundamental concepts of Intuitionism. Or in other words, Intuitionism encompasses some, perhaps informal, theory of constructions T. We shall further assume that in T there are (possibly partial) predicates of the form:

$\pi(c, \ulcorner\theta\urcorner)$ read: *the construction* c *proves* θ

$\pi_A(c)$ read: *the construction* c *proves* A,

and (possibly partial) operations of the form:

c'd = *the result of applying the construction* c *to* d ,

c:d = *the ordered pair of the constructions* c *and* d *(also a construction)*.

1.1. Intuitionistic sentential connectives.

The conditional is usually the most problematic of the connectives. However in classical logic, once one accepts the truth tables then the mysticism of the conditional, as well as of the other connectives, disappears.

Similarly for intuitionistic logic. The intuitionistic conditional $(A \supset B)$ is explained by giving the conditions under which a construction proves $(A \supset B)$; i.e.

$$\pi_{(A \supset B)}(c{:}d) \text{ iff } \pi(c, \ulcorner\pi_A(x) \to \pi_B(d'x)\urcorner).$$

And correspondingly for the other connectives $\vee$, $\wedge$, $\perp$. For "new" connectives we can then proceed as follows:

Suppose that P is a sentential parameter and that $C_P(a)$ is a formula of the theory **T** of constructions in which there is at least one occurence of π_P. Then $C_P(a)$ can be used to define a unary sentential connective $\underset{\sim}{C}$ by stipulating that for all formulae A of the extended language:

$$\pi_{\underset{\sim}{C}_A}(c:d) \quad \text{iff} \quad \pi(c, \ulcorner C_A(d) \urcorner),$$

where $C_A(d)$ is the formula obtained from $C_P(a)$ by replacing all occurrences of P by A, and all occurrences of a by d.

Given a language L, then by L + $\underset{\sim}{C}$ we understand the extension of L obtained by adding the (unary) sentential connective $\underset{\sim}{C}$; **SC** is the sentential language with the connectives $\wedge$, $\vee$, $\supset$, and $\perp$.

NISC is the natural deduction axiomatizacion of ISC, in the language **SC** (see Prawitz [1]).

1.1.1. DEFINITION. *A sentential connective* $\underset{\sim}{C}$ *is* axiomatizable w.r.t. **T** *iff there is a finite set* **R** *of rules such that for every formula* A *of* **SC** + $\underset{\sim}{C}$; *if* **NISC** + R $\vdash$ A *then there is a construction* c *such that in* **T**, $\pi_A(c)$.

1.1.2. DEFINITION. *Suppose that* $\underset{\sim}{C}$ *is an axiomatizable connective*, **R** *its associated rules and* **H** *an extension of intuitionistic logic not containing the connective* $\underset{\sim}{C}$. *Then the connective* $\underset{\sim}{C}$ *is an* intuitionistic sentential connective w.r.t. H *iff* (H+R) *is a conservative extension of* **H**.

1.1.3. DEFINITION. $\underset{\sim}{C}$ *and its associated rules, is an* intuitionistic sentential connective *iff* $\underset{\sim}{C}$ *is an intuitionistic sentential connective* w.r.t. *every intuitionistic logic* **H** *such that* **H** *does not contain* $\underset{\sim}{C}$.

2. THE SENTENTIAL CONNECTIVES $\ulcorner$, $\Box$, $\Diamond$ AND MODELLINGS.

In the **H-B PC** the unary connective $\ulcorner$ can be defined by $\ulcorner A = ((\perp\supset\perp)\dot{-}A)$ and the corresponding condition in the Kripke models is:

$$K,N \Vdash \ulcorner A[\![s]\!] \quad \text{iff} \quad \exists N'_{N' \leqslant N}(K,N' \nVdash A[\![s]\!]),$$

where s is an assignment of the individual parameters of A to the individuals *at the node* N. It is almost immediate from the above that for it to make sense the Kripke model must be one of constant domains, otherwise $K,N' \Vdash A[\![s]\!]$ might fail for the wrong reasons. And it is well known that the formulae valid in all Kripke models with constant domains is *not* a conservative extension of the **IPC**.

Thus any attempt to discover, through the use of Kripke models, if there is an intituitionistic connective corresponding to $\ulcorner$ appears to be doomed from the start. Nevertheless the Kripke models give us a hint of what to look for. The interpretation of Kripke models as stages of positive research (see [6], page 36) give us: *to assert* $\ulcorner A$ *at stage* N *we need to know that there exists an earlier stage* N' *such that our information about* A *is not sufficient to verify* A *at stage* N'.

Since under most interpretations, c and d come before c:d, the above remarks suggest that a possible formula $C_p(a)$ for the connective $\ulcorner$ is $\neg\pi_p(a)$ so that:

$$\pi_{\ulcorner A}(c{:}d) \quad \text{iff} \quad \pi(c, \ulcorner\neg\pi_A(d)\urcorner). \qquad (*)$$

Unfortunately, if we wish to use (*) in order to discover an axiomatization for $\ulcorner$ we must first develop part of the theory T of constructions and the currently available theories of constructions are quite complicated.

Thus we shall use a more ad hoc method for obtaining an axiomatization. Namely, we take the Beth semantics, which is complete and sound for **IPC** and *which uses constant domains*, and try to accomodate the connective $\ulcorner$. The semantics then suggest a set **R** of rules so that **NISC+R** is sound and complete. Once we have a set **R** of rules we can return to the theory **T** of constructions and verify that **R** is indeed an axiomatization (in the sense of definition 1.1.1).

As a matter of fact, the Beth semantics leads us to another connective (which is inter definable with $\ulcorner$) and which, in certain respects, is much more natural.

2.1. Beth models for SC+$\ulcorner$.

We extend the usual definition of satisfaction (forcing) in Beth structures to formulae of SC+$\ulcorner$ by adding the clause:

$$B,N \Vdash \ulcorner A[\![s]\!] \quad \text{iff} \quad \exists N'_{N' \leqslant N}(B,N' \nVdash A[\![s]\!]).$$

Then we define:

$$VAL = \{A \mid \text{for all Beth structures B and all assignments s in B: } B \Vdash A[\![s]\!]\}.$$

An induction on the complexity of the formula A of SC+$\ulcorner$ give us the following:

2.1.1. LEMMA. *For all Beth structures* B,

(1) *If* $B,N \Vdash A$ *and* $N \leqslant N'$ *then* $B,N' \Vdash A$

(2) $B,N \Vdash A$ *iff* $\forall\beta_{N\in\beta}\exists t(B,\bar{\beta}t \Vdash A)$,

where β *ranges over paths through* B, "N ∈ β" *expresses that the node* N *belongs to the path* β *and* $\bar{\beta}t$ *is the node* $\langle\beta0,\beta1,\ldots,\beta(t-1)\rangle$.

2.1.2. DEFINITIONS. $\Box A = \neg\ulcorner A$, $\quad \Diamond A = \ulcorner\neg A$.

2.1.3. COROLLARIES.

(1) $B,N \Vdash \ulcorner A$ *iff* $B \nVdash A$ *iff* $B \Vdash \ulcorner A$.

(2) $B,N \Vdash \Box A$ *iff* $\forall N'(B,N' \Vdash A)$ *iff* $B \Vdash A$ *iff* $B \Vdash \Box A$.

(3) $B,N \Vdash \Diamond A$ *iff* $\exists N'(B,N' \Vdash A)$ *iff* $B \Vdash \Diamond A$ *iff* $B \Vdash \Box\Diamond A$.

2.1.4. LEMMA. *The following schema belong to* VAL:

(1) $\Box A \supset A$.

(2) $A \supset \Diamond A$.

(3) $\Box(A \supset B) . \supset . \Box A \supset \Box B$.

(4) $\Box A \supset \Box\Box A$.

(5) $\Diamond A \supset \Box\Diamond A$.

(6) $\ulcorner A \equiv \neg\Box A$.

(7) $A \vee \ulcorner A$.

(8) $\neg\Box\neg A \equiv \Diamond A$.

(9) $\Diamond\Box A \supset \Box A$.

(10) $\Box A \vee \neg\Box A$.

2.1.5. LEMMA. *There are instances of the following schemas which do not belong to* VAL:

(1) $A \supset \Box A$.

(2) $\Diamond A \supset A$.

(3) $\Box(A \vee B) . \supset . \Box A \vee \Box B$.

(4) $\Box\exists x A x \supset \exists x \Box A x$.

(5) $(\Box A \supset \Box B) \supset \Box(A \supset B)$.

(6) $\neg\Box A \supset \Box\neg A$.

(7) $\neg\Diamond\neg A \supset \Box A$.

2.1.6. DEFINITION. *A formula* A *of* SC+$\ulcorner$ *is* essentially modal (e.m.) *iff either:*

(i) *for some* B, $A = \Box B$, *or*

(ii) *for some* B, $A = \Diamond B$, *or*

(iii) *for some* e.m. B *and* C, $A = (B \vee C)$ *or* $A = (B \wedge C)$, *or*

(iv) *for some* e.m. B, $A = \exists x B$ *or* $A = \forall x B$, *or*

(v) A *is* $\bot$, *or*

(vi) *for some* e.m. B, $A = \neg B$ $(= B \supset \bot)$.

2.1.7. LEMMA. *If* A *is an e.m. formula then* $(A \supset \Box A)$, $A \vee \neg A$ *and* $A \equiv \Box A$, *all belong to* VAL.

From the above results we see that the sentential combinations $\neg\ulcorner$ and $\ulcorner\neg$ behave as modal operators. Since modal operators are better understood than weak (paraconsistent) negations and since according to lemma 2.1.4 $\ulcorner A$ is (semantically) equivalent to $\neg\Box A$, we shall now change to the language PC($\Box$) in which $\Box$ is a sentential connective and $\ulcorner$, $\Diamond$ are abbreviations for $\neg\Box$, $\neg\Box\neg$, respectively. The definition for the satisfaction of $\Box A$ in Beth models is given by 2.1.

2.2. Rules of inference for $\Box$.

$$
(\ulcorner E)\ \frac{\begin{array}{cc} [\Box A] & [\neg\Box A] \\ \vdots & \vdots \\ B & B \end{array}}{B}
$$

$(\Box E)\ \dfrac{\Box A}{A}$ $\qquad$ $(\Box I)\ \dfrac{A}{\Box A}$

$$(\Diamond E)\ \dfrac{\Diamond A \quad \begin{matrix}\not{A} \\ \vdots \\ B\end{matrix}}{B}$$

Restriction on the □I *rule*: Every undischarged assumption formula on which A depends must be an essentially modal formula.
Restriction on the ◇E *rule*: B and every undischarged assumption formula on which B depends (except possibly A) must be an essentially modal formula.

2.3. Some theorems of NMPC.

By **NMPC** we understand the system of natural deduction obtained by adjoining the rules 2.2. to the intuitionistic system.

2.3.1. THEOREM. *The following schemas are provable in* **NMPC**:

(1) $A \vee \ulcorner A$.
(2) $\neg\neg\Box A \supset A$.
(3) $\Diamond A \wedge \Box(A \supset B) . \supset . B$ *provided* B *is an e.m.f.*
(4) $\Diamond A \supset \neg\Box\neg A$.
(5) $\Box\neg B \supset \neg\Box B$.
(6) $\Diamond A \equiv \neg\Box\neg A$.
(7) $\Diamond\Box A \supset \Box A$.
(8) $(A \supset \neg\Box A) \supset \neg\Box A$.
(9) $\neg\neg A \supset \Diamond A$
(10) $\Box A \vee \neg\Box A$
(11) $A \vee \neg A$, *if* A *is an e.m.f.*
(12) $A \equiv \Box A$, *if* A *is an e.m.f.*

2.4. Soundness theorem for NMPC.

For every formula A *of* PC(□), *every Beth structure* $\mathcal{L}$ *and every assignment* S *in* $\mathcal{L}$ *:*

$$\textit{if}\ \ \mathbf{NMPC} \vdash A\ \ \textit{then}\ \ \mathcal{L} \Vdash A\,[\![S]\!].$$

Proof. By induction on the length of the derivation in **NMPC**.

2.5. Completeness theorem for NMPC.

We prove the (weak) completeness theorem in the form that if a sentence S of PC(□) is not provable in **NMPC** then there is a Beth structure $\mathcal{L}_S$ such that $\mathcal{L}_S \nVdash S$.

2.5.1. DRAMATIS PERSONAE.

(1) S, an unprovable sentence of **NMPC**.

(2) $S^* = \ulcorner S$ $(= \neg\Box S)$,

(3) QS = the (finite) set of quasi-subformulae of S^*,

(4) $\mathcal{F} = QS \cup \{\neg G: G \in QS\} \cup \{\Box G: G \in QS\} \cup \{\neg\Box G: G \in QS\} \cdot$
$\cup \{\Box\neg G: G \in QS\} \cup \{\neg\Box\neg G: G \in QS\} \cup \{\bot\}$

(5) L = {A: A is a formula and A is a substitution instance of some $G \in \mathcal{F}$},

(6) Par (θ) = the set of individual parameters ocurring in θ,

(7) $L_k = \{A: A \in L$ and $Par(A) \subseteq \{a_o,..,a_{k-1}\}\}$,

(8) EM = {A: $A \in L$ and A is an e.m. formula} ,

(9) $EM_k = EM \cap L_k$,

(10) NM = L − EM,

(11) $NM_k = L_k - EM_k$,

(12) $A_o, A_1, \ldots$ is an enumeration with infinite repetition of NM, such that for each k, $A_k \in NM_k$,

(13) $\Gamma = \{A: A \in EM$ and $S^* \vdash A\}$,

(14) $\Gamma_1 \supseteq \Gamma$ is such that $\Gamma_1 \subseteq EM$ is consistent and for each formula $\exists xBx \in \Gamma_1$ there corresponds an individual parameter a_{2i} such that $B(a_{2i}) \in \Gamma_1$. Furthermore if $A \in EM$ and $\Gamma_1 \vdash A$ then $A \in \Gamma_1$,

(15) $\Gamma_2 = \Gamma_1 \cup \{\neg G: G \in EM$ and $G \notin \Gamma_1\}$.

2.5.2. CONSTRUCTION OF THE TREE $\sum$.

Basis: $\sum(0) = \sum(< >) = \Gamma_2$.

Recursion step: Suppose $\sum(\vec{u}) = \sum(<u_o,\ldots,u_{k-1}>)$ has already been defined. Consider then the formula A_k $(\in L_k)$.

Case 1: A_k is neither a disjunction nor an existential formula then we have 3 subcases to consider.

Subcase 1: $\sum(\vec{u}) \cap L_k \vdash A_k$.
Then we define $\sum(\vec{u}\frown<1>) = \sum(\vec{u}) \cup \{A_k\}$.

Subcase 2: $\sum(\vec{u}) \cap L_k \vdash \neg A_k$.
Then we define $\sum(u\frown<0>) = \sum(u)$.

Subcase 3: Neither subcase 1 nor subcase 2.
Then we define $\sum(\vec{u}\frown<0>) = \sum(\vec{u})$, $\sum(\vec{u}\frown<1>) = \sum(\vec{u}) \cup \{A_k\}$.

Case 2: $A_k = B_1 \vee B_2$. Then the definition of $\sum$ is changed as follows:

Subcase 1: $\sum(\vec{u}\frown<1>) = \sum(\vec{u}) \cup \{A_k, B_1\}$, if consistent,
$\sum(\vec{u}\frown<2>) = \sum(\vec{u}) \cup \{A_k, B_2\}$, if consistent.

Subcase 2: No change.

Subcase 3: $\sum(\vec{u}\frown<0>) = \sum(\vec{u})$,
$\sum(\vec{u}\frown<1>) = \sum(\vec{u}) \cup \{A_k, B_1\}$, if consistent,
$\sum(\vec{u}\frown<2>) = \sum(\vec{u}) \cup \{A_k, B_2\}$, if consistent.

Case **3**: $A_k = \exists xB(x)$. The definitions are then:

Subcase 1: $\sum(\vec{u}\frown<0>) = \sum(\vec{u}) \cup \{A_k, B(a_{2k+1})\}$,

Subcase 2: No change.

Subcase 3: $\sum(\vec{u}\frown<0>) = \sum(\vec{u})$,

$\sum(\vec{u}\frown<1>) = \sum(\vec{u}) \cup \{A_k, B(a_{2k+1})\}$.

2.5.3. PROPERTIES OF THE TREE $\sum$.

(1) If $A \in L_k$, $\ell th(\vec{u}) = k$ then
$\sum(\vec{u}) \cap L_k \vdash A$ iff $\forall \beta_{\vec{u} \in \beta \in \sum} \exists t[\sum(\bar{\beta}t) \cap L_t \vdash A]$

(2) If $A \in L_k$, $\ell th(\vec{u}) = k$, then
$\sum(\vec{u}) \cap L_k \vdash \neg A$ iff $\forall \beta_{\vec{u} \in \beta \in \sum} \forall t \geqslant k[\sum(\bar{\beta}t) \cap L_t \nvdash A]$

(3) If $(A \supset B) \in L_k$, $\ell th(\bar{u}) = k$, then
$\sum(\vec{u}) \cap L_k \vdash (A \supset B)$ iff $\forall \beta_{\vec{u} \in \beta \in \sum} \forall t \geqslant k[$if $\sum(\bar{\beta}t) \cap L_t \vdash A$ then $\sum(\bar{\beta}t) \cap L_t \vdash B]$

(4) If $A \vee B \in L_k$. $\ell th(\bar{u}) = k$ then
$\sum(\vec{u}) \cap L_k \vdash (A \vee B)$ iff $\forall \beta_{\vec{u} \in \beta \in \sum} \exists t \geqslant k[$Either $\sum(\bar{\beta}t) \cap L_t \vdash A$ or $\sum(\bar{\beta}t) \cap L_t \vdash B]$.

2.5.4. DEFINITION OF THE BETH MODEL $\mathcal{L}_S$. $\mathcal{L}_S$ is the Beth model $<\sum, \leqslant, V>$, where $\sum$ is the tree just defined and V is the function defined on the nodes of $\sum$ such that $V(<u_o,\ldots,u_{k-1}>) =$

$\{A$: A is an atomic formula, $A \in L_k$ and $\sum(<u_o,\ldots,u_{k-1}>) \cap L_k \vdash A\}$.

2.5.5. RELATION BETWEEN THE TREE $\sum$ AND $\mathcal{L}_S$. If $k = \ell th(\vec{u})$, then for all formulae $A \in L_k$, $\sum(u) \cap L_k \vdash A$ iff $\mathcal{L}_S$, $\vec{u} \Vdash A\|S\|$, where S is the identity assignment.

2.5.6. COROLLARY. $\mathcal{L}_S \Vdash \neg\Box S$ *and thus* $\mathcal{L}_S \nVdash S$.

2.5.7. THEOREM. **NMPC** *is a conservative extension of* **NPC** *(i.e. of the intuitionistic predicate calculus).*

2.7. The connectives $\ulcorner$, $\Box$ and theories of constructions.

Now that we have a set of rules of inference for the new connectives we must consider if they are an axiomatization, in the sense of 1.1.1, w.r.t. a theory of constructions.

The most problematic rule is ($\ulcorner$E), which in presence of the rules for v is equivalent to the axiom schema $A \vee \ulcorner A$. Now the schema $A \vee \ulcorner A$ is forced upon us because we are considering satisfaction in Beth structures and, at least from a classical viewpoint, A is valid in the Beth structure B or it is not, from which it follows that $B \Vdash A \vee \ulcorner A$.

One way to avoid such possible unfaithful results would be to use an intuitionistic metatheory on the Beth structures, however we feel that would be counterproductive since one of the pleasant aspects of Beth (and Kripke) semantics is that one can operate on them using classical techniques to obtain results about intuitionistic theories.

So, for the time being, we are stuck with the axiom schema $A \vee \ulcorner A$. Is there then any reasonable interpretation of $\ulcorner$ in terms of constructions?

When the predicates π_A, π were first introduced by Kreisel it was stipulated that they be decidable (the argument being that one always knew if one had a proof or not). Unfortunately too liberal use of that principle quickly leads to a contradiction; nevertheless it is a useful heuristic principle, so we shall temporarily adopt it.

Originally we had stated that a construction e proved $\ulcorner A$ just in case that $e = c{:}d$ and $\pi(c, \ulcorner \neg \pi_A(d) \urcorner)$. Now if π_A is decidable then we need not give explicitly the construction that proves $\neg \pi_A(d)$. In other words, it suffices that $\pi_{\ulcorner A}$ satisfy the conditions:

$$\pi_{\ulcorner A}(d) \quad \text{iff} \quad \neg \pi_A(d).$$

Now the decidability of π_A also gives us:

$$\pi_A(d) \vee \neg \pi_A(d).$$

Hence for all proof constructions d we have that:

$$\pi_A(d) \vee \pi_{\ulcorner A}(d).$$

From which it follows that for all proof constructions d there corresponds a proof construction d^* such that:

$$\pi_{A \vee \ulcorner A}(d^*).$$

$\Box A$ was originally introduced as $\neg \ulcorner A$, so $\pi_{\Box A}(c{:}d)$ iff $\pi(c, \ulcorner \neg \pi_{\ulcorner A}(d'x) \urcorner)$ iff $\pi(c, \ulcorner \neg \neg \pi_A(d'x) \urcorner)$ iff $\pi(c', \ulcorner \pi_A(d'x) \urcorner)$.

It now is routine to verify that relative to the informal theory of constructions considered above, the rules of inference given in 2.2 form an axiomatization for $\Box$ in the sense of definition 1.1.1.

Section 2.5 then give us that the connective $\Box$ (with the associated rules 2.2) is a intuitionistic connective w.r.t. the Intuitionistic Predicate Calculus.

CONJECTURE. *There is an extension* H *of the* IPC *such that the connective* $\Box$ *(with the rules 2.2) is not a intuitionistic connective* w.r.t. H (see §1).

3. CONCRETE MODELS.

A concrete model for a theory of (proof) constructions is an arithmetically definable model, with the natural numbers as the domain of constructions and decidable predicates P, P_A as interpretations of π, π_A respectively.

In Troelstra [8] a concrete model is given for **IIL**, the intuitionistic implicational logic (with the rules ($\supset$I) and ($\supset$E)). In this section we show how to extend the model to the extension **IIL**$_\Box$ of **IIL** obtained by the addition of the unary connective $\Box$ and the rules ($\Box$I) and ($\Box$E) of 2.2.

3.1. The simple model for IIL$_\Box$.

3.1.1. SOME PRELIMINARY DEFINITIONS.

(1) PR a formal (intuitionistic) number theory including at least primite recursive arithmetic. Prf(x,y) is the canonical, primitive recursive, proof predicate for PR.

(2) 0^n, the n-th numeral (in the language of PR).

(3) "A", the numeral corresponding to the Gödel number of A.

(4) $\langle\!\langle A\rangle\!\rangle(x)$ the primitive recursive term such that:

$$PR \vdash \langle\!\langle A\rangle\!\rangle(0^n) = 0^k,$$

where k is the Gödel number of A_0^x.

(5) Der(x,y) is the canonical, primitive recursive, proof predicate for **IIL**$_\Box$ such that PR $\vdash$Der(0^n,"A") iff n is the Gödel number of a derivation in **IIL**$_\Box$ of A.

(6) μ is the canonical, primitive recursive term such that if n is the Gödel number of the derivation Π_1 of (A $\supset$ B), m is the Gödel number of the derivation Π_2 of A, and k is the Gödel number of the derivation of B obtained by ($\supset$E) from Π_1 and Π_2, then

$$PR \vdash \mu(0^n,0^m) = 0^k.$$

(7) δ is the canonical, primitive recursive term such that if n is the Gödel number of the derivation Π of $\Box$A and k is the Gödel number of the derivation of A obtained from Π by ($\Box$E), then PR $\vdash \delta(0^n) = 0^k$.

(8) (j,j_1,j_2) are primitive recursive terms forming an onto pairing system.

3.1.2. DEFINITION OF C_p. To each sentential parameter p of **IIL**$_\Box$ we assign a primitive recursive predicate C_p such that PR $\vdash C_p(x) \supset$ Der(x,"p").

3.1.3. DEFINITION OF T_A. To each formula A of **IIL**$_\Box$ we assign a (primitive recursive) predicate T as follows:

(i) if A is a sentential parameter p, then $T_A(x) \equiv C_p(x)$,

(ii) if A is not a sentential parameter then, then $T_A(x) \equiv$ Der(x,"A").

3.1.4. DEFINITION OF Pr_A AND f_A. To each formula A of **IL□** we assign a primitive recursive predicate Pr_A and primitive recursive function f_A such that:

$$PR \vdash T_A(x) \supset Pr_A(f_A x)$$

and

$$PR \vdash Pr_A(y) \supset Der(j_2 y, "A").$$

(i) If A is the sentential parameter p, then

$$Pr_p(x) \equiv T_p(j_2 x) \wedge (j_1 x = j_2 x)$$
$$f_p(x) = j(x,x).$$

(ii) If A = (B ⊃ C) and Pr_B, Pr_C, f_B and f_C have already been defined, then we proceed as follows:

Assume that $T_{(B\supset C)}(0^n)$ and $Pr_B(y)$. Then $Der(0^n, "(B \supset C)")$, $Der(j_2 y, "B")$. Hence $Der(\mu(0^n, j_2 y), "C")$ and thus $T_C(\mu(0^n, j_2 y))$. Therefore $Pr_C(f_C(\mu(0^n, j_2 y)))$. In other words, we have shown that:

$$PR \vdash T_{(B\supset C)}(0^n) \supset [Pr_B(y) \supset Pr_C(t(0^n, y))],$$

where $t(x,y) = f_C(\mu(x, j_2 y))$. From the latter we obtain

$$PR \vdash T_{(B\supset C)}(0^n) \supset \forall y[Pr_B(y) \supset Pr_C(t(0^n, y))]. \qquad (*)$$

Furthermore, the Gödel number of the proof of (*) is primitive recursive in n so that there is a term g such that

$$PR \vdash Prf(g(x), \ll T_{(B\supset C)}(x) \supset \forall y[Pr_B(y) \supset Pr_C(t(x,y))]\gg(x)).$$

Also, using the fact that $T_{(B\supset C)}(x)$ is primitive recursive, we obtain that there is a term h such that

$$PR \vdash T_{(B\supset C)}(x) \supset Prf(h(x), \ll T_{(B\supset C)}(x)\gg(x)).$$

Combining the last two observations and using the μ function, we obtain a primitive recursive θ such that:

$$PR \vdash T_{(B\supset C)}(x) \supset Prf(\theta(x), \ll \forall y[Pr_B(y) \supset Pr_C(t(x,y))]\gg(x)).$$

Thus we define:

$$Pr_{(B\supset C)}(x) \equiv T_{(B\supset C)}(j_2 x) \wedge Prf(j_1 x, \ll \forall y[Pr_B(y) \supset Pr_C(t(x,y))]\gg(x))$$
$$f_{(B\supset C)}(x) = j(\theta(x), x).$$

(iii) If A = □B and we have at hand Pr_B and f_B then we proceed as follows:

Assume that $T_{\Box B}(0^n)$. Then $T_B(\delta(0^n))$ and hence $Pr_B(f_B(\delta(0^n)))$. Let s be the primitive recursive term such that $s(x,y) = f_B(\delta(x))$, then what we have shown is that

$$PR \vdash T_{\Box B}(0^n) \supset \forall y \; Pr_B(s(0^n, y)).$$

Procceding as in case (ii) we then obtain a primitive recursive φ such that

$$PR \vdash T_{\Box B}(x) \supset Prf(\varphi(x), \langle\!\langle \forall y\ Pr_B(s(x,y)) \rangle\!\rangle (x)) .$$

Thus we define:

$$Pr_{\Box B}(x) \equiv T_{\Box B}(j_2 x) \supset Prf(j_1 x, \langle\!\langle \forall y\ pr_B\ (s(x,y)) \rangle\!\rangle (x)) .$$

$$f_{\Box B}(x) = j(\varphi(x), x).$$

3.1.5. DEFINITION OF P AND P_A.

$P(x,y) \equiv Der(j_2 x, y) \wedge Prf(j_1 x, y)$

$P_A(x) \equiv Pr_A(x)$.

3.1.6. THEOREM. *For each* B, C *of* $\mathbf{IIL}_{\Box}$ *and natural number* n:

(1) $PR \vdash P_{(B \supset C)}(0^n) \equiv P(j_1 0^n, \text{"}\forall y [P_B(y) \supset P_C(t_n(y))]\text{"})$.

(2) $PR \vdash P_{\Box B}(0^n) \equiv P(j_1 0^n, \text{"}\forall y P_B(s_n(y))\text{"})$.

(3) *If* $\mathbf{IIL}_{\Box} \vdash B$ *then for some* m, $PR \vdash P_B(0^m)$.

3.2. Extension to $\mathbf{IIL}_{\Box\ulcorner}$.

Since Pr_A is a decidable predicate , so is $\neg Pr_A$ and thus we may trivially extend the concrete model to the extension $\mathbf{IIL}_{\Box\ulcorner}$ of $\mathbf{IIL}_{\Box}$ obtained by adding the unary connective $\ulcorner$ and the following rules of inference (suggested by 2.2):

$$\dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix} \qquad \begin{matrix}[\ulcorner A] \\ \vdots \\ B\end{matrix}}{B} \qquad\qquad \dfrac{\begin{matrix}[A] \\ \vdots \\ \ulcorner A\end{matrix}}{\ulcorner A}$$

$$\dfrac{\ulcorner \ulcorner A}{A} \ .$$

The extension in the concrete model is as follows:

$$T_{\ulcorner A}(x) \equiv Der(x, \text{"}\ulcorner A\text{"}),$$

$$Pr_{\ulcorner A}(x) \equiv T_{\ulcorner A}(j_2 x) \wedge \neg Pr_A(x).$$

We still obtain that there is a primitive recursive $f_{\ulcorner A}$ such that:

$$PR \vdash T_{\ulcorner A}(x) \supset Pr_{\ulcorner A}(f_{\ulcorner A} x);$$

in fact, $f_{\ulcorner A}(x) = j(x,x)$. For suppose that $T_{\ulcorner A}(x)$. Then clearly $T_{\ulcorner A}(j_2 f_{\ulcorner A} x)$. Now suppose, for reductio ad absurdum, that $Pr_A(f_{\ulcorner A} x)$. That is, suppose that $Pr_A(j(x,x))$. Then $Der(x, \text{"}A\text{"})$, but $Der(x, \text{"}\ulcorner A\text{"})$. Thus $\neg Pr_A(f_{\ulcorner A} x)$. In other words, we have shown that $Pr_{\ulcorner A}(f_{\ulcorner A} x)$.

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