ABSTRACT. For some problems which are defined by combinatorial properties good complexity bounds cannot be found because the combinatorial point of view restricts the set of solution algorithms. In this paper we present a phenomenon of this type with the classical word problem for finitely presented groups. A presentation of a group is called $E_n$-derivation-bounded ($E_n$-d.b.), if a function $k \in E_n$ exists which bounds the derivations of the words defining the unit element. For $E_n$-d.b. presentations a pure combinatorial $E_n$-algorithm for solving the word problem exists. It is proved that the property of being $E_n$-d.b. is an invariant of finite presentations, but that the degree of complexity of the pure combinatorial algorithm may be as far as possible from the degree of complexity of the word problem itself.

The complexity of logical theories and of algorithmic problems in algebraic structures has been object of intensive studies during the last years ([Av], [Av-Mad1], [Can], [Can-Cat], [Fer-Rae], [Gat], [Madl]). One interesting aspect in the proofs of good lower and upper bounds is the fact that some of these results were achieved not only by using combinatorial methods but also by using algebraic arguments. Even more, for some problems which are defined by combinatorial properties good complexity bounds cannot be found because the combinatorial point of view restricts the set of solution algorithms.

In this paper we want to present a phenomenon of this type within the classical word problem for finitely presented groups ([M-K-S]).

Let $\Sigma = \{s_1, \ldots, s_m\}$ be a finite alphabet, $\hat{\Sigma} = \{s_1^*, \ldots, s_m^*\}$ a disjoint copy of $\Sigma$ ($s_i^*$ is the formal inverse of $s_i$), $\Sigma^* = \Sigma \cup \hat{\Sigma}$, and $\Sigma^*$ the set of words over $\Sigma$. For $w = a_1 \ldots a_n \in \Sigma^*$, $a_i \in \Sigma$, let be $w^{-1} = \bar{a}_n \ldots \bar{a}_1$ ($\bar{s} = s$), let $n = |w|$ be the length of $w$, $e$ the empty word, and $L \subseteq \Sigma^*$.

The group $G$ given by the presentation $\langle \Sigma; L \rangle$ can be viewed as the set of equivalence classes of the Thue system...

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where \( u \sim v \) if there is a derivation from \( u \) to \( v \) in \( T \). The set of equivalence classes forms a group with the operations \([u] \cdot [v] = [uv]\) and \([u]^{-1} = [u^{-1}]\), \([e]\) being the unit element.

\( E \) is the set of generators, and \( L \) is the set of defining relators of this presentation. If \( E \) is finite, \( \langle E; L \rangle \) is a finitely generated (f.g) presentation of \( G \), and \( G \) is called f.g. If \( L \) is finite, too, then \( \langle E; L \rangle \) is a finite presentation of \( G \), and \( G \) is finitely presented (f.p.).

The word problem for the presentation \( \langle E; L \rangle \) of \( G \) is the problem of deciding for an arbitrary word \( w \in \Sigma^* \) whether \( w \) defines the unit element of \( G \) or not, i.e. the membership to the set \( \{w \in \Sigma^* | \exists \omega \in \Gamma \omega \neq e \text{ there is a derivation from } w \text{ to } e \in T\} \). It is well known that the complexity of the word problem for \( G \) is independent of the chosen f.g. presentation for \( G \), and we can speak therefore about the complexity of the word problem for \( G \).

We call an algorithm solving the word problem for \( \langle E; L \rangle \) a natural algorithm (n.a.) if for \( w \in \Sigma^* \) it produces a derivation \( w = w_0 \rightarrow \ldots \rightarrow w_m = e \) in the Thue system \( T \). Of course the length of a produced derivation is a lower bound for the complexity of a n.a..

From each solution of the word problem for \( \langle E; L \rangle \) we can define a n.a. simply by generating all derivation in \( T \) for the words \( w \) with \( w \in \Gamma \), in some ordering.

Some questions concerning the n.a. arise. Does the complexity of any n.a. give information about the complexity of the word problem? Of course, it gives an upper bound, but does it give a lower bound in any way, too? Starting with an algorithm which solves the word problem can we produce a n.a. of the same complexity? Given two presentations of the same group, what is the relation between the complexities of natural algorithms in both presentation?

We introduce the concept of derivation bounded presentations to formulate these questions more precisely and also to give the answers. Let \( K \) be any complexity class of word functions. We will restrict ourselves to the Grzegorzyk classes \( \mathbb{E}_n \) which are well known ([Weih]). A finite presentation \( \langle E; L \rangle \) is called \( K \)-derivation-bounded (K-d.b.) if there is a function \( k \in K \) such that every word \( w \in \Sigma^* \) which defines the unit element of \( \langle E; L \rangle \) can be derived to \( e \) in \( T \), within no more than \(|k(w)|\) steps.

For a K-d.b. presentation there is always a standard n.a. for solving the word problem. In order to decide for a word \( w \in \Sigma^* \) whether \( w \not\in \Gamma \), just produce all possible derivation in \( T \) which start with \( w \), of length bounded by \(|k(w)|\), and test whether \( e \) has been derived. If \( K = \mathbb{E}_n \) \( (n > 3) \) this is an \( \mathbb{E}_n \)-algorithm. In particular the word problem for an \( \mathbb{E}_n \)-d.b. finite presentation is decidable.
On the other hand if there is a natural $E_n$-algorithm solving the word problem for $<\Sigma;L>$ then $<\Sigma;L>$ is $E_n$-d.b..

We will prove the following results.

(a) If a f.g. group has an $E_n$-d.b. finite presentation for some $n \geq 1$ then every finite presentation of this group is $E_n$-d.b. So the standard n.a. is an $E_n$-algorithm for all finite presentations of this group, for $n \geq 3$.

(b) Every f.g. group $G$ with $E_n$-decidable word problem ($n \geq 3$), and hence any countable group with $E_n$-decidable word problem ([Ott]), can be embedded into a f.p. group having an $E_n$-d.b. presentation. This means that a n.a. of the same complexity can effectively be constructed from an algorithm solving the word problem for $G$, but in general for a larger group only. The restriction of this n.a. solves the word problem for $G$, but is general it is not a n.a. for $G$.

These two facts give the hope that at least for f.p. $E_n$-d.b. groups with $n \geq 3$ an optimal n.a. exists. But this hope is disappointed by the following fact.

(c) For every $n \geq 4$ there is a f.p. $E_n$- but not $E_{n-1}$-d.b. group $G$ having an $E_3$-decidable word problem. So $G$ has no natural $E_{n-1}$-algorithm for solving the word problem although there is an $E_3$-algorithm for solving it. Thus the complexity of any n.a. may be as far as possible from the complexity of the word problem. These results show that combinatorial properties of a Thue system are not sufficient to prove good complexity bounds for the word problem. Similar results can be proved for semigroups.

Since there is a f.p. group with $E_3$-decidable word problem such that none of its finite presentations allows a natural $E_3$-algorithm, the following question seems to be natural: is there an infinite "easy" presentation of this group for which a natural $E_3$-algorithm exists?

Of course one could take all relators of the group as defining relators of a presentation, which then trivially is $E_0$-d.b., since each derivation is of length 1. But such a presentation is not "easy" because the full complexity of the word problem is contained in the defining relators and so in the presentation. Let an easy presentation of a group be one for which the set of defining relators is $E_1$-decidable. Then we have:

(d) Every f.g. group $G$ with $E_n$-decidable word problem ($n \geq 3$) has a f.g. presentation with an $E_1$-decidable set of defining relators which allows a natural $E_n$-algorithm for solving the word problem.

Similar questions may be posed for finitely axiomatized (f.a.) theories.

Are natural decision algorithms for f.a. theories optimal, or are there easily decidable theories for which the optimal proofs in any finite axiomatization are too long?
1. EN-DERIVATION-BOUNDED GROUPS.

1.1. DEFINITION. Let $G = \langle \Sigma; L \rangle$ be a group, and let $w \in \Sigma^*$ be such that $w \subseteq \Sigma$.

a) A derivation from $w$ is a sequence of words $w = w_0, w_1, \ldots, w_k \subseteq \Sigma$ such that $w_{i+1}$ is formed by insertion of a word $u$ between any consecutive symbols of $w_i$, or before $w_i$, or after $w_i$, or by deletion of a word $u$ if it forms a block of consecutive symbols of $w_i$. In both cases $u$ must be a member of $\mathbb{L} \cup \{s \bar{s} : s \in \Sigma\}$. Here $\mathbb{L}^{-1}$ is defined as $\{w^{-1} : w \in L\}$, where $e^{-1} = e$, $(w^{-1})^{-1} = sw^{-1}$, and $\varepsilon$ denotes the identity of the free monoid $\Sigma^*$.

$k$ is the length of this derivation.

b) Let be $n \geq 1$, $\langle \Sigma; L \rangle$ is $E_n$-derivation-bounded ($E_n$-d.b.) if there is a function $k \in E_n(\Sigma)$ satisfying for all $w \in \Sigma^*$:

$w \subseteq \Sigma$ implies that there is a derivation from $w$ of length $\leq |k(w)|$, where $| |$ denotes the length of a word, i.e. the number of letters. Then $k$ is called an $E_n$-bound for $\langle \Sigma; L \rangle$.

Of course a natural algorithm for solving the word problem exists for a finite $E_n$-d.b. presentation.

1.2. LEMMA. Let $n \geq 1$, and $k \in E_n(\Sigma)$ be such that $k(e) = e$. Then there is a monotonous function $k_1 \in E_n(\Sigma)$ satisfying: $|k_1(u)| + |k_1(v)| \leq |k_1(uv)|$ and $|k_1(w)| \leq |k_1(w)|$ for all $u, v, w \subseteq \Sigma$.

Proof. $n = 1$. Let $k \in E_1(\Sigma)$ with $k(e) = e$. Then there is a monotonous function $k' \in E_1(\Sigma)$ satisfying $|k(w)| \leq |k'(w)|$ and $k'(e) = e$. Define $k_1$ by $k_1(w) = w$ for every $w \subseteq \Sigma$. Let $u, v \subseteq \Sigma$ then $|k_1(u)| + |k_1(v)| = |uv| = |k'(uv)|$. $n \geq 2$. Let $k \in E_n(\Sigma)$ with $k(e) = e$. Then there is a monotonous function $k' \in E_n(\Sigma)$ satisfying $|k(w)| \leq |k'(w)|$ and $k'(e) = e$. Define $k_1(v) = v$, $k_1(ws) = v k(k_1(w), k'(ws))$, where the function $v k \in E_1(\Sigma)$ denotes the concatenation of two words. Then

$k_1(s_0^{\tilde{\beta}_1} \cdots s_r^{\tilde{\beta}_r}) = \bigoplus_{j=1}^r k(s_0^{\tilde{\beta}_j}, s_r^{\tilde{\beta}_j})$

and therefore $|k_1(w)| \leq |w| \cdot |k'(w)|$; since $k'$ is monotonous and $n \geq 2$, $k_1 \in E_n(\Sigma)$ is also monotonous, and $k_1$ a bound for $k$. Now,

$|k_1(u)| + |k_1(v)| = \sum_{j=1}^r |k'(s_j)| + |v| |k'(s_j)|$

$\leq \sum_{j=1}^r |k'(s_j)| + |v| |k'(s_j)| = |u| |k'(us_j)| = |k_1(uv)|$.

This proves Lemma 1.2.

1.3. REMARK. If $k$ is an $E_n$-bound it may be assumed that $k(e) = e$. But then because of 1.2 it may be assumed that $k$ is monotonous and satisfies $|k(u)| + |k(v)| \leq |k(uv)|$.

Now we give an example of an $E_0$-d.b. presentation.
1.4. LEMMA. \( F = \langle \Sigma; \emptyset \rangle \), the free group generated by \( \Sigma \), is \( E_0 \)-d.b.

Proof. Define \( k(w) \equiv w \), then \( k \in E_0(\Sigma) \). Now let \( w \in \Sigma^* \) such that \( w \notin \emptyset \). This means \( \gamma_F(w) \equiv e \), where \( \gamma_F \) denotes the function calculating the free reduction. But the execution of the free reduction gives a derivation from \( w \) of length \( \frac{1}{2}\|w\| \). So \( k \) is an \( E_0 \)-bound for \( \langle \Sigma; \emptyset \rangle \).

The following three propositions give technics to construct \( E_n \)-d.b. presentations of groups from given \( E_n \)-d.b. presentations, such that the groups defined by the given presentations are embedded into the groups defined by the constructed presentations.

1.5. PROPOSITION. Let \( H_1 = \langle \Sigma_1; L_1 \rangle \) and \( H_2 = \langle \Sigma_2; L_2 \rangle \) be groups such that \( \langle \Sigma_1; L_1 \rangle \) and \( \langle \Sigma_2; L_2 \rangle \) are \( E_n \)-d.b. for some \( n \geq 2 \). Then

a) the presentation \( \langle \Sigma_1 \cup \Sigma_2; L_1, L_2 \rangle \) of \( H_1 \ast H_2 \) is \( E_n \)-d.b., and

b) the presentation \( \langle \Sigma_1 \cup \Sigma_2; L_1, L_2, \alpha \beta \mid \alpha \in \Sigma_1, \beta \in \Sigma_2 \rangle \) of \( H_1 \ast H_2 \) is \( E_n \)-d.b.

Proof. Without loss of generality it may be assumed that \( \Sigma_1 \) and \( \Sigma_2 \) are disjoint alphabets. Let \( k_1 \in E_n(\Sigma_1) \) and \( k_2 \in E_n(\Sigma_2) \) be \( E_n \)-bounds for \( \langle \Sigma_1; L_1 \rangle \) and \( \langle \Sigma_2; L_2 \rangle \), respectively, and let \( w \equiv u_0 \circ \ldots \circ u_i v_j \ldots u_1 v_1 u \in \Sigma_1^* \), where \( u_i \) and \( v_j \) are the syllables of \( w \).

a) \( w \equiv e \) in \( H_1 \ast H_2 \). Then there is an \( i \in \{0, \ldots, l\} \) such that \( e \notin u_i \mathbb{P}_1 e \) or \( e \notin v_j \mathbb{P}_2 e \). So within no more than \( |k_1(u_i)| \), respectively \( |k_2(v_j)| \), steps \( w \) can be derived to a word \( w' \) containing less syllables than \( w \). Hence there is a derivation from \( w \) of length

\[
\mu \leq |k_1 \circ \mathbb{P}_1 (w) + |k_2 \circ \mathbb{P}_2 (w)|,
\]

where \( \mathbb{P}_1 \equiv E_1(\Sigma_1 \cup \Sigma_2) \) denotes the projection onto \( \Sigma_1^* \).

Define for \( s \in \Sigma_1 \cup \Sigma_2 \), \( U_1(s) = s|w| \), which is an \( E_1 \)-function. Let \( k(w) = \text{vk}(k_1 \circ U_1(w), k_2 \circ U_2(w)) \) for some \( a_1 \in \Sigma_1, b_1 \in \Sigma_2 \). Then \( k \in E_n(\Sigma_1 \cup \Sigma_2) \) with

\[
|k(w)| = |k_1 \circ U_1 (w)| + |k_2 \circ U_2 (w)| = |k_1 \circ \mathbb{P}_1 (w)| + |k_2 \circ \mathbb{P}_2 (w)|
\]

since \( |\mathbb{P}_1 (w)| \leq |w| = |U_1 (w)| \) and \( |\mathbb{P}_2 (w)| \leq |w| = |U_2 (w)| \). Hence \( k \) is an \( E_n \)-bound for \( \langle \Sigma_1 \cup \Sigma_2; L_1, L_2 \rangle \).

b) \( w \equiv e \) in \( H_1 \ast H_2 \). Then \( w = \mathbb{P}_1 (w) \mathbb{P}_2 (w) \) in \( H_1 \ast H_2, \mathbb{P}_1 (w) \mathbb{P}_1 e, \) and \( \mathbb{P}_2 (w) \mathbb{P}_2 e \).

There is a derivation from \( \mathbb{P}_1 (w) \) of length not exceeding \( |k_1 \circ \mathbb{P}_1 (w)| \) in \( \langle \Sigma_1; L_1 \rangle \), and there is a derivation from \( \mathbb{P}_2 (w) \) of length not exceeding \( |k_1 \circ \mathbb{P}_2 (w)| \) in \( \langle \Sigma_2; L_2 \rangle \). \( w \) can be derived to \( \mathbb{P}_1 (w) \mathbb{P}_2 (w) \) by sequences of the form \( \alpha \beta \alpha \beta \beta \beta \alpha \beta \). Therefore \( \mathbb{P}_1 (w) \mathbb{P}_2 (w) \) can be derived from \( w \) within no more than \( 3|\mathbb{P}_1 (w)| \cdot |\mathbb{P}_2 (w)| \) steps. Define \( \text{VK}(w, e) \equiv e, \text{VK}(w, u_1) \equiv \text{vk}(\text{VK}(w, u_1), u_2) \).

Then

\[
\text{VK}(w, u) \equiv w|u| \text{ and } \text{VK} \in E_2(\Sigma_1 \cup \Sigma_2).
\]

Now let \( k(w) \equiv \text{vk}((\text{VK}(w, w))^3, \text{vk}(k_1 \circ U_1 (w), k_2 \circ U_2 (w))) \). Since \( n \geq 2 \),

\[
k \in E_n(\Sigma_1 \cup \Sigma_2)
\]
136

\[ |k(w)| > 3|w|^2 + |k_1 \circ \$E_1 (w)| + |k_2 \circ \$E_2 (w)| > 3|\$E_1 (w)| + |\$E_2 (w)| + |k_1 \circ \$E_1 (w)| + |k_2 \circ \$E_2 (w)|. \]

Hence \( k \) is an \( E_n \)-bound for \( \langle z_1 \cup z_2; z_1, z_2 \rangle \), \( a \in z_1, b \in z_2 \).

1.6. PROPOSITION. Let \( H = \langle z_1; l, z_2 \rangle \) be an \( E_n \)-d.b. for some \( n \geq 3 \).

a) If \( H^* = \langle H, t; t u^i t v^j ; i = 1, \ldots, \ell \rangle \) is an HNN-extension of \( H \) with rewriting functions \( \omega_u \) for \( \langle u_1, \ldots, u_\ell \rangle_H \) and \( \omega_v \) for \( \langle v_1, \ldots, v_\ell \rangle_H \) bounded by polynomials, then the given presentation of \( H^* \) is \( E_n \)-d.b.

b) If \( H^* = \langle \ell, t_1, \ldots, t_k; t_1 t v^j, \ldots, t_k t v^j ; j = 1, \ldots, \ell \rangle \), \( i = 1, \ldots, k \) is an HNN-extension of \( H \) with rewriting functions \( \omega_{u_1} \) for \( \langle u_1, \ldots, u_\ell \rangle_H \) and \( \omega_{v_1} \) for \( \langle v_1, \ldots, v_\ell \rangle_H \), \( i = 1, \ldots, k \), bounded by polynomials, then the given presentation of \( H^* \) is \( E_n \)-d.b. (See [Lyn-Sch] for the definition of HNN-extension).

Proof. As part (b) is nothing else than a finite iteration of part (a) it suffices to prove part (a).

Define \( \varphi : U = \langle u_1, \ldots, u_\ell \rangle_H \rightarrow V = \langle v_1, \ldots, v_\ell \rangle_H \) as follows: If \( w \in \Sigma^* \cap U \), then \( w \mapsto \omega_u (w) \equiv \prod_{j=1}^{\ell} u_{ij} \). Let \( \varphi (w) \equiv \prod_{j=1}^{\ell} v_{ij} \). Define \( \varphi : V \rightarrow U \) analogously. Now \( \varphi \) and \( \varphi \) realize the isomorphisms used for constructing the HNN-extension \( H^* \) of \( H \). \( \omega_u \) and \( \omega_v \) are bounded by polynomials, and so are \( \varphi \) and \( \varphi \). Therefore \( c \geq 1 \) and \( d \geq 2 \) can be chosen in such a way that for all \( w \in \Sigma^* \), \( |\omega_u (w)|, |\omega_v (w)|, |\varphi (w)|, |\varphi (w)| < \epsilon |w|^d \) are valid.

Define \( f(e) = e \), \( f(ws) = f(w) s, s \in \Sigma \).

\[
\begin{align*}
\text{f}(w t) &= \begin{cases} 
\text{u} \varphi (v) & \text{if } f(w) \equiv utv, v \in \Sigma^* \cap U \\
(\text{f}(w) t & \text{ otherwise}
\end{cases} \\
\text{f}(w t) &= \begin{cases} 
\text{u} \varphi (v) & \text{if } f(w) \equiv utv, v \in \Sigma^* \cap V \\
(\text{f}(w) t & \text{ otherwise}
\end{cases}
\end{align*}
\]

According to [Av-Madl] 3.2, p.94, \( f \) is a \( t \)-reduction function for \( H^* \) satisfying

\[ \forall w \in (\Sigma U (t))^* \mid f(w) \mid < 2^2 c d |w|. \]

Let \( k \) be an \( E_n \)-bound for \( \langle z_1; L \rangle \), and let \( \omega \in \Sigma^* \cap (1) \) such that \( w \equiv e \). Then \( f(w) \in \Sigma^* \) and \( f(w) \equiv e, f(w) \) results from \( w \) by pinching out \( k_1 |w| \) \( t \)-pinches, and subsequently \( f(w) \) can be derived to \( e \) in \( \langle z_1; L \rangle \) within no more than \( k_1 |f(w)| \) steps. Let

\[ w \equiv w_0 u_1 \cdots u_k w_0 \cdots w_k \in \Sigma^* \], \( u_0, \ldots, u_k \in \Sigma^* \), \( u_1, \ldots, u_k \in \{ \pm 1 \} \)

and

\[ t^{-2} \]

be the lefmost \( t \)-pinch contained in \( w \).

\[ u_i = -1 : w \equiv w_0 u_1 \cdots u_{i-1} t w_i t w_{i+1} \cdots w_k (1) \]

\[ w_0 \cdots w_{i-1} t w_i (u (w_i))^{-1} u (w_i) t w_{i+1} \cdots w_k (2) \]
\[ w_0 \cdot w_{i-1} \cdot \tilde{w}_u(w_i) \cdot t\tilde{w}_{i+1} \cdot \ldots \cdot w_k \quad (3) \]

\[ w_0 \cdot \tilde{u}_1 \cdot w_{i-1} \cdot (w_i) \cdot \tilde{w}_{i+1} \cdot \ldots \cdot w_k \equiv w' \]

**ad (1)**, \(|\omega_u(w_i)|\) trivial relators are inserted.

**ad (2)**, \( w_i(\omega_u(w_i))^{-1} \equiv e \), and so \( w_i(\omega_u(w_i))^{-1} \) can be derived to \( e \) in \( <\Sigma; L> \) within at most \( |k_H(\omega_u(w_i))^{-1}| \) steps.

**ad (3)**, \(|\omega_u(w_i)|_u\) = the number of generators \( u_1, \ldots, u_\ell \) in \( \omega_u(w_i) \). Now (3) can be realized by \(|\omega_u(w_i)|_u\) steps of the following kind:

(a) Insertion of \( \tilde{t} \).
(b) Insertion of \( \tilde{v}_j t \tilde{v}_j \) by using trivial relators.
(c) Deletion of \( \tilde{t} \).

Hence within at most

\[ m_1 = |\omega_u(w_i)|_u \cdot |k_H(\omega_u(w_i))^{-1}| + |\omega_u(w_i)|_u \cdot (2 + \max_{j=1, \ldots, \ell} |v_j|) \]

steps the first \( t \)-pinch of \( w \) can be pinched out.

\[ m_1 \leq |\omega_u(w_i)| \cdot (3 + \max_{j=1, \ldots, \ell} |v_j|) + k_H(\omega_u(w_i))^{-1} | =: m_2 \]

since

\[ |\omega_u(w_i)|_u \leq |\omega_u(w_i)|. \]

**ad (1)**, \(|\omega_v(w_i)|\) trivial relators are inserted.

**ad (2)**, \( w_i(\omega_v(w_i))^{-1} \equiv e \), and so \( w_i(\omega_v(w_i))^{-1} \) can be derived to \( e \) in \( <\Sigma; L> \) within no more than \( |k_H(\omega_v(w_i))^{-1}| \) steps.

**ad (3)**, by \(|\omega_v(w_i)|_v\) steps of the following kind (3) can be realized:

(a) Insertion of \( \tilde{t} u_j t \tilde{u}_j \) by using trivial relators.
(b) Deletion of \( \tilde{t} u_j t \tilde{u}_j (\equiv (\tilde{t} u_j t \tilde{v}_j)^{-1}) \) and of \( \tilde{t} \).

In this way \( u_j t \) is derived from \( v_j \). Hence within at most

\[ m^+_1 = |\omega_v(w_i)| + |k_H(\omega_v(w_i))^{-1}| + |\omega_v(w_i)|_v \cdot (4 + \max_{j=1, \ldots, \ell} |u_j|) \]

steps the first \( t \)-pinch of \( w \) can be pinched out.

\[ m^+_1 \leq |\omega_v(w_i)| \cdot (5 + \max_{j=1, \ldots, \ell} |u_j|) + |k_H(\omega_v(w_i))^{-1}| =: m^+_2 \]

since

\[ |\omega_v(w_i)|_v \leq |\omega_v(w_i)|. \]
Let \( A = \max \{|u_j|, |v_j|\} \), and \( a \in \Sigma \). Now the first \( t \)-pinch of \( w \) can be pinched out in at most \( c |w|^d (5A + |k_H(a(c+1)|w|^d) \) steps. Let \( w_1' \) be the word formed by pinching out the first \( t \) pinches of \( w \).

**Assertion.** Let \( i \in \{1, 2, \ldots, t |w|_t \} \). Then \( |w_i'| \leq (c+1)^{d2i-1} |w|^d \), and \( w_i' \) can be derived from \( w_{i-1}' \) within \( m_i' \) steps where \( m_i' \) satisfies

\[
m_i' \leq (5A) \cdot (c+1)^{d2i-1} |w|^d + |k_H(a(c+1)|w|^d) |
\]

**Proof.** By induction on \( i \).

\( i = 1 \): \( w_1' \equiv w' \), then \( |w_1'| = |w| - |w_1'| - 2 + |q^H(w_1)| \)

\[
\leq |w| + c |w|^d \leq (c+1)|w|^d \leq (c+1)^d |w|^d.
\]

\[
m_1' \leq c |w|^d (5A + |k_H(a(c+1)|w|^d) | \leq (5A) (c+1)^d |w|^d.
\]

\( i > 1 \): \( w_{i+1}' \) is formed from \( w_i' \) by pinching out a \( t \)-pinch, then

\[
|w_{i+1}'| \leq |w_i'| + c |w_i'| |d \leq (c+1) |w_i'| |d
\]

\[
\leq (c+1) \cdot (c+1)^{d2i-1} |w|^d = (c+1)^{d2i+1} |w|^d + 1
\]

\[
\leq (c+1)^{d2i+1} |w|^d + 1,
\]

and

\[
m_{i+1}' \leq (5A) c |w|^d |k_H(a(c+1)|w_i'| |d)
\]

\[
\leq (5A) \cdot (c+1)^{d2i-1} |w|^d |d + |k_H(a(c+1)|w|^d) |
\]

\[
= (5A) c (c+1)^{d2i+1} |w|^d |d + |k_H(a(c+1)|w|^d) |
\]

\[
\leq (5A) (c+1)^{d2i+1} |w|^d |d + |k_H(a(c+1)|w|^d) |
\]

Let \( w^+ \) be the word formed by pinching out all \( t \)-pinches of \( w \). Then \( w^+ \equiv w_{2t} |w|_t \), and hence

\[
|w^+| \leq (c+1)^d |w|^d |w|_t \leq ((c+1)^d |w|_t q |w|
\]

The derivation from \( w \) to \( w^+ \) can be performed within

\[
m = \frac{1}{2} m_i' \leq \frac{1}{2} \left\{ (5A)(c+1)^{d2i-1} |w|^d |d + |k_H(a(c+1)|w|^d) | \right\}
\]

steps. At last, \( w^+ \) is derived to \( e \) in \( \Sigma^*L^* \) within at most

\[
|k_H(w^+)| \leq |k_H(a(c+1)|w|^d)|
\]

Hence there is a derivation from \( w \) in the given presentation of \( H^* \) of length not exceeding

\[
m_w = m^+ + |k_H(w^+)| \leq |w| \left\{ (5A)(c+a)|w|) q |w| + |k_H(a(c+1)|w|^d) | \right\}
\]

Define \( d_1(w) \equiv a |d| |w|, \), \( d_2(w) \equiv VK(w, a^{c+1}) \), and

\[
d_3(w, e) \equiv a, d_3(w, w') \equiv VK(d_3(w, w), w).
\]
Then \( d_1 \in E_3(\Sigma U\{t\}) \), \( d_2 \in E_2(\Sigma U\{t\}) \), \( d_3 \in E_3(\Sigma U\{t\}) \), \( d_2(w) = w^{c+1} \)

\[ |d_2(w)| \equiv (c+1)|w| \]
and \( d_3(w,u) = a|w||u| \). 

\( d_4(w) \equiv d_3(d_2(w),d_1(w)) \) is a function from \( E_3(\Sigma U\{t\}) \) satisfying

\[ d_4(w) = a((c+1)|w|d|w|) \]

and \( k(w) \equiv VK(vK(d_4(w),a^{5+A}), k_H d_4(w)) \) is from \( E_n(\Sigma U\{t\}) \) satisfying:

\[ |k(w)| = |w|((5+A)((c+1)|w|d|w|) + k_H a((c+1)|w|d|w|)) \]

Hence \( k \) is an \( E_n \)-bound for the given presentation of \( H^* \). Thus this presentation is \( E_n \)-b.d.

1.7. PROPOSITION. The \( H = <\Sigma;L> \) be \( E_n \)-b.d. for some \( n \geq 2 \). If \( H^* = \langle H; t; t_1t^{-1}; i = 1, \ldots, t \rangle \) is an HNN-extension of \( H \) with the identity as isomorphism and with a rewriting function \( w \in E_n(\Sigma) \) for \( <u_1, \ldots, u_\ell>_H \), then the given presentation of \( H^* \) is \( E_n \)-b.d.

**Proof.** Define \( f(e) \equiv e \), \( f(ws) \equiv f(w)s \), \( s \in \Sigma \),

\[
    f(wt^u) \equiv \begin{cases} 
    uv & \text{if } f(w) \equiv ut^v, \ v \in \Sigma^* \cap <u_1, \ldots, u_\ell>_H \\
    f(w)t^u & \text{otherwise} 
    \end{cases}
\]

\( f \) is a \( t \)-reduction function for \( H^* \) satisfying \( |f(w)| \leq |w| \). Let \( w \in (\Sigma U\{t\})^* \)

with \( w \equiv e \). Then \( f(w) \in \Sigma^* \) and \( f(w) \equiv e \). Therefore \( w \) can be derived to \( e \) by

first pinching out all the \( t \)-pinches of \( w \) and thereafter deriving the resulting word to \( e \) in \( \langle \Sigma;L \rangle \). \( \omega(e) \equiv e \) may be assumed. Then according to Lemma 1.2 there

is a monotonous function \( \omega \in E_n(\Sigma) \) satisfying \( |\omega(w)| \leq |\omega_2(w)| \) and

\[ |\omega_2(u)| + |\omega_2(v)| \leq |\omega_2(uv)| \] for every \( w,u,v \in \Sigma^* \).

Let \( k_H \in E_n(\Sigma) \) be an \( E_n \)-bound for \( \langle \Sigma;L \rangle \), and let \( w \equiv w_0 t^u_1 \ldots t^u_i w_r \),

\( w_0, \ldots, w_r \in \Sigma^* \), \( u_1, \ldots, u_i \in \{t\} \), with \( w \equiv e \) contain the \( t \)-pinch \( t^{u_i}w_i^t t^{u_i+1} \). This \( t \)-pinch can be pinched out by the following sequence of operations:

\[
    w \equiv w_0 t^{u_1} w_1 \ldots w_{i-1} t^{u_i} w_i t^{u_i+1} w_{i+1} \ldots w_r \quad (1)
\]

\[
    w_0 \ldots w_{i-1} t^{u_i} w_i (\omega(w_i))^{-1} \omega(w_i) t^{u_i+1} w_{i+1} \ldots w_r \quad (2)
\]

\[
    w_0 \ldots w_{i-1} t^{u_i} \omega(w_i) t^{u_i+1} w_{i+1} \ldots w_r \quad (3)
\]

\[
    w_0 \ldots t^{u_i-1} w_i^{-1} \omega(w_i) w_i t^{u_i+2} \ldots w_r \quad (4)
\]

\[
    w_0 \ldots t^{u_i-1} w_i^{-1} \omega(w_i) w_i t^{u_i+2} \ldots w_r \quad (5)
\]

\[
    w_0 \ldots t^{u_i-1} w_i^{-1} w_i t^{u_i+2} \ldots w_r \equiv : w'.
\]

ad (1), \( |\omega(w_i)| \) trivial relators are inserted.
\textit{ad} (2), \( w_1'(\omega(w_1))^{-1} \) can be derived to \( e \) in \( \langle \Xi; l \rangle \) within at most \( |k_H(w_1'(\omega(w_1))^{-1})| \) steps.

\textit{ad} (3), \( |\omega(w_1)| \) steps of the following form:

\[ u_1 = -1: (a) \text{ Insertion of } t_u^{-1}u_j \text{ by using trivial relators} \]

\[ (b) \text{ Deletion of } t_u^{-1}u_j \]

In this way \( u_1 \) is derived from \( t_u \).

\[ u_1 = 1: (a) \text{ Insertion of } t_u^{-1}u_j \text{ by using trivial relators} \]

\[ (b) \text{ Deletion of } u_jt_u^{-1} \text{ and of } t_i: \]

\[ t_u + t_u^{-1}t_u \rightarrow t_u^{-1}t_u \rightarrow u_1.t \]

\textit{ad} (4), \( w_1'(\omega(w_1))^{-1} \) can be derived from \( e \) by inverting the derivation of \( (2) \).

\textit{ad} (5), \( |\omega(w_1)| \) trivial relators are deleted.

Hence the t-pinch of \( w \) can be pinched out within

\[ m' \leq |\omega(w_1)| + |k_H(w_1'(\omega(w_1))^{-1})| + |\omega(w_1)| \max_{j=1, \ldots, |u_j|} \]

\[ (4+\max_{j=1, \ldots, |u_j|}) \]

steps, since \( |\omega(w_1)| \leq |\omega(w_1)| \). Let \( A = \max_{j=1, \ldots, |u_j|} \). Then

\[ m' \leq |\omega(w_1)| + 2|k_H(w_1'(\omega(w_1))^{-1})| \leq 2|k_H(w_1,w_2(w_1))| + (6+A)|w_2(w_1)|, \]

since \( |\omega(w_1)| = |\omega(w_1)| \leq |\omega(w_2)| \), and \( k_H \) being monotonous,

\[ \leq 2|k_H(a[w] + \omega_2(a[w])}| + (6+A)|\omega_2(a[w])| \]

since \( |w_1| \leq |w| \), and \( k_H \) and \( \omega_2 \) being monotonous.

\( \omega(w_1)^* \) t-pinches must be pinched out. Of course \( |w'| \leq |w| \). Hence \( w \) can be derived to \( e \) in the given presentation of \( H^* \) within \( m \) steps where \( m^* \) satisfies:

\[ m^* \leq \frac{\omega(w_1)^*}{l} + (2|k_H(a[w] + \omega_2(a[w])}| + (6+A)|\omega_2(a[w])|} \]

\[ \leq |w||k_H(a[w] + \omega_2(a[w])}| + (3+A)|\omega_2(a[w])|} \].

\( f(w) \) is derived to \( e \) in \( \langle \Xi; l \rangle \) within at most \( m \leq |k_H^o f(w)| \leq |k_H(a[w])| \) steps, as \( |f(w)| \leq |w| \) and \( k_H \) being monotonous. Hence \( w \) can be derived to \( e \) in the given presentation of \( H^* \) within \( m \) steps where \( m \) satisfies:

\[ m = m^* + m \leq |w||k_H(a[w] + \omega_2(a[w])}| + (3+A)|\omega_2(a[w])|} + k_H(a[w])|} . \]

Define

\[ k(w) = \omega(v_k(V_k(k_H^o v_k(U_{a(w)}), U_{\omega_2(w)}U_{a(w)}), V_k(\omega_2U_{a(w)}, a^{3+A}), w), k_H^o U_{a(w)}). \]

Then \( k \in E_n(\Xi \cup \{t\}) \) and \( k \) satisfies:

\[ |k(w)| = |w||k_H(a[w] + \omega_2(a[w])}| + (3+A)|\omega_2(a[w])|} + k_H(a[w])|} . \]

Therefore \( k \) is an \( E_n \)-bound for the given presentation of \( H^* \), which is \( E_n \)-d.b. herewith.
2. AN EMBEDDING INTO DERIVATION-BOUNDED GROUPS.

The proposition in Sec. 1 gives examples of embeddings of d.b. groups into d.b. groups. But now the question arises whether a group possessing no $E_n$-d.b. presentation can be embedded into a $E_n$-d.b. group. The answer to this question is given by the next theorem and its corollary.

2.1. THEOREM. Let $G = \langle \Sigma; L \rangle$ be f.g. with $\text{WP}_G \in E_n(\Sigma)$, i.e. the word problem for the given presentation $\langle \Sigma; L \rangle$ of $G$ is $E_n$-decidable, for some $n \geq 3$. Then there is a finite $E_n$-d.b. presentation $\langle \Sigma; M \rangle$ of a group $H$ such that $G$ can be embedded in $H$.

Proof. Starting with $\langle \Sigma; L \rangle$ we construct $\langle \Sigma; M \rangle$ in a few number of steps. Let $\hat{L} = \{ w \in \Sigma^* | w = e \}$, and $\hat{G} = \langle \Sigma; \hat{L} \rangle$. Then $\hat{G}$ is f.g., $\text{WP}_G \in E_n(\Sigma)$, $\hat{G} \cong G$, via the identity mapping, and for each word $w \in \Sigma^*$ with $w \neq e$ there is a derivation of length 1 in $\langle \Sigma; L \rangle$, because $w = e$ in $\hat{G}$ implies $w \in \hat{L}$.

Let $\hat{\Sigma} = \{ \hat{s} | s \in \Sigma \}$ be a copy of $\Sigma$ satisfying $\hat{\Sigma} \cap \Sigma = \emptyset$, $\Sigma_0 = \Sigma \cup \hat{\Sigma}$, and let $\hat{\Sigma}^* \rightarrow \Sigma_0^*$ be defined by $\hat{\Sigma}(s) \equiv \hat{s}$, $\hat{\Sigma}(\hat{s}) \equiv s$. Let $L_0 = \{ w \in \Sigma_0^* | \exists u \in \hat{L}: \hat{\Sigma}(u) = w \}$ and $G_0 = \langle \Sigma_0; L_0 \rangle$, then $G_0$ is f.g., $\text{WP}_{G_0} \in E_n(\Sigma_0)$, $G_0 \cong G$ via $\hat{\Sigma}$, the defining relations of $G_0$ do contain only positive letters, and for every word $w \in \Sigma_0^*$ with $w \neq e$ there is a derivation of length $\leq 2|w| + 1$ in $\langle \Sigma_0; L_0 \rangle$, because at first all letters of the form $\hat{s}$ (s \in \Sigma) contained in $w$ must be substituted by $s$ by means of the derivation $\hat{s} \rightarrow \hat{s}s\hat{s} \rightarrow s$, as $\hat{s}s\hat{s} \in L_0$, and at last the produced word $w' \in \Sigma_0^*$ can be deleted in one step.

$L_0$ is an $E_n$-decidable subset of $\Sigma_0^*$. Hence there is a Turing Machine $T = (E_0, Q_T, q_0, \beta)$, where $Q_T$ is a finite set of states, $q_0 \in Q_T$ is the initial state of $T$, and $\beta$ is the transition function of $T$, and a function $g \in E_n(\Sigma_0)$ such that $T$ computes the characteristic function of the set $L_0$ and $g$ is a time bound for $T$.

Now $T$ can be modified to get a Turing Machine $\tilde{T} = (E_0, Q_T, q_0, \beta)$, where $\tilde{E}_0$ is a finite alphabet including $\Sigma_0$, satisfying the following two conditions:

1. There is a special state $q_a \in Q_T$ called the accepting state such that starting at $q_0^w$, $\tilde{T}$ eventually reaches the state $q_a$ if and only if $w \in L_0$.

2. There is a function $k_T \in E_n \subseteq (\tilde{E}_0^* \cup Q_T^*)$ satisfying for all $u, v \in \tilde{E}_0^*$, $q_j \in Q_T$: starting at the configuration $u_j v$, $\tilde{T}$ halts within $|k_T(u_j v)|$ steps if $\tilde{T}$ reaches the accepting states $q_a$ after all.

Especially it is $E_n$-decidable whether starting at $u_j v$, $\tilde{T}$ eventually reaches the state $q_a$. For that $\tilde{T}$ works as follows:
The tape is divided into four tracks. The input is copied onto track $N^0 1$. Below the leftmost letter of the copied input a "1" is printed onto track $N^0 4$.

Loop:

Track $N^2 1$ is copied onto track $N^0 2$, and track $N^0 4$ is copied onto track $N^0 3$. If a letter $a \in \Sigma^* - \Sigma_0$ is contained in $w$, or if a letter $a \neq 1$ is contained in the inscription of track $N^2 4$, $\tilde{T}$ halts at the state $q_1$, a nonaccepting state. Otherwise $\tilde{T}$ simulates $T$ starting at $q_0 w$ on its track $N^0 2$. Ahead of each step of this simulation a "1" is erased from track $N^0 3$. If $T$ halts and accepts, then $\tilde{T}$ halts at state $q_a$. If $T$ halts without accepting, then $\tilde{T}$ halts at state $q_{-}$. If the whole inscription of track $N^0 3$ is erased before reaching the end of the computation of $T$, then $\tilde{T}$ breaks off the simulation of $T$, cleans track $N^0 2$, adds a "1" to the inscription of track $N^0 4$, and starts the loop again.

For carrying out this computation, $\tilde{T}$ needs two additional tracks as scratch paper to note the direction of the beginning of the inscription of track $N^0 1$, and with it the beginning of the inscriptions of tracks $N^0 3$ and $N^0 4$, and the direction in which the actual cell of track $N^0 2$ is situated in relation to the position of the head of $\tilde{T}$.

Now the following is satisfied: for $w \in L_0$ starting at $q_0 w$, $T$ halts and accepts. Hence starting at $q_0 w$ $\tilde{T}$ reaches the state $q_a$. On the other hand if starting at $q_0 w$, $\tilde{T}$ reaches the state $q_{-}$, then $w \in \Sigma_0^*$, and $T$ halts and accepts starting at $q_0 w$, i.e. $w \in L_0$.

With the input $w \in \Sigma_0^*$, $T$ does not carry out more than $\lceil g(w) \rceil$ steps, $\tilde{T}$ simulates $T$ step by step. Each step of this simulation takes $\tilde{T}$ at most $0(\lceil g(w) \rceil)$ steps, for $\tilde{T}$ must erase a "1" from track $N^0 3$. Altogether, $\tilde{T}$ simulates $\sum_{i=1}^{g(w)} \lceil i \rceil$ steps of $T$.

Hence $\tilde{T}$ needs $0(\lceil g(w) \rceil^3)$ steps to carry through the simulation of $T$ with input $w$. If $\tilde{T}$ is started at an arbitrary configuration $u_{ij} v$, it simulates $T$ starting at a configuration depending on $u_{ij} v$ on track $N^0 2$ for as many steps as the inscription of track $N^0 3$ tells. This takes no more than $0(\lceil uv \rceil^2)$ steps. Afterwards $\tilde{T}$ simulates $T$, starting at a well defined initial configuration $q_0 w$,
where \( w \) is the inscription of track \( N^2 \) 1 of the configuration \( uq_jv \), if \( w \in \Sigma^\ast \).
Otherwise \( T \) halts at state \( q \). As \( n \geq 3 \) there is a function \( k_n \in \mathbb{E}_n \) (\( \Sigma_o \cup \{q\} \)) satisfying condition (2).

According to [Av-Madl], p.89, a semigroup \( \Delta_T = (SUQ;\pi) \) where \( S = \Sigma_o \cup \{h\} \), \( Q = Q_T \cup \{q\} \), and

\[
\pi = \{ F_i q_i G_i = H_i q_i K_i | q_i, q_i \in Q, F_i, G_i, H_i, K_i \in S^\ast, \ i = 1, \ldots, N \}
\]
can be constructed from \( T \), satisfying:

(3) \( \forall w \in \Sigma_o^\ast \) (\( \Sigma_T \) \( w \) \( \Leftrightarrow w \in L_o \)).

(4) If \( uq_jv = q \), then there is a derivation from \( uq_jv \) to \( q \) in \( \Delta_T \) of length not exceeding \( 2|k_T(uq_jv)| + |uq_jv| \), because it may be assumed that \( k_T \) is non-decreasing ([Weih]).

Let \( u, v \in S^\ast \) with \( uq_jv \neq q \). Then \( uq_jv \neq q \), or \( u \neq hu' \), \( v \neq v'h \), \( q_j \neq q \), and starting at \( u'q_jv' \), \( T \) reaches the accepting state \( q \). But for doing so, \( T \) does not need more than \( |k_T(u'q_jv')| \) steps. Hence \( uq_jv \equiv hu'q_jv'h \) can be derived to \( hq_a vh \) in \( \Delta_T \), within at most \( |k_T(u'q_jv')| \) steps. Of course \( |uv| < |u'v'| + |k_T(u'q_jv')| \), since \( T \) can increase the length of its tape inscription by at most one per step. It takes \( \Delta_T \) \( |uv| \) steps to derive \( hq_a vh \) from \( hq_a vh \) by erasing \( uv \); \( hq_a vh \) can be derived to \( q \) within one step. Hence \( \Delta_T \) can derive \( q \) from \( uq_jv \) within at most

\[
2|k_T(uq_jv)| + |u'v'| + 1 \leq 2|k_T(uq_jv)| + |uq_jv| \text{ steps.}
\]

Define \( k_\Delta(w) = vk(vk(k_T(w), k_T(w)), w) \). Then \( k_\Delta \in \mathbb{E}_n (S UQ) \), and \( k_\Delta \) bounds the derivation of words \( w \in (S UQ)^\ast \) with \( w \in \Sigma^\ast \) to \( q \) in \( \Delta_T \).

Now a Brîton tower of groups is constructed:

- \( D_0 = <x;\emptyset> \)
- \( D_1 = <x, y; \tilde{s}x = x^2 (s \in S)> \)
- \( D_2 = <D_1, Q; \emptyset> \)
- \( D_3 = <D_2, r_i q_i, G_i = H_i q_i K_i | i = 1, \ldots, N> \)
- \( D_4 = <D_3, t; t = r (r \in R)> \)
- \( D_5 = <D_4, k_x k_x; k_x = r (r \in R), k_x t q_k = q_k t> \)
- \( D_6 = <D_5, s_i t; s_i = t (s_i \in S)> \)
- \( D_7 = <D_6, k; s_i t = t (s_i \in S)> \)
- \( D_8 = <D_7, \Sigma'; \Sigma' = \Sigma \cup \{s, t\}> \)
- \( G' = G_0 \) via \( \Sigma' \).

Then \( G' = G_0 \) via \( \Sigma' \).

- \( H_0 = D_0 \ast G' = <S_0 \cup \Sigma_0; m_0, l_0, a s' = s a (a \in S_0, s' \in \Sigma' > \)
- \( H_1 = D_1 \ast G' = <D_1, d; d s t' = s d k_0 s k_0 d = k_0 s k_0 (s \in \Sigma_0), d t_0 d = t_0, k_0 t_0 k_0 d = k_0 t_0 k_0 \)
\[ H_2 = \langle H_1, z; \tilde{z}sz = s, \tilde{z}k_0 s_k z = k_0 s_k (s \in \Sigma_0), \tilde{z}t_o z = t_o d, \tilde{z}k_0 t_k o z = k_0 t_k o d \rangle \]

\[ \Delta = S_0 \cup \Sigma_0 \cup \{ d, z \} \]

Let \( M \) be the set of defining relators of the given presentation of \( H_2 \), and \( M = M - \{ L'-\{ s's', \tilde{s}'s' \} | s' \in \Sigma' \} \) where \( \Sigma' = (\Sigma) ' \subseteq \Sigma' \)

**REMARK.** \( \forall s \in \Sigma(s_s, \tilde{s}_s \in \tilde{L}) \Rightarrow \forall s \in \Sigma(s_s, \tilde{s}_s \in \Sigma_0) \Rightarrow \forall s' \in \Sigma'(s'_s, \tilde{s}'s' \in \Sigma'') \)

Let \( H = \langle \Delta; M \rangle \). In [Av-Madl] Satz 1.1, p.184, Avenhaus and Madlener prove: \( H \) is f.p., \( W_P H = E_n(\Delta) \), and \( G \) embeds in \( H \). It remains to show that \( \langle \Sigma; M \rangle \) is \( E_n \)-d.b.

According to [Ott]§15, pp.156-173, the following assertions are valid:

- \( D_0 \) is \( E_n \)-d.b.
- \( D_1, D_2, D_3, \) and \( D_4 \) are \( E_3 \)-d.b.
- \( D_5 \) is \( E_n \)-d.b.

For proving these assertions propositions 1.4 until 1.7 are used. At the last part one has to construct a rewriting function \( \omega \in E_n((x,t) U S U Q U R) \), for \( x, \tilde{q}tq, R \triangleright D_1 \). After that, proposition 1.7 can be applied. Analogously there is an \( E_n \)-rewriting function for \( \langle \tilde{h}d, \tilde{h}h, \tilde{h}d, \tilde{h}d, \tilde{h}d, \tilde{h}d, \tilde{h}d, \tilde{h}d, \tilde{h}d \rangle \) in \( D'_4 = \langle D_5, t_0 ; (\tilde{q}_0 \tilde{q}_0 h) \cdot a(\tilde{q}_0 \tilde{q}_0 h) = a (a \in (x) U R) \rangle \) where \( D'_4 \) is \( E_5 \)-d.b. just like \( D_4 \). Hence \( D_0 \) is \( E_n \)-d.b., too.

\((5) \langle \Sigma; M \rangle \) is \( E_n \)-d.b.

*Proof.* a) Let \( w' \in L' = L'_0 - \{ s's', \tilde{s}'s' \} | s' \in \Sigma' \} \), and \( w \equiv (w') ('-1) \subseteq L_0 \subseteq \Sigma_0 \).

Then \( \tilde{h}_0 \tilde{w}_h \tilde{w}_h \equiv q \), and hence \( k_0 (w^{-1} t_0 w) = k_0 (w^{-1} t_0 w) k_0 \), due to [Av-Madl], p.185.

\( w' \in L'_0 \subseteq L_0 \Rightarrow w' \equiv e \Rightarrow w' \equiv e \), since \( H \equiv H_2 \) via the identity. Now \( w' \) can be derived to \( e \) in \( \langle \Delta; M \rangle \) as follows:

\[ w' \overset{1}{\rightarrow} w^{-1} t_0_{o} w w_{d}^{-1} t_0_{w w} \overset{2}{\rightarrow} w^{-1} t_0_{w w}^{-1} t_0_{d w d} \overset{3}{\rightarrow} \]

\[ w^{-1} t_0_{o w z}^{-1} t_0_{w z d} \overset{4}{\rightarrow} w^{-1} t_0_{w z k o}^{-1} t_0_{w k o z d} \overset{5}{\rightarrow} \]

\[ w^{-1} t_0_{o w k o}^{-1} t_0_{w k o d} \overset{6}{\rightarrow} w^{-1} t_0_{o w w}^{-1} t_0_{w d} \overset{7}{\rightarrow} e. \]

ad (1), 2) |\( w' \) + 2 trivial relators are inserted.

ad (2), by using the commutation relators of \( H_0 \) \( w \) and \( w' \) can be mixed within at most \( 3|w|^2 \) steps:

\[ w w' + s_i s_i s_i s_i s_i s_i s_i s_i \]

After that:

\[ s_i s_i s_i s_i s_i s_i s_i s_i s_i d = d s_i s_i s_i d d s_i s_i s_i s_i s_i s_i s_i s_i s_i d \]

(Insertion of \( \lambda = |w| \) trivial relators)

\[ d s_i s_i s_i s_i d = d w \]

(Insertion of \( d s_i s_i s_i s_i s_i s_i d \) and deletion of \( d s_i s_i s_i s_i s_i d \)).

Taken altogether this derivation doesn't need more than \( 3|w|^2 + 3|w| \) steps.
ad (3), \( w^{-1}t_o w = s_1 \lambda t_0 s_1 t_0 d s_1 \ldots s_i \lambda \)
\[\begin{split}
&+ (z s_i \lambda z)(z s_i \lambda z s_i \lambda) \ldots (z s_i \lambda z)(z s_i \lambda z s_i \lambda)(t_0 d t_0 z)(z t_o z) \\
&(s_1 \lambda z s_i \lambda z)(z s_i \lambda z)(s_1 \lambda z s_i \lambda z)(z s_i \lambda z)
\end{split}\]

(Insertion of \(6|w|+3\) trivial relators),
\[\begin{split}
&+ z s_i \lambda z \ldots z s_i \lambda z (z t_o z) z s_i \lambda z
\end{split}\]

(Deletion of \(2|w|+1\) relators of the form \(z s_i \lambda z, s_i \lambda z t_0 d t_0 z\),
\[\begin{split}
&+ z w^{-1}t_o w z
\end{split}\]

(Deletion of \(2|w|\) trivial relators).
Altogether (3) needs at most \(|10|w +4\) steps.

ad (4), \( w^{-1}t_o w + (w^{-1}t_o w k_o w^{-1}t_o w k_o)(k_o w^{-1}t_o w k_o)\)

(Insertion of \(2|w|+3\) trivial relators).
Let \(k_6 \in E_n(S_o)\) be an \(E_n\)-bound for \(<S_o;M_o>\). Then \(w^{-1}t_o w k_o w^{-1}t_o w k_o\) can be de-

cribed to \(e \in <S_o;M_o>\) within at most \(|k_6(w^{-1}t_o w k_o w^{-1}t_o w k_o)| = |k_6(x^4|w|+4)|\)
steps. Hence;

\[\begin{split}
(w^{-1}t_o w k_o w^{-1}t_o w k_o)(k_o w^{-1}t_o w k_o) &\rightarrow k_o w^{-1}t_o w k_o \text{ in } <\Delta;M> \\
\text{within at most }|k_6(x^4|w|+4)| \text{ steps, and (4) can be carried out within not more }\non-
\text{than }|k_6(x^4|w|+4)|+2|w|+3 \text{ steps.}
\end{split}\]

ad (5), \(z k_o w^{-1}t_o w k_o z = z k_o s_i \lambda \ldots s_i t_o s_i \ldots s_i \lambda k_o z\)
\[\begin{split}
&+ (k_o s_i \lambda k_o)(k_o s_i \lambda k_o z k_o s_i \lambda k_o z)(k_o s_i \lambda k_o z k_o s_i \lambda k_o z) \ldots \\
&(z k_o t_o z k_o t_o k_o)(k_o t_o k_o)(z k_o s_i \lambda k_o z k_o s_i \lambda k_o z) \\
&(k_o s_i \lambda k_o)(z k_o s_i \lambda k_o z k_o s_i \lambda k_o z) \ldots (z k_o s_i \lambda k_o z k_o s_i \lambda k_o z)(k_o s_i \lambda k_o z k_o s_i \lambda k_o z)
\end{split}\]

(Insertion of \(10|w|+6\) trivial relators),
\[\begin{split}
&+ k_o s_i \lambda k_o s_i \lambda k_o s_i \lambda k_o \ldots k_o s_i \lambda k_o s_i \lambda k_o s_i \lambda k_o \ldots k_o s_i \lambda k_o s_i \lambda k_o \\
&\text{(Deletion of }2|w|+1\text{ relators of the form }z k_o s_i \lambda k_o s_i \lambda k_o s_i \lambda k_o, k_o s_i \lambda k_o s_i \lambda k_o s_i \lambda k_o, k_o s_i \lambda k_o s_i \lambda k_o s_i \lambda k_o, k_o s_i \lambda k_o s_i \lambda k_o s_i \lambda k_o\text{')}
\end{split}\]
\[\begin{split}
&(s_i j \in E_o), z k_o t_o k_o d k_o c o k_o c o k_o,}
\[\begin{split}
&+ k_o w^{-1}t_o k_o d k_o s_i k_o \ldots k_o s_i \lambda k_o \\
&\text{(Deletion of }|w|+2\text{ trivial relators),}
\end{split}\]
\[\begin{split}
&+ k_o w^{-1}t_o k_o d k_o s_i k_o d k_o s_i \lambda k_o \ldots (k_o s_i \lambda k_o d k_o s_i \lambda k_o d)(d k_o s_i \lambda k_o d). \ldots (k_o s_i \lambda k_o d k_o s_i \lambda k_o d)(d k_o s_i \lambda k_o d). \\
&\text{(Insertion of }5|w|\text{ trivial relators),}
\end{split}\]
\[\begin{split}
&+ k_o w^{-1}t_o k_o d k_o s_i k_o \ldots d k_o s_i \lambda k_o \\
&\text{(Deletion of }|w|\text{ relators of the form }k_o s_i j k_o d k_o s_i j k_o \ldots (s_i j \in E_o)).}
\end{split}\]
\[\begin{split}
&+ k_o w^{-1}t_o w k_o d. \\
&\text{(Deletion of }2|w|\text{ trivial relators).}
\end{split}\]
Hence (5) can be carried out within $21|w| + 9$ steps.

ad (6), $k_0 w^{-1} t_0 w k_0^{-1} t_0 w (w^{-1} t_0 w)$ (Insertion of $2|w| + 1$ trivial relators).

Hence $k_0 w^{-1} t_0 w k_0^{-1} t_0 w$ can be derived to $e$ in $<S_6; M_0>$ within $|k_0(x^4|w| + 4)|$ steps, and so $(k_0 w^{-1} t_0 w k_0^{-1} t_0 w)(w^{-1} t_0 w) \rightarrow w^{-1} t_0 w$ in $<\Delta; M>$ within at most $|k_0(x^4|w| + 4)|$ steps. Hence (6) doesn't need more than $|k_0(x^4|w| + 4)| + 2|w| + 1$ steps altogether.

ad (7), $2|w| + 2$ trivial relators are deleted.

Taken altogether, there is a derivation from $w'$ in $<\Delta; M>$ of length not exceeding $2|k_0(x^4|w| + 4)| + 3|w|^2 + 4|w| + 21$. Define

\[ k'(w) \equiv v k(v k(x^8), k_0 U(x^8), V K(V k(x^8), x^6)). \]

Then $k' = E_5(\Delta^*)$, $k'$ is nondecreasing, for all $u, v \in \Delta^*$ \(|k'(u)| + |k'(v)| \leq |k'(uv)|\) and for every $w' \in L'_0$ there is a derivation from $w'$ in $<\Delta; M>$ of length bounded by $|k'(w')|$. 

b) Let $w \in \Delta^*$ with $|w| = 0$ and $w \not\in e$, and so $w \not\in e$. According to the proof of Proposition 1.6 (a), $w$ can be derived to $e$ in $H_1$ in the following way:

\[
\begin{align*}
 w & \overset{(1)}{\rightarrow} w' \overset{(2)}{\rightarrow} w_0(w') \overset{(3)}{\rightarrow} w_0(w') \overset{(4)}{\rightarrow} e \\
& \text{(d-pinches are pinched out in } H_1, \text{ in step (1))}
\end{align*}
\]

This derivation can be simulated in $<\Delta; M>$:

ad (1), d-pinches are pinched out in the following way:

\[
\begin{align*}
& d^{-1} u d^1 \rightarrow d^{-1} u (\omega^1(u))^{-1} \omega^1(u) d^1 \\
& \text{(Insertion of $|\omega^1(u)|$ trivial relators),}
\end{align*}
\]

(Within $3(|u| + |\omega^1(u)|)^2$ steps $u(\omega^1(u))^{-1}$ can be transformed into $u_1 u_2$ where $u_1 \in S_6^*$ and $u_2 \in L_0^*$),

\[
\begin{align*}
& d^{-1} u_1 u_2 \omega^1(u) d^1 \\
& \text{(u(\omega^1(u))^{-1} \not\in e, and so } u_1 \not\in S_6^* \text{ and } u_2 \not\in L_0^*. \text{ But then } u_1 \text{ can be derived to } e \text{ in } <S_6; M_0> \text{ within at most } |k_0(u_1)| \text{ steps),}
\end{align*}
\]

\[
\begin{align*}
& d^{-1} u_2 \omega^1(u) d^1 \\
& \text{(In } u_2, s' \text{ is substituted by } s', \text{ and } \tilde{s}' \text{ is substituted by } s': s' \rightarrow s's's's' \rightarrow s', \text{ and } \tilde{s} \rightarrow s's's's' \rightarrow s'. \text{ Let } \tilde{u}_2 \text{ be the result of these substitutions. Then } \tilde{u}_2 \text{ can be derived from } u_2 \text{ within at most } 2|u_2| \text{ steps. Since } e \not\in u_1 u_2 \not\in u_2, \text{ } \tilde{u}_2 \in L_0^*, \text{ and because of (a), } \tilde{u}_2 \text{ can be derived to } e \text{ in } <\Delta; M> \text{ within no more than } |k'(\tilde{u}_2)| \text{ steps),}
\end{align*}
\]

\[ v^1(u) \]
Let $A_1 = \langle s's', k_0sk_0(s \in S_0), t_0t_0t_0t_0t_0 > H_0$, $B_1 = \langle s's', k_0sk_0(s \in S_0), t_0t_0t_0t_0t_0 > H_0$, and $\Phi$ and $\Psi$ denote function realizing the isomorphisms $A_1 \to B_1$ and $B_1 \to A_1$, respectively. According to [Av-Madl] Lemma 1.4, p.187, there are constants $c > 1$ and $d > 2$ satisfying $|\omega_{A_1}(w)|$, $|\omega_{B_1}(w)|$, $|\Phi(w)|$, $|\Psi(w)| < c|w|^d$. Hence for pinching out the $d$-pinch $d'|w|d'$ one doesn't need more than

$$8c|u|^d + 3(c+1)^2|u|^d + k_6(x(c+1)|u|^d) + 2(c+1)|u|^d + |k'(x(c+1)|u|^d)|$$

$$\leq 13(c+1)^2|u|^{2d} + k_6(x(c+1)|u|^d) + |k'(x(c+1)|u|^d)|$$

steps in $\Delta;M$. Let $w_1$ be the word formed from $w$ by pinching out $i$ $d$-pinches. Then by the proof of Prop. 1.6 (a),

$$|w_1| \leq (c+1)d^2|w|^d.$$ 

Therefore every $d$-pinch $d'|w|d'$ pinched out at (1) is bounded by

$$|u| \leq (c+1)d^2|w|^d.$$ 

Hence there is a function $k_1' \in E_n(N)$ bounding the number of steps needed for carrying out (1), since $n > 3$. Of course $w'$ satisfies $|w'| \leq ((c+1)|w|)d^2|w|$.

ad (2), by using the commutation relators of $H_0$ and some trivial relators, $w'$ can be transformed into $\pi_0(w')\pi_0(w')$, within at most $3|w'|^2$ steps. So this transformation can be bounded by a function $k_2' \in E_n(N)$.

ad (3), there is a derivation from $\pi_0(w')$ in $<\Delta;M>$ consisting of no more than $|k_6\cdot\pi_0(w')| \leq |k_6(x|w'|)|$ steps, and so there is a function $k_3' \in E_n(N)$ bounding this derivation.

ad (4), within at most $2|\pi_0(w')|$ steps each $s'$ and each $s'$ contained in $\pi_0(w')$ can be substituted by $s'$ or $s'$, respectively. In this way $\pi_0(w')$ is transformed into a word $\tilde{w} \in L_0$ which can be derived to $e$ in $\Delta;M$ within at most $|k'(x|\tilde{w}|)| \leq |k'(x|w'|)|$ steps because of (a). Hence (4) is bounded by a function $k_4' \in E_n(N)$, too.

So there is a function $k \in E_n(N)$ bounding the derivations from $w$ to $e$ in $\Delta;M$ for all $w \in \Delta$ satisfying $|w|_z > 0$ and $w_H = e$.

c) Let $w \in \Delta$ with $|w|_z > 0$ and $w \equiv H e$, and so $w \equiv H e$. According to the proof of Prop. 1.6 (a), $w$ can be derived to $e$ in $H_2$ as follows:

$$w \overset{(1)}{\longrightarrow} w' \overset{(2)}{\longrightarrow} e$$

(3-pinches are pinched out in $H_2$, in steps (1)).

This derivation can be simulated in $\Delta;M$.

ad (1), $z$-pinches are pinched out in the following way

$$\tilde{z}^Hw^z + \tilde{z}^Hu(w_\mu(u))^-1w_\mu(u)z^H$$

(Insertation of $|w_\mu(u)|$ trivial relators),
(u(\omega(w(u)))^{-1}]_z = 0. Hence u(\omega(w(u)))^{-1} can be derived to e in <\Delta;M> within at most k(|u|+|\omega(w(u))|) steps because of (b).

\[ \omega_1(w) \rightarrow \omega_2(w) \rightarrow \omega_3(w) \rightarrow e \]

(8|\omega_2(w)| steps of the form: insertion of trivial relators, deletion of a z-relator, and deletion of trivial relators). Let

\[ A_2 = \langle s, k_0 s k_0, t_0 k_0 \rangle_{H_1}, B_2 = \langle s, k_0 s k_0, t_0 d, k_0 t_0 k_0 \rangle_{H_1}, \]

and \( \psi \) and \( \phi \) denote functions realizing the isomorphisms \( A_2 \rightarrow B_2 \) and \( B_2 \rightarrow A_2 \), respectively. Because of [Av-Madl] Lemma 1.5, p.187, there are constants \( a, \beta > 2 \) satisfying:

\[ |\omega_2(w)|, |\omega_3(w)|, |\phi(w)|, |\phi(w)| \leq a|w|^{\beta}. \]

Hence for pinching out the z-pinch \( zuz \) one only needs \( a|u|^{\beta + k((a+1)|u|^{\beta}} \) steps.

Let \( w' \) denote the word formed from \( w \) by pinching out \( i \) z-pitches. By the proof of Prop. 1.6 (1), \( |w'| \leq ((a+1)|w|)^{\beta}|w| \), and therefore the number of steps necessary to realize (1) can be bounded by a function \( k_1', e \in F_1(N) \). Furthermore \( |w'| \leq ((a+1)|w|)^{\beta}|w| \).

ad (2), \( |w'| \leq 0 \) and \( e = w \equiv w' \). Hence, because of (b), \( w' \) can be derived to \( e \) in <\Delta;M> within at most \( \tilde{K}(|w'|) \) steps and so, there is a function \( k_1' \in F_1(\Delta) \) bounding the derivations of all words \( w \in \Delta \) with \( w \equiv e \) in <\Delta;M>.

Therefore <\Delta;M> is En-d.b.

2.2. COROLLARY. Every countable group \( G \) having an \( E_n \)-decidable word problem for some \( n \geq 3 \) can be embedded into a f.p. group \( H \) possessing a finite \( E_n \)-d.b. presentation.

Proof. Every countable group \( G \) having an \( E_n \)-decidable word problem for some \( n \geq 2 \) can be embedded into a f.g. group \( G_1 \) having an \( E_n \)-decidable word problem too ([Ott] Thm. 12.1, p.117).

3. F.P. \( E_n \)-DERIVATION BOUNDED GROUPS AND THE WORD PROBLEM.

For finite \( E_n \)-d.b. presentations of groups there is a standard natural algorithm for solving the word problem. But of what degree of complexity is this algorithm, and how is this degree of complexity related to the selected finite presentation?

3.1. THEOREM. Let \( H = <\Sigma;L> \) be f.p. and \( E_n \)-d.b. for some \( n \geq 3 \). Then the standard natural algorithm for <\Sigma;L>, as it is described in the introduction,
is an Eₙ-algorithm. In particular the word problem for <L;L> is Eₙ-decidable.

Proof. Let Σ = {s₁, ..., sₘ}, L = {w₁, ..., wₘ} ∈ Σ*, and k ∈ Eₙ(Σ) be an Eₙ-bound for <L;L>. Without loss of generality m > 3 may be assumed, for otherwise auxiliary generators and defining relations can be added.

If w ∈ Σ* with w ≡ₑ e, then there is a derivation from w in <L;L> of length not exceeding |k(w)|. During each step of this derivation a word u ∈ Rel = L⁻¹U{s₁, ..., sₘ} is inserted or deleted. L contains l, and Σ contains m elements only. Hence there are only 2(ℓ+μ) possible choices for u. Define λ as the length of the longest possible word u. Then every word v found in that bounded derivation from w satisfies |v| < |w| + ₃₄ |k(w)|, where ₃₄ denotes the least natural number greater than or equal to ₃₄, because in order to derive a word of greater length from w more than ₃₄ |k(w)| steps are necessary, but then in order to derive this word to e more than ₃₄ |k(w)| steps are needed, again contradicting the fact that the derivation from w is bounded by |k(w)|. Define

\[ u_w = |w| + ₃₄ |k(w)|. \]

A step of a derivation can be encoded as a triple (i₁, i₂, i₃) of natural numbers such that i₁ ∈ {0, 1}, i₂ ∈ {1, 2, ..., 2(ℓ+μ)}, and i₃ ∈ {0, 1, 2, ..., ₃₄}. Here i₁ = 0 stands for "insertion", i₁ = 1 for "deletion" of the relator with the number i₂ at the position described by i₃. Hence there are \( ν_w = 2(2(ℓ+μ)) + ₁₃ |w| + ₁₃ |k(w)| \) different steps which can be chosen in a derivation of w. Therefore there are not more than \( ν_m |k(w)| \) possible derivations from w of length |k(w)|. In order to decide w ≡ₑ e, it is sufficient to apply these derivations one after another to w, and to test whether one of these derivations produces e. Define \( f₁(ₑ) = e, f₂(ₑ) = s₁, f₁(ws) = f₂(w), f₂(ws) = νk(f₁(w), s₁) \) then \( f₁, f₂ \in E₁(Σ) \), satisfying

\[ f₁(w) ≡ s₁, f₂(w) ≡ s₁. \]

Let \( ML(w) = νk(U₁(w), νK(U₁(w), f₁(s₁))) \) where \( λ = \max_{u ∈ Rel} |u| \). Then \( ML ≡ Eₙ(Σ) \) and

\[ |w| + ₃₄ |k(w)| \equiv s₁. \]

Each step in a derivation is described by a triple

\[ (i₁, i₂, i₃) ∈ \{0, 1\} × \{1, 2, ..., 2(ℓ+μ)\} × \{0, 1, ..., ₃₄\}, \]

and so it can be encoded as a word over Σ, namely as

\[ s₁^{i₁+1} s₂^{i₂+1} s₃^{i₃+1} \]

which is a word of length not exceeding \( ₂²(2(ℓ+μ)) + ₁₃ + ₁₃ |w| + ₃₄ |k(w)| \). Hence a derivation of w can be described by a word of length at most

\[ (2(ℓ+μ)) + ₃₄ |w| + ₃₄ |k(w)| \cdot |k(w)| \]

Let \( LDA(w) = νK(νk(s₁²(2(ℓ+μ)) + ₃₄ , ML(w)), k(w)) \).
then $LDA \in E_1(\mathbb{Z})$ satisfying

$$LDA(w) = \sum_{i} s_i$$

In order to decide whether $w \models e$ is valid or not one only has to check whether there is a word $u$ of length at most $|LDA(w)|$ describing a derivation from $w$ to $e$ in $<\Sigma;L>$. Now a Turing Machine $M$ will be defined to test for a pair $(w,u) \in (\mathbb{Z}^*)^2$ whether $u$ is the description of a derivation from $w$, by trying to apply $u$ to $w$. In an initial part of $u$ is the description of a derivation from $w$ to $e$, then $M$ will halt with its output tape being empty, but if $u$ doesn't meet this condition, then $M$ will print the letter "$s_1$" and halt.

Let $M$ have two input tapes, one output tape, and four auxiliary tapes.

1) $w$ is the inscription of the first input tape, and $u$ is the inscription of the second one.

2) $w$ is copied onto the first auxiliary tape, while $u$ is copied onto the second one. This can be done within $2|w|+2|u|+3$ steps. i.e. amount of time $(A.t.) = 2|w|+2|u|+3$.

3) The elements of the set $Rel$ are printed onto the third auxiliary tape separated by a "b", respectively.

$$A.t. \leq 2(\lambda+1) \cdot 2 \cdot (\lambda+1) \leq 8\lambda(\lambda+1).$$

<table>
<thead>
<tr>
<th>I1:</th>
<th>..b w b..</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>↑</td>
</tr>
<tr>
<td>I2:</td>
<td>..b u b..</td>
</tr>
<tr>
<td></td>
<td>↑</td>
</tr>
<tr>
<td>A1:</td>
<td>..b w b..</td>
</tr>
<tr>
<td></td>
<td>↑</td>
</tr>
<tr>
<td>A2:</td>
<td>..b u b..</td>
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<tr>
<td></td>
<td>↑</td>
</tr>
<tr>
<td>A3:</td>
<td>..b w_1 b w_1^{-1} b w_2 b..b w_1^{-1} b s_1 s_1 b..b s_m s_m b..</td>
</tr>
<tr>
<td></td>
<td>↑</td>
</tr>
</tbody>
</table>

4) If $u$ starts with a letter $s \neq s_1$, then outputs $s_1$ and halts. $A.t. = 3$.

If $u$ starts with $s_1$, then mind "insertion". $A.t. = 3$.

If $u$ starts with $s_2^i$, then mind "deletion". $A.t. = 4$.

If $u$ starts with $s_i^j$ for an $i > 2$, then outputs $s_1$ and halts. $A.t. = 4$.

$$A2: \quad ..b b' u' b.. \quad u \equiv s_i^j u' \quad \text{for some} \quad i \in \{1,2\}.$$
If $u'$ starts with $s_2^i$, then for $i-1$ times $M$ puts the head of its third auxiliary tape onto the next symbol "b" to the right of the actual position of the head. After that this head performs one step to the right. $A.t. \leq i(\lambda+1)+1$.

If $M$ reads a "b" on its third auxiliary tape, then output $s_1$ and halts. $A.t. = 2$.

Otherwise, the head of $A_3$ is pointing to the relator which shall be inserted or deleted from $w$.

$A_2$: \[ \ldots b b' u' b \ldots \quad u' = s_2^j u'' \text{ for some } i \in \{1, \ldots 2(\ell+m)\} \]

$A_3$: \[ \ldots b w_1 b \ldots b w_\mu b \ldots b s_\text{rel} b \operatorname{rel} b \ldots \]

If $u''$ starts with a letter $s \neq s_3$, then output $s_1$ and halts. $A.t. = 2$.

If $u''$ starts with $s_3^j$, then the operation $R$ (i.e. make a step to the right) is executed on $A_1$, $j-1$ times. $A.t. = j$.

If the head of $A_1$ is now pointing at a cell containing "b", and if $M$ has to delete the relator marked on $A_3$, then $M$ prints "$s_1^j"$ and halts. $A.t. = 2$.

If the head of $A_1$ is pointing at a cell containing "b", if $j \geq 2$, and if $M$ has to insert the relator marked on $A_3$, then $M$ prints "$s_1^j"$ and halts. $A.t. = 2$.

Otherwise, the head of $A_1$ is pointing at the first letter of $w$ which shall be erased or behind which the indicated relator shall be inserted.

$A_1$: \[ \ldots b b w' s w'' b \ldots \quad w = w'sw'' \]

5) **Insertion**: The indicated relator is copied from $A_3$ onto $A_4$, subsequently $w''$ is appended at the right end of this copy, and at last $w''$ is erased from $A_1$. $A.t. \leq \lambda + |w| + 1$.

If $j = 1$, then the inscription of $A_4$ is copied onto $A_1$, in the course of which it is erased from $A_4$. Otherwise the inscription of $A_4$ is appended to the inscription of $A_1$ ($w'$s), at which it is erased from $A_4$. The head of $A_1$ is put onto the first "b" to the left of the inscription of $A_1$.

$A.t. \leq |w| + 2(|w|+\lambda+1)+|w|+\lambda+1 = 4|w|+3\lambda+3$.

$A_1$: \[ \ldots b b w' s w_\mu w'' b \ldots \]

$A_2$: \[ \ldots b u b \ldots \quad u'' = s_3^i u \]

$A_3$: \[ \ldots b w_1 b \ldots b w_\mu b \ldots b s_\text{rel} s \operatorname{rel} b \ldots \]

$A_4$: \[ \ldots b b b \ldots \]
Deletion. The indicated relator is compared to the subword of \( w \), beginning at the position the head of \( A_1 \) is pointing at. By doing so, the subword of \( w \) is erased. If this subword of \( w \) and the indicated relator do not coincide, then \( M \) prints "s" and halts. Otherwise an initial part or an internal segment of \( w \) has been erased. In the first case the head of \( A_1 \) performs one step to the left, in the second case \( M \) appends the remained end of \( w \) to the remained initial part by using the tape \( A_4 \) as scratch paper. At last \( M \) puts the head of \( A_1 \) onto the first "b" to the left of the inscription of \( A_1 \). 

\[
\begin{align*}
A_1: & \quad \cdot b b w' w'' b. \quad w \equiv w'w'w'' \\
A_2: & \quad \cdot b \bar{u} b. \quad u'' \equiv s_3^j \bar{u} \\
A_3: & \quad \cdot b w_1 \underline{b.} b w_\mu b \underline{b.} s_m s_m b. \\
A_4: & \quad \cdot b b b. \\
\end{align*}
\]

6) The head of tape \( A_3 \) returns to the left. 

\[A. t. \equiv 2.\]

If the inscription of tape \( A_1 \) is \( e \), then \( M \) halts because \( e \) has been derived from \( w \). Otherwise \( M \) continues with step (4). 

\[A. t. = 2.\]

Of course \( M \) eventually halts satisfying \( f_M(w,u) \equiv e \) iff an initial part of \( u \) is describing a derivation from \( w \). Altogether \( M \) has the following amount of time.

\[
T_M(w,u) \leq 2|w|+2|u|+3+8\lambda(\ell+m)+|u|\cdot(4+|u|)(\lambda+1)+1+2+|u|+2+
5(|w|+\lambda|u|)+4\lambda+4+4\lambda(\ell+m)+2+2).
\]

(In the course of the computation \( w \) may grow, but it cannot become larger than \(|w|+|u|\))

\[
= 2|w|+2|u|+3+8\lambda(\ell+m)+|u|\cdot(5|w|+(6\lambda+2)|u|+4\lambda(\ell+m+1)+17).
\]

But \( \lambda, \ell, m \) are constants, and so \( f_M \in E_2(\Sigma) \) because of [Weih] Kap. 4.3, Satz 2. Now we have:

\[
w \models e \iff \exists u \in \Sigma^* (|u| \leq \text{LDA}(w) \text{ and } f_M(w,u) \equiv e)
\]

\[
\iff \exists u \leq \text{vk}(\text{LDA}(w), s_1) (f_M(w,u) \equiv e).
\]

But as \( n \geq 3 \), \( E_n(\Sigma) \) is closed under bounded quantification and therefore \( w \models e \) is \( E_n \)-decidable by the standard n. a. implemented above. Hence, \( \text{WP}_{H} \models E_n(\Sigma) \).
Next we prove that $E_n$-derivation boundedness is an invariant of finite presentations.

3.2. THEOREM. Let $H = \langle \Sigma; L \rangle$ be f.p. and $E_n$-d.b. for some $n \geq 1$. Then every finite presentation for $H$ is $E_n$-d.b., too.

Proof. Let be $\Sigma, L$, and $k$ as in the proof of Theorem 3.1, and let $\langle \Delta; M \rangle$, $\Delta = \{ t_1, \ldots, t_r \}$, $M = \{ u_1, \ldots, u_s \} \subseteq \Sigma^*$, be another finite presentation for $H$. Then, for all $s_i \in \Sigma$ there is $v_i \in \Delta^*$ such that $s_i$ and $v_i$ define the same element of the group $H$. Define $f(e) \equiv e$, $f(w_i \mu) \equiv v_k(f(w), v_i \mu)$. Then for all $w \in \Sigma^*$, $w$ and $f(w)$ define the same element of the group $H$, and there is a constant $c_1 > 0$ such that $|f(w)| \leq c_1 \cdot |w|$. $v_t \in \Delta \exists x_j \in \Sigma^*$, $t_j$ and $x_j$ define the same element of $H$. Moreover, $g(e) \equiv e$, $g(w_t \mu) \equiv v_k(g(w), x_j \mu)$. Then for all $w \in \Sigma^*$, $w$ and $g(w)$ define the same element of $H$, and there is a constant $c_2 > 0$ such that $|g(w)| \leq c_2 \cdot |w|$.

Then $\beta(w) \equiv \alpha \circ g(w) \equiv \beta \circ f \circ g(w)$, and so $w \equiv f \circ g(w)$. Also $|f \circ g(w)| \leq c_1 \cdot |g(w)| \leq c_1 \cdot c_2 \cdot |w|$. Especially $t_j^H(f \circ g(t_j^H))^{-1} \equiv e$. Hence for each $t_j \in \Delta$ there is a derivation from $t_j^H(f \circ g(t_j^H))^{-1}$ to $e$ in $\langle \Delta; M \rangle$ of length $\ell_j$. If $c_3 = \max(\ell_j, u | j = 1, \ldots, r, u \in \{ \pm1 \})$, then $f \circ g(t_j)$ can be derived from $t_j^H$ in $\langle \Delta; M \rangle$ within at most $c_4 = c_3 + 1$ steps by the following sequence:

$$t_j^H \leftarrow t_j^H f \circ g(t_j^H) \leftarrow (f \circ g(t_j^H))^{-1} \equiv f \circ g(t_j^H).$$

Hence every word $w \in \Delta^*$ can be derived to $f \circ g(w)$ within $c_4 \cdot |w|$ steps.

For every $u \in \text{Rel}$, $f(u) \equiv e$, and therefore there is a derivation from $f(u)$ to $e$ in $\langle \Delta; M \rangle$ of length $\ell_u$. If $c_5 = \max(\ell_u | u \in \text{Rel})$, then $f(u)$ can be derived to $e$ in $\langle \Delta; M \rangle$ within no more than $c_5$ steps. Let $w \in \Delta^*$ with $w \equiv e$, then $g(w) \equiv e$, too. Hence there is a derivation from $g(w)$ to $e$ in $\langle \Sigma; L \rangle$ of length not exceeding $|k \circ g(w)| \leq |k(s_1^C \cdot |w|)|$.

$$g(w) \equiv u_0 + u_1 + \cdots + e.$$ 

But then

$$f \circ g(w) \equiv g(u_0) \leftarrow f(u_1) \leftarrow \cdots \leftarrow f(e) \equiv e$$

in $\langle \Delta; M \rangle$, i.e. there is a derivation from $f \circ g(w)$ to $e$ in $\langle \Delta; M \rangle$ of length not exceeding $c_5 \cdot |k(s_1^C \cdot |w|)|$. Now $w$ can be derived to $e$ in $\langle \Delta; M \rangle$ in the following manner:

$$w \leftarrow f \circ g(w) \leftarrow \cdots \leftarrow f(e) \equiv e.$$
Of course there is an $E_n$-function bounding this derivation. Hence $<\Sigma; M>$ is $E_n$-d.b.

The last theorem shows that the property of being $E_n$-d.b. does not depend on the chosen finite presentation. It merely depends on the group. Hence a f.p. group is called $E_n$-d.b. if one, and therewith each, of its finite presentations is $E_n$-d.b. A conclusion of the proof of the last theorem is the fact that even every f.g. presentation of a f.p. $E_n$-d.b. group is $E_n$-d.b. But of course each f.p. $E_n$-d.b. group has a f.g. $E_0$-d.b. presentation, i.e. $<\Sigma; \{w \in \Sigma^* \mid w \equiv_0 e\}>$ for example. Therefore the property of being $E_n$-d.b. does depend on the chosen f.g. presentation of a group.

It remains to answer the question whether for f.p. $E_n$-d.b. groups with $n \geq 3$ an optimal n.a. exists. The following theorem gives an answer in the negative sense.

3.3. THEOREM. For every $n \geq 4$ there is a f.p. group $G_5 = <S_5; L_5>$ such that the word problem for $G_5$ is $E_5$-decidable, but $<S_5; L_5>$ is only $E_n$, but not $E_{n-1}$-d.b. Especially there is no finite $E_3$-d.b. presentation for $G_5$.

Proof. Let $n \geq 4$. The f.p. group $G_5$ will now be constructed in the same manner as the group $D_5$ has been constructed in the proof of Theorem 2.1. Only the underlying Turing Machine will be modified. Let $S' = \{s_1, s_2, s_3\}$ and $L = S'$, and let $T = (S', Q_T, q_0, \beta)$ be a single tape machine acting as follows. For every $w \in S'^*$, starting at $q_0 w$, $T$ computes $A_n(w, w)$ where $A_n \in E_n(S')$ denotes the $n$-th Ackermann function over $S'$ ([Weih]). After that $T$ enters the accepting state $q_a$ and halts. For carrying out this computation $T$ has to execute more than $|A_n(w, w)|$ steps. On the other hand, $T$ can be chosen in such a way that there exists a function $g \in E_n(S')$ which bounds the time, i.e. the number of steps $T$ needs for its computation ([Weih] Kap. 4.4, Satz 1).

Now $T$ can be modified to get $\tilde{T} = (\tilde{S}, Q_{\tilde{T}}, q_0, \tilde{\beta})$, where $\tilde{S}$ is a finite alphabet containing $S'$ such that there is a function $k_\tilde{T} \in E_n(\tilde{S} U Q_{\tilde{T}})$ satisfying.

$w u, v \in S^* W_{q_j} \in Q_{\tilde{T}}$, starting at the configuration $u q_j v$, $\tilde{T}$ halts in the accepting state $q_a$ within at most $|k_\tilde{T}(u q_j v)|$ steps.

This modification is done in the same way as the one used in the proof of theorem 2.1, with the only exception that the non-accepting state $q_-$ is omitted, i.e. instead of entering $q_-$, $\tilde{T}$ enters the accepting state $q_a$. Since for every $w \in S'^*$, starting at $q_0 w$, $T$ halts in the state $q_a$, $\tilde{T}$ also halts in the state $q_a$, starting at any configuration $u q_j v$. The execution time of $T$ is bounded by the function $g \in E_n(S')$. Hence there is a function $k_\tilde{T} \in E_n(\tilde{S} U Q_{\tilde{T}})$ satisfying the condition formulated above. Of course, starting at $q_0 w$, $\tilde{T}$ has to carry out more than $|A_n(w, w)|$ steps for every $w \in S'^*$, too.
CLAIM. Let $S = S \cup \{h\}$, $Q = Q_1 \cup \{q\}$ and $\Delta = (S \cup Q; \pi)$, where $\pi = \{F_i q_j G_i = H_i q_i G_i K_i, |q_i q_j| = Q, F_i G_i H_i K_i \in S^*, i = 1, \ldots, N\}$ is the semigroup constructed from $T$ according to [Av-Madl], p. 89. Then the following three conditions are satisfied:

1. $\forall u, v \in S^* q_j \in Q (u q_j v \equiv q \iff u q_j v \equiv q \text{ or } u \equiv h u', v \equiv v' h$, with $u', v' \in S^*$ and $q_j \not\equiv q$).

2. $\forall w \in S^* q_j \in Q (u q_j v \equiv q \implies \exists \text{ derivation from } u q_j v \text{ to } q \text{ in } \Delta, \text{ of length not exceeding } 2|k_\Delta(u_q_j v)| + |u q_j v|)$.

3. $\forall w \in S^* (h q_0 w h \equiv q, \text{ but there is no derivation from } h q_0 w h \text{ to } g \text{ in } \Delta \text{ of length } \leq |A_n(w, w)|)\).

Proof.

ad (1) "\iff". Let $u q_j v \equiv q$, but $u q_j v \not\equiv q$. Then $q_j \not\equiv q$, $u \equiv h u'$ and $v \equiv v' h$ for some $u', v' \in S^*$. "\Leftarrow". Let $u', v' \in S^*$, $q_j \in Q$. Then $u' q_j v' \equiv q_d$, and so $h u' q_j v' h \equiv h q_d h \equiv q$. $\Rightarrow$

ad (2) This can be proved in exactly the same way as the corresponding statement in the proof of Theorem 2.1 was proved. Hence there is a function $k_\Delta \in E_n(S \cup Q)$ which bounds the derivations from $w \in (S \cup Q)^*$ to $q$ in $\Delta$ if $w \not\equiv q$.

ad (3) $\Delta$ simulates $\bar{T}$, step by step. But starting at $q_0 w$, $\bar{T}$ has to execute more than $|A_n(w, w)|$ steps before reaching $q_a$. Therefore $\Delta$ has a carry out more than $|A_n(w, w)|$ steps to reach $q$, too, when started at $h q_0 w h$.

Now a Britton tower of groups is constructed:

$G_0 = \langle x, \emptyset \rangle$, $S_0 = \{x\}$, $G_1 = \langle G_0, S; x s x = x^2(s \in S) \rangle$, $S_1 = S_0 \cup U S$, $G_2 = \langle G_1, G_\emptyset \rangle$, $S_2 = S_1 \cup U Q$, $G_3 = \langle G_2, R_1; s x = x s x, s \in S, t \in S \rangle$, $S_3 = S_2 \cup R$, $G_4 = \langle G_3, t; \emptyset t x = x, \emptyset t r t = r (r \in R) \rangle$, $S_4 = S_3 \cup \{t\}$, $G_5 = \langle G_4, k; k \emptyset a (a \in ) x, q t q \emptyset \cup R \rangle$, $S_5 = S_4 \cup \{k\}$, $R = R \cup \{x\}$.

Of course $G_0, G_1, \ldots, G_5$ are f.p. Furthermore they satisfy ([Av-Madl]):

- (a) For $i = 1, \ldots, 4$, $G_i$ is an HNN-extension of $G_{i-1}$, there is a reduction function $f_1 \in E_3(S_1)$ for $G_1$, and the word problem for $G_1$ is $E_3$-decidable.

- (b) There is a function $g \in E_3(S_3)$ satisfying:

- $\forall w \in S_3^* (g(w) \equiv u w$ for some $u \in R_3^*$. $\forall w \in S_3^*$ is $R$-reduced, there is no $u \in R_3^*$ such that there is a $R$-pinch in $u q_j u$ just on the border $u - g(w)$.

- $\forall w \in S_3^*$ is $R$-reduced, and if $g(w) \equiv u r_1 v$ where $u \in S_2^*$, $v \in S_3^*$, then $w$ has the form $u r_1 v$ for some $u' \in S_3^*$.

- (y) Over $S_3^*$ define the predicate: $P(u) \iff \exists w_1, w_2 \in R_3^* (w_1 u w_2 \equiv G_3 q)$. $\forall u \in S_3^*$ is reduced and $v \equiv g((g(u))^{-1})^{-1}$, then: $P(u)$ if $v \in S_2^*$ and $P(v)$. $P(u)$ if $v \in S_2^*$ is reduced and $v'$ is the result of deleting all $x$ and $\bar{x}$ symbols of $v$, then:
\[ P(v) \text{ iff } 3X,Y \in S^*, q_j \in Q (v' = Xq_j Y \text{ and } P(Xq_j Y)). \]

- \( P(v) \text{ iff } 3X,Y \in S^*, q_j \in Q (P(Xq_j Y)) \text{ iff } Xq_j Y \in Q \).

**Assertion.** \( \tilde{P} \in E_3(S_3) \).

**Proof.** Let \( u' \in S_3^* \). Then \( u \equiv f_3(u') \) satisfies \( u' \equiv u \), and so \( \tilde{P}(u') \) iff \( \tilde{P}(u) \). Let \( v \equiv g((g(u))^{-1})^{-1} \). Then because of \((\gamma)\), \( \tilde{P}(u) \) iff \( v \in S_2^* \) and \( \tilde{P}(v) \), since \( u \) is reduced. Let \( \tilde{v} \equiv f_2(v) \), and \( v' \equiv \pi_{S \cup Q}((\gamma)). \) If \( v \in S_2^* \), then the following is true because of \((\gamma)\):

\[ \tilde{P}(v) \text{ iff } 3X,Y \in S^*, q_j \in Q (v' = Xq_j Y \text{ and } P(Xq_j Y)). \]

Altogether we have thus:

\[ \tilde{P}(u') \text{ iff } \tilde{P}(u) \text{ iff } v \in S^* \text{ and } P(v) \text{ iff } v \in S^* \text{ and } P(v'). \]

But \( u,v,\tilde{v}, \) and \( v' \), and therewith also \( Xq_j Y \), are \( E_3 \)-computable from \( u' \). \( Xq_j Y \in \mathbb{Q} \) is \( E_1 \)-decidable because of \((1)\). Hence \( \tilde{P} \in E_3(S_3) \).

Now let \( u \in S_4^* \) be such that \( f_4(u) \equiv u_0 t_1 u_1 \ldots t_n u_n \), \( u \equiv S_3^* \), \( u \equiv S_3^* \).

According to the proof of [Av-Mad] Lemma 4.9, p. 102, the following assertion is satisfied:

\[ u = (x,q_{tq},R)_G \text{ iff } u_0 u_1 \ldots u_m \in (x,R)_G \text{ and } \bigwedge_{i=0}^{m-1} \tilde{P}((u_0 u_1 \ldots u_1)^{-1}). \]

But \( (x,R)_G \) is \( E_3 \)-decidable because of the proof of [Av-Mad] Lemma 4.6, p. 100. Hence \( (x,q_{tq},R)_G \) is \( E_3 \)-decidable and so \( G_5 \) is an \( E_3 \)-admissible HNN-extension of \( G_4 \). Hence \( WP_{G_5} = E_3(S_3) \).

According to [Ott] §15, pp. 156-173, the presentation \( <S_5;L_5> \) of \( G_5 \) is \( E_\infty \)-d.b.

Now let \( w \in S^* \), then \( q_0 w \equiv q_0 a_0 \ldots q_0 a_n \), and therefore \( hq_0 w \in \mathbb{F} \). \( \tilde{w} \equiv q_0 h\tilde{q}_0 w \) according to [Rot] Lemma 12.13, p. 229. Therefore, there is a derivation from \( \tilde{w} h q_0 h\tilde{q}_0 w \) to \( e \) in \( <S_5;L_5> \). During this derivation \( k \) and \( k \) must be eliminated by using relators of the form \( k a^-1 \) (\( a \equiv (x,q_{tq}) \cup R \)). But for that, \( \tilde{w} h q_0 h\tilde{q}_0 w \) must be rewritten into a word \( u \equiv (x,q_{tq},R)^* \). Let \( u = u_0 \tilde{q}_1 u_1 \ldots \tilde{q}_m u_m, u \equiv R_x, u_i \equiv (\pm 1) \) such that \( \tilde{w} h q_0 h\tilde{q}_0 w \equiv u \equiv u_0 \tilde{q}_1 u_1 \ldots \tilde{q}_m u_m \) where \( \gamma_f \) denotes the free reduction. Then \( \tilde{w} h q_0 h\tilde{q}_0 w \) can be rewritten into a word \( v_1 \equiv \gamma_f(u_0 \ldots u_m) \) and \( v_2 \equiv \gamma_f(u_0 \ldots u_m) \). So, \( \tilde{w} h q_0 h\tilde{q}_0 w \) is \( \tilde{w} h q_0 h\tilde{q}_0 w \) and \( v_3 \equiv \gamma_f(u_0 \ldots u_m) \). Hence there is a \( v_3 \equiv \beta_x \) freely reduced with \( hq_0 w \in \mathbb{Q} \). But \( v_3 hq_0 w \in \mathbb{Q} \) with \( v_3^{-1}, v_2^{-1} \equiv \beta_x \) freely reduced. So \( |v_3^{-1}|_R = |v_2^{-1}|_R \). According to the proof
of [Rot] Lemma 12.18, p.304, $\pi_R(v_2^{-1})$ describes a derivation from $whq_0wh$ to $q$ in $\Delta$. Because of (3) such a derivation contains more than $|A_n(w,w)|$ steps. This means $|v_2^{-1}|_R > |A_n(w,w)|$, and therefore $|A_n(w,w)| < |v_2^{-1}|_R < |v_2^{-1}| < |u_1..u_1| < |u| - 3$. Therefore, a word of length $2|w|+7$, namely $hw^{-1}q_0hthq_0wh$, is substituted by a word of a length $>|A_n(w,w)|+5$, namely $u$.

Let $\alpha = \max \{|y|: y \in L_5U_{s}\{s\bar{s},\bar{s}s|s \in S_5\}\}$. Then in order to construct a word of length $>|A_n(w,w)|+4$ from a word of length $2|w|+7$, at least $r_{\alpha\alpha}(|A_n(w,w)|-2|w|-3)1$ steps are necessary. Hence a derivation from $hw^{-1}q_0hthq_0wh$ to a word $u \in ((x_5q_0x_5)UR)^*\alpha$ needs at least $r_{\alpha\alpha}(|A_n(w,w)|-2|w|-3)1$ steps. Therefore every derivation of $hw^{-1}q_0hthq_0wh$ to $e$ in $<S_5;L_5>$ needs at least $r_{\alpha\alpha}(|A_n(w,w)|-2|w|-3)1$ steps, i.e. in order to derive a word of length $3|w|+16$ to $e$ in $<S_5;L_5>$ at least $r_{\alpha\alpha}(|A_n(w,w)|-2|w|-3)1$ steps are necessary.

Hence $<S_5;L_5>$ is not $E_{n-1}$-d.b., which proves Theorem 3.3.

3.4. COROLLARY. For every $n \geq 4$ there is a f.p. group having an $E_3$-decidable word problem such that each finite presentation of this group is $E_n$, but not $E_{n-1}$-d.b.

Proof. Theorem 3.3 and Theorem 3.2.

3.5. COROLLARY. For every $4 \leq m < n$ there is a f.p. group such that the word problem for this group is $E_m$, but not $E_{m-1}$-decidable, and each finite presentation of this group is $E_n$, but not $E_{n-1}$-d.b.

Proof. Let $G_1 = <E_1;L_1>$ be f.p. having an $E_3$-decidable word problem and being $E_n$, but not $E_{n-1}$-d.b. (3.3). Let $H = <\Delta;M>$ be f.g. having an $E_m$, but not $E_{m-1}$-decidable word problem. Then there is a group $G_2 = <E_2;L_2>$ which is f.p. and $E_{m}$-d.b. s.t. $H \leq G_2$ (2.1). According to 3.1, $G_2$ has an $E_m$-decidable word problem. The word problem of $G_2$ is not $E_{m-1}$-decidable since the word problem of $H$ is not either. Hence $G_2$ is not $E_{m-1}$-d.b. because of 3.1. Let $G = G_1^*G_2 = <E_1U_{E_2;L_1,L_2}>$. Then $G$ is f.p., the word problem for $G$ is $E_m$, but not $E_{m-1}$-decidable, and the given presentation of $G$, and therewith each finite presentation of $G$, is $E_{n-1}$, but not $E_{n-1}$-d.b. (1.5 a)).

This last corollary shows that even for f.p. groups the complexity of a n.a. for solving the word problem can be of an arbitrarily higher degree than the complexity of the word problem itself.

3.6. REMARK. According to a remark in [Av-Madl], p.93, the word problem of the group $G_5$ constructed in the proof of Theorem 3.3 is even $E_2$-decidable, since the special word problem of the underlying semigroup $\Delta$ is $E_1$-decidable.
because of (1), p. 155. Hence for every \( n \geq 3 \) there is a f.p. group having an \( E_2 \)-decidable word problem and being \( E_n \), but not \( E_{n-1} \)-d.b.

4. NATURAL \( E_n \)-ALGORITHMS FOR \( E_n \)-DECIDABLE GROUPS.

For f.p. groups the property of \( E_n \)-derivation-boundedness leads to a natural \( E_n \)-algorithm for solving the word problem of the group. If a presentation has infinitely many relators we have infinitely many possibilities of inserting a relator in each step of a derivation, but only a finite number of deletions of a defining relator are possible, since only subwords are deleted. For non-f.p. groups a stronger concept of derivation-boundedness is therefore needed which guarantees the existence of a natural algorithm of the same complexity. There are several different possible definitions of d.b. group presentations for non-f.p. groups. We choose the following one, in which the allowed derivations are restricted.

4.1. DEFINITION. Let \( G = \langle \Sigma; \Lambda \rangle \) f.g. The presentation \( \langle \Sigma; \Lambda \rangle \) is strongly \( E_n \)-derivation bounded (s. \( E_n \)-d.b.) if there is a function \( k \in E_n(\Sigma) \) such that for any \( w \in \Sigma^* \), there is a derivation \( w = w_0 \rightarrow w_1 \rightarrow \ldots \rightarrow w_L = e \) in \( \langle \Sigma; \Lambda \rangle \) such that (i) \( L \leq |k(w)| \), (ii) only trivial relators are inserted. Such a derivation is called a strongly \( E_n \)-bounded derivation.

4.2. OBSERVATION. a) Let \( G = \langle \Sigma; \Lambda \rangle \) f.p. Then for all \( n \geq 1 \), \( \langle \Sigma; \Lambda \rangle \) is s.-\( E_n \)-d.b. iff \( \langle \Sigma; \Lambda \rangle \) is \( E_n \)-d.b. (The insertion of a relator \( u \) can be simulated by the insertion of \( uu_1 \) by using trivial relators and the deletion of \( u_1 \). So the length of the derivation is at most increased by the factor \( \mu = (\max\{|u| : u \in \Lambda L\})+1 \).

b) Let \( n, p \geq 0 \), and \( g = \max(n, p, 3) \). If \( G = \langle \Sigma; \Lambda \rangle \) is s.\( E_n \)-d.b. with \( L \in \Sigma^* \), \( E_p \)-decidable, then there is a natural algorithm \( x \in E_p(\Sigma) \) for the word problem of \( \langle \Sigma; \Lambda \rangle \), i.e.

\[
x(w) = \begin{cases} 
(w_0, w_1, \ldots, w_L) & \text{if } w \in \epsilon, \text{ and } w = w_0 \rightarrow w_1 \rightarrow \ldots \rightarrow w_L = e \text{ is a strongly } E_n \text{-bounded derivation from } w \text{ to } e \text{ in } \langle \Sigma; \Lambda \rangle. \\
\# & \text{if } w \notin \epsilon.
\end{cases}
\]

The proof of this fact is similar to the proof of Theorem 3.1. The only difference is that only strongly \( E_n \)-bounded derivations are considered.

c) The property of being strongly \( E_n \)-d.b. is dependent on the chosen presentation of the group. Let \( n \geq 2 \), \( \Delta = \{a_i | i \geq 1\} \), and

\[
G = \langle \Delta; a_1^{i+1}(i \geq 1), a_1 a_1^{i+1}(i \geq 2) \rangle,
\]

where \( A_{i+1} \) is the \( n+1 \)st Ackermann-function ([Rit]Def.1.1, p.1028).
For all $w \in \Delta^*$, $w \in G$, i.e. $G \leq <e>$, and so $WP_G \leq E_1(\Delta)$. Let $F = <b,c;\emptyset>$ and $X = F*G \leq F$. Then $WP_X \leq E_1(\Delta \cup \{b,c\})$. Finally let

$$H = <K,t;b^n c^n b^n c^n t = b^n c^n a_n b^n c^n : n > 1>$$

where $b,c,t, (bcbcbc bcb bcb bcb bcb bcb bcb bcb bcb)$ for $i > 1$.

$$= <\epsilon; l_{n+1}>$$

$$= <b; b^i b^i b^i t b^i c^i b^i t b^i c^i b^i : i > 1> =: <\Sigma; L'>.$$

Then $<\Sigma; L'>$ is s.E$_2$-d.b. and $<\Sigma; l_{n+1}>$ is s.E$_n+1$-d.b. but not s.E$_n$-d.b.

Since there are f.p. groups with E$_3$-decidable word problem for which no finite presentation allows a natural E$_3$-algorithm (the group $G_S = <S_5; L_5>$ in 3.3 has this property), we ask whether there is an infinite strongly E$_3$-d.b. presentation for this group, and further on whether this is the case for all E$_n$-decidable f.g. groups.

For the group $G_S$ we get that the presentation $<S_5; L_5; k, q_1^{-1} x_1^{-1} h e x q_j y h k h y y^{-1} q_j h^{-1} e x q_j y h : e \in \{t,1\}, x,y \in S^*, q_j \in Q-\{q}\>$. has an E$_1$-decidable set of defining relators, and that it is in fact s.E$_3$-d.b. So a natural E$_3$-algorithm exists for this special presentation. We want to prove that such easy presentation can be constructed for all E$_n$-decidable f.g. groups $(n > 3)$. Therefore we need the following technical lemma, which is proved by standard methods.

4.3. LEMMA. Let $\Sigma$ with $|\Sigma| > 1$, $t \in \Sigma$, be a finite alphabet, and $\emptyset \neq L \subseteq \Sigma^*$ be E$_n$-decidable for some $n > 3$. Then there is a function $g \in E_1(\Sigma)$ such that

(a) $g(t^i | i > 0) = L$.

(b) There exists a function $k \in E_n(\Sigma)$ satisfying:

$$\forall w \in \Sigma^* \wedge (w \in L \Rightarrow \exists i < |k(w)| : g(t^i) \equiv w),$$

i.e. $L$ is enumerated by an E$_1$-function $g$ such that for each word $w$ an index can be calculated by an E$_n$-function.

4.4. THEOREM. Let $G = <\Sigma; L>$ be f.g. with E$_n$-decidable word problem for some $n > 3$, and let $t \not\in \Sigma$. Then $G$ has a non-finite presentation $<\Sigma; t; L>$ such that

(1) $L \leq (\Sigma \cup \{t\})^*$ is E$_1$-decidable.

(2) $<\Sigma; t; L>$ is strongly E$_n$-d.b.

Proof. Let $\tilde{L} = \{w \in \Sigma^+ | w \notin \Sigma \}$. $\tilde{L}$ is E$_n$-decidable in $\Sigma^*$, and so $\tilde{L}$ is E$_n$-decidable in $(\Sigma \cup \{t\})^*$. Because of Lemma 4.3 there is a function $g \in E_1(\Sigma \cup \{t\})$ such that $g(t^i | i > 0) = \tilde{L}$ and there exists a function $k \in E_n(\Sigma \cup \{t\})$ satisfying:

$$\forall w \in (\Sigma \cup \{t\})^* (w \in \tilde{L} \Rightarrow \exists i < |k(w)| : g(t^i) \equiv w)).$$
Let \( L_g = \{ t, t_i g(t_i) : i > 0 \} \). Then
\[
\langle \Sigma, t; L \rangle = \langle \Sigma, t; t_i g(t_i) : i > 0 \rangle = \langle \Sigma; g(t_i) : i > 0 \rangle = \langle \Sigma; L \rangle = G,
\]
and so \( \langle \Sigma, t; L \rangle \) is a f.g. presentation of \( G \).

a) **Claim.** \( L_g \) is \( E_1 \)-decidable in \((\Sigma \cup \{ t \})^*\). We have \( w \in L_g \) iff \( w = t \) or \( w = t^i v \) with \( v \in \Sigma^* \) and \( v \equiv g(t^i) \).

b) **Claim.** \( \langle \Sigma, t; L \rangle \) is strongly \( E_n \)-d.b. Let \( w \in L \). Then we have the following derivation, where \( w' \in \Sigma^* : w \rightarrow^* w' \rightarrow^* t^i w' \rightarrow^* e \).

\( \quad \quad \quad ad \; 1, \) all \( t \in \Sigma \) which appear in \( w \) are deleted. This takes \( |w| + |w| \) steps, and \( w' \equiv \pi_e(w) \) satisfies \( |w'| \leq |w| \) and \( w' \in L \).

\( \quad \quad \quad ad \; 2, \) if \( w' \equiv e \) then we are ready. Let \( w' \neq e \). Then \( w' \in L \) and because of (b) there is an \( i \leq k(w') \) with \( g(t^i) \equiv w' \). Insertion of \( i \) trivial relators \( t^i \) and deletion of \( i \) relators \( t \) result in \( t^i w' \). Here \( 2i \leq 2|k(w')| \) steps are sufficient.

\( \quad \quad \quad ad \; 3, \) \( t^i w' \equiv t^i g(t^i) \in L_g \), and so \( t^i w' \) can be deleted within one step. Thus we have a derivation of \( w \) to \( e \) in \( \langle \Sigma, t; L \rangle \) of length \( m \leq |w| + 2|k(w')| + 1 \) in which only trivial relators are inserted. Hence the presentation \( \langle \Sigma, t; L \rangle \) is s.\( E_n \)-d.b.

4.5. **COROLLARY.** Let \( G = \langle \Sigma; L \rangle \) be f.g. with \( E_n \)-decidable word problem for some \( n \geq 3 \). Then there exists a f.g. presentation for \( G \) with an \( E_1 \)-decidable set of defining relators such that the word problem for this presentation can be solved by a natural \( E_n \)-algorithm.

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