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LANGUAGES EXTENDING L(Q)

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ABSTRACT. We present a survey of the model theory of the quantifiers Q^n and $Q^{m,n}$, where $M \models_\kappa Q^n_\kappa \bar{x}\phi\bar{x}$ means that there is a κ-powered subset X of M such that $M \models_\kappa \phi\bar{a}$ whenever $a_1, \ldots a_n \in X$, and $M \models_\kappa Q^{m,n}\bar{x}\bar{y}\phi\bar{x}\bar{y}$ means that there is a κ powered subset X of M such that $M \models \phi\bar{a}\bar{b}$ whenever $a_1,\ldots,a_m \in X$ and $b_1,\ldots,b_n \notin X$. Some recent results are announced and several open problems are given.

1. INTRODUCTION.

Over the past several years, there has been considerable work done in the model theory of languages more expressive than the first order predicate calculus L. Our interests have centered about the languages L^n and $L^{m,n}$ introduced in [MM]$_1$. The first adds the quantifier Q^n to L, where $M \models_\kappa Q^n\bar{x}\phi$ means that there is κ powered subset X of M such that $M \models \phi\bar{a}$ whenever $a_1,\ldots,a_n \in X$. The second adds $Q^{m,n}$ to L, with $M \models_\kappa Q^{m,n}\bar{x}\bar{y}\phi\bar{x}\bar{y}$ meaning that some κ powered subset X of M exists such that $M \models_\kappa \phi\bar{a}\bar{b}$ for all $a_1,\ldots,a_m \in X$ and all $b_1,\ldots,b_n \notin X$. While considerable progress has been made in the study of these languages over the past few years, many fundamental questions remain open. Our intention here is to present a survey of known results, some recent unpublished results, and some of the open problems.

Section 2 is devoted to preliminaries, notation and definitions that will be used throughout the paper.

Section 3 is concerned with compactness questions for the L^n languages.

Section 4 considers the relative expressive power of these languages.

Section 5 is concerned with decidability questions arising in the context of the L^n languages.

Some recent results for $L^{m,n}$ are presented in section 6.

Some open problems are described in section 7.

This survey is in no way comprehensive, either in the results stated or the problems mentioned. Rather, the material represents the personal interests of the author.

2. PRELIMINARIES.

We use i,j,k,ℓ,m,n to denote natural numbers; $\alpha,\beta,\gamma,\delta$ to denote ordinals; κ, λ,μ,ν to denote uncountable cardinals; κ^+ is the cardinal successor of κ, cX is the cardinality of X, and ${}^nX = \{(x_1,\ldots,x_n): x_i \in X \text{ for } i = 1,\ldots,n\}$.

n termed sequences $(x_1,\ldots,x_n)$ will be denoted by $\bar{x}$.

In [J], Jensen introduced the combinatorial principle $\Diamond_\kappa$: *there is set of subsets of* κ, $\{S_\alpha: \alpha < \kappa\}$ *such that for all* $X \subseteq \kappa$, $\{\alpha \in \kappa: X \cap \alpha = S_\alpha\}$ *is stationary* (i.e. meets every closed bounded subset of κ). He proved that if $V = L$ then $\Diamond_\kappa$ holds for every regular κ. $\Diamond_\kappa$ will appear in the hypotheses of several of the theorems we shall mention.

M and N will be used to denote structures. tM is the type of M. $|M|$ is the universe of M. If s is a type then $M{\restriction}s$ is the reduct of M to s. If $A \in tM$, A a unary relation symbol, then $M{\restriction}A$ is the relativization of M to A.

Let $\mathcal{Q}$ be a set of quantifiers, L the first order predicate calculus. $L(\mathcal{Q})$ is the language obtained by adjoining the quantifiers in $\mathcal{Q}$ to L, i.e, to the definition of fm for L we add the clause:

if $Q \in \mathcal{Q}$ and Q binds n variables and $\bar{v}$ is a sequence of n variables and $L(\mathcal{Q})$ then $Q\bar{v}\phi \in L(\mathcal{Q})$.

For $\mathcal{Q} = \{Q\}$ we write $L(Q)$.

The language L^n is $L(Q^n)$ where Q^n binds n variables. For each κ, $M \models Q^n\bar{v}\phi$ is given a κ interpretation: there is a κ powered subset X of $|M|$ such that $M \models \phi\bar{a}$ for all $\bar{a} \in {}^nX$.

$L^{m,n}$ is $L(Q^{m,n})$ where $Q^{m,n}$ binds m+n variables. The κ interpretation of $M \models Q^{m,n}\bar{u}\bar{v}$ is that for some κ powered proper subset X of $|M|$ and all $\bar{a} \in {}^mX$, $\bar{b} \in {}^n\tilde{X}$ we have $M \models \phi\bar{a},\bar{b}$. (The restriction that X be proper is necessary to avoid vacuous satisfaction of $Q^{m,n}\bar{u}\bar{v}\phi$).

$L^{<\omega} = L(\mathcal{Q})$ where $\mathcal{Q} = \{Q^1, Q^2, \ldots\}$.

We may write L^n_κ, Q^n_κ, $\models_\kappa$, etc. when the κ interpretarion is intended.

If Σ is a set of sentences then Mod Σ is the set of models of Σ. $Th_\Sigma M = \{\sigma \in \Sigma: M \models \sigma\}$, $Th_\Sigma K = \{\sigma \in \Sigma: M \models \sigma$ for all $M \in K\}$. $M \equiv_\Sigma N$ means that $Th_\Sigma M = Th_\Sigma N$.

Given two languages L_1 and L_2 we write $L_1 \leqslant L_2$ if for all $\Sigma_1 \subseteq L_1$ there is some $\Sigma_2 \subseteq L_2$ such that $\text{Mod}\,\Sigma_1 = ((\text{Mod}\Sigma_2){\restriction}A){\restriction}t\Sigma_1$. $L_1 < L_2$ means $L_1 \leqslant L_2$ but $L_2 \nleqslant L_1$. If the Σ_2 can always be chosen to be of type $t\Sigma_1$ we write $L_1 \leqslant\!\cdot L_2$.

$\text{Val}(L_1)$ is the set of valid L_1 sentences. L_1 is axiomatizable if $\text{Val}(L_1)$ is recursively enumerable.

A language L_1 is κ-compact if whenever $\Sigma \subseteq L_1$, $c\Sigma < \kappa$, and $\text{Mod}\,\Sigma = \emptyset$ then there is some finite subset $\Delta \subseteq \Sigma$ such that $\text{Mod}\Delta = \emptyset$. We say L_1 is countably compact if it is ω_1-compact.

3. COMPACTNESS, AXIOMATIZABILITY, AND LOWENHEIM-SKOLEM RESULTS FOR $L^{<\omega}$.

Clearly, L^1_κ is not fully compact. Indeed if $\Sigma = \{\neg Q_\kappa v(v = v)\} \cup \{c_\alpha \neq c_\beta : \alpha < \beta < \kappa\}$ then every subset of Σ of power $< \kappa$ has a model but Σ does not.

In [K], Keisler proves that L^1_κ is κ compact for all uncountable κ and axiomatizable for regular uncountable κ. His proof for $\kappa = \omega_1$ provided a starting point for our proof of compactness of $L^{<\omega}_{\omega_1}$. Recently, considerable progress has been made in the study of compactness for the L^n languages but many fundamental questions are still open.

THEOREM 3.1.1. ($\Diamond_{\omega_1}$) $L^{<\omega}_{\omega_1}$ *is countable compact and axiomatizable.*

3.1.2. ($\Diamond_{\kappa^+}, \Diamond_{\kappa^{++}}$) $L^{<\omega}_{\kappa^{++}}$ *is* κ^{++} *compact and axiomatizable.*

The first result was proved in $[MM]_1$. The κ^{++} compactness of $L^{<\omega}_{\kappa^{++}}$ is asserted in [S]. The axiomatizability of $L^{<\omega}_{\kappa^{++}}$ is not found in the literature but can be obtained as in Theorem 9.5 of $[MM]_1$.

The assumption $\Diamond_{\omega_1}$ in 3.1 is not necessary as was shown in $[MM]_1$ p.257, and similar arguments show that it is not necessary for 3.1.2 either.

THEOREM 3.2.1. ($\Diamond_{\omega_1}$) *If* $\sigma \in \mathrm{Val}(L^{<\omega}_{\omega_1})$ *then* $\sigma \in \mathrm{Val}(L^{<\omega}_\kappa)$ *for every regular* κ.

3.2.2. ($\Diamond_{\kappa^+}, \Diamond_{\kappa^{++}}$) *If* $\sigma \in \mathrm{Val}(L^{<\omega}_\lambda)$ *then* $\sigma \in \mathrm{Val}(L^{<\omega}_{\kappa^{++}})$ *for all regular* λ.

This first clause is found in $[MM]_1$ the second is a consequence of [S] but is not found there.

THEOREM 3.3. *If* κ *is weakly compact then* $L^{<\omega}_\kappa$ *is* κ *compact and axiomatizable.*

In fact if $\lambda < \kappa_\alpha$ for all $\alpha \in \lambda$ and each κ_α is weakly compact then $L(\{Q^n_{\kappa_\alpha} : n \in \omega, \alpha \in \lambda\})$ is λ compact. This is a straightforward generalization of 3.3 which appears in $[MM]_1$.

THEOREM 3.4. *If* κ *is weakly compact then* $\mathrm{Val}(L^{<\omega}_\kappa) \supseteq \mathrm{Val}(L^{<\omega}_\lambda)$ *for all* λ.

This is found in $[MM]_1$. Notice that the sentence

$$\forall uv[Ruv \to Rvu] \to [Q^2 uvRuv \vee Q^2 uv \neg Ruv]$$

is in $\mathrm{Val}(L^{<2}_\kappa)$ just in case κ is weakly compact.

When κ is a limit cardinal there is a natural alternative interpretation for Q_n: $M \models_\kappa Q^n \bar{v}\phi$ means that for all $\lambda < \kappa$ there is a λ powered subset $X \subseteq |M|$

such that $M \models_\kappa \phi\bar{a}$ for all $\bar{a} \in {}^nX$. In [MM]$_1$ we prove

THEOREM 3.5. *If κ is a strong limit cardinal and if $\lambda < \kappa$ then L^n_κ in the limit interpretation is λ compact.*

The problem of compactness for languages of the form $L(Q^m_\kappa, Q^n_\lambda)$ has for the most part been intractable to date. However, the following result appears in [Mo]$_1$.

THEOREM 3.6. *Let $\lambda < \kappa$ with κ weakly compact and $L^{<\omega}_\lambda$ λ-compact. Let $\mathcal{Q} = \{Q^n_\kappa\colon n = 1,2,\ldots\} \cup \{Q^n_\lambda\colon n = 1,2,\ldots\}$. Then $L(\mathcal{Q})$ is λ-compact and axiomatizable.*

In [MR] the Q^n quantifiers are generalized to higher order suggested by writing $\exists X \forall v_1,\ldots,v_n \in X$ instead of $Q^n v_1,\ldots,v_n$. We let:

$$P^1_\kappa(R) = R$$

$$P^2_\kappa(R) = \{S\colon S \subseteq R \text{ and } cS \geqslant \kappa\}$$

$$P^{2+n}_\kappa(R) = P^2_\kappa(P^{n+1}_\kappa(R)) \quad n = 1,2,\ldots .$$

Let X^n_i, $i = 0,1,2,\ldots$ be variables ranging over $P^n_\kappa(|M|)$ in the κ interpretation for $n > 1$. An *n-order properly descending* quantifier is one of the form

$$B_n B_{n-1} \ldots B_2 B_1$$

where

$$B_n \text{ is } \exists X^n_j \text{ for some } j$$

and for $m < n$

$$B_m \text{ is a sequence } B_{m,1},\ldots,B_{m,k_m}$$

where

$$B_{m,i} \text{ is } \exists X^m_{i_1}, X^m_{i_2},\ldots,X^m_{i_{n(i)}} \in X^{m+1}_{\ell_i}$$

for some $X^{m+1}_{\ell_i}$ occurring in B_{m+1}. We identify X^1_i with the first order variable v_i. As an example,

$$\exists X^3_0 \,\forall X^2_0, X^2_1 \in X^3_0 \,\forall v_0, v_1 \in X^2_0 \,\forall v_2 \in X^2_1 [Rv_0v_1 \wedge \neg Rv_0v_2]$$

asserts the existence of a subset of the universe partitioned into κ many κ powered equivalence classes by R.

Let $\mathcal{Q}^*$ be the collection of all n-th order properly descending quantifiers for all n. Let $L^* = L(\mathcal{Q}^*)$. In [MR] it is shown that

THEOREM 3.7. $(\Diamond_{\omega_1})$ L^* *is countably compact and axiomatizable in the ω_1 interpretation.*

The results in [S] can be used to generalize this to interpretations κ^{++} when $\diamondsuit_{\kappa^+}$ and $\diamondsuit_{\kappa^{++}}$ both hold, to give: L^* is κ^{++} compact and axiomatizable in the κ^{++} interpretation. Moreover, the analogs of 3.2.1, 3.2.2, 3.3, 3.4. all hold for L^*.

Fix a similarity type t with a unary relation symbol B. Vaught defined the two cardinal type of a structure M to be $(c|M|, cB^M)$. In [Mor] and [V] it is proved that if for all $n > 0$ there is a κ such that Σ has a model M of two cardinal type $(2^\kappa_n, \kappa)$, then Σ has a model of any two cardinal type (λ, μ) where $\lambda \geqslant \mu \geqslant c\Sigma + \omega$ (Here $2^\kappa_1 = \kappa$, $2^\kappa_{n+1} = 2^\lambda$ where $\lambda = 2^\kappa_n$). The following theorem from $[MM]_1$ generalized this.

THEOREM 3.8. *Let* $\Sigma \subseteq L$ *and let* R *be an* n-*ary relation symbol in* $\tau\Sigma$. *Suppose for each* n *there is a* κ *and a model* M *of* Σ *such that* $c|M| = 2^\kappa_n$ *and* $M \models \neg Q^n_x \bar{v} R \bar{v}$. *Then for every* $\kappa > \lambda \geqslant c\Sigma + \omega$ *there is a model* M *of* Σ *such that* $c|M| = \kappa$ *and* $M \models \neg Q^n_\lambda \bar{v} R \bar{v}$.

4. RELATIVE EXPRESSIVE POWER OF THE L^n LANGUAGES.

In $[MM]_1$ we showed that $L^1_\kappa \leqslant L^2_\kappa$ for all regular κ. In an unpublished paper, S. Garavaglia proved that $L^n_\kappa <\cdot\, L^{n+1}_\kappa$. Recently, using a forcing argument, it is shown in [RS] that

THEOREM 4.1.1. *Assuming* $\diamondsuit_{\omega_1}$, $L^n_{\omega_1} < L^{n+1}_{\omega_1}$ *for all* n.

Combining this result with the techniques in [S] one easily obtains

THEOREM 4.1.2. *Assuming* $\diamondsuit_{\kappa^+}$ *and* $\diamondsuit_{\kappa^{++}}$, $L^n_{\kappa^{++}} < L^{n+1}_{\kappa^{++}}$.

P. Rothmaler and P. Tuschik [RT] give sufficient conditions for the elimination of the L^n quantifiers for a countable first order theory. So elementary classes whose theories satisfy the conditions can not be split by means of L^n sentences.

5. DECIDABLE QUESTIONS.

Here we mention a few results about the decidability of models, decidability of theories, and the decidability of sentences with respect to theories. In several of these instances one can view the results as showing the expressive strength of L^n over L^1.

It is easy to find structures whose L theories are decidable but whose L^1_κ

theories are not, for example, take $M = \langle A, B_n \rangle_{n \in \omega}$ where for some nonrecursive set X, B_n is countably infinite iff $n \in X$. On the other hand, for $n \geqslant 1$ we do not know of such structures whose L^n_κ theory is decidable but whose L^{n+1}_κ theory is not.

In [R] one finds an example of a "natural" class of structures whose $L^1_{\omega_1}$ theory is decidable but whose $L^2_{\omega_1}$ theory is not.

THEOREM 5.1. (Rubin). *The* L^1 *theory of boolean algebras is decidable but the* L^2 *theory is not.*

The decidability of the L^1 theory of boolean algebras was discovered independently by M. Weese [W].

A number of other decidability results of this nature are mentioned and an extensive bibliography is given in D. Seese [Se]. Many of the decidability results can be found in [BSTW]. In particular, the reader should see H. Tuschik [T] for results on the decidability of L^n theories of linear orderings.

In another direction Macintyre [Ma], Morgenstern $[Mo]_2$, and Schmerl and Simson [SS] turn their attention to L^2 extensions of Peano's arithmetic. The axiomatization given in $[MM]_1$ (correct and, with $\Diamond_{\omega_1}$, complete for validities in the ω_1 interpretation) is correct for the ω interpretation. When the usual first order version of the Peano arithmetic is enriched by adding all instances of the induction schema involving L^2 formulas we get the theory P^2 (Morgenstern observes that the Q^1 quantifier can be defined in arithmetic using L and that the quantifiers Q^n for $n > 2$ can be defined in arithmetic using L^2). In $[Mo]_2$ and [Ma] it is shown that truth for first order formulas in arithmetic can be defined in P^2, which leads to

THEOREM 5.2. *The Harrington Paris combinatorial principle is provable in* P^2.

Simson and Schmerl broaden this to show that even stronger combinatorial principles considered by Friedman, McAloon and Gunison are also provable in P^2. This leads naturally to the problem of finding a "meaningful" statement of P^2 or Peano's arithmetic that is undecidable in P^2 (of course by Gödel's 2nd theorem there are undecidable L statements in P^2). Morgenstern has noticed that Kruskals theorem [K] is statable in P^2 and this is a candidate.

6. THE $L^{m,n}$ LANGUAGES.

The languages $L^{m,n}$ were introduced in $[MM]_1$, being called $L^\#$ there. It was shown there that even $L^{1,1}$ is *not* countably compact in any infinite power. The purpose of presenting this language there was to show that the L^n languages could not be generalized in this direction without losing compactness properties. However, in [Ma] we began to investigate the model theory of $L^{m,n}$. Regarding the relative expressive power of these languages we have

THEOREM 6.1. $L^m_\kappa \leqslant L^{m,n}_\kappa \leqslant L^{2,2}_\kappa$ *for all* m,n. *When* L^m_κ *is countably compact then* $L^m_\kappa < L^{m,n}_\kappa$.

All questions about relative expressive power not answered by 6.1 are open. The only other bit of information on these languages is

THEOREM 6.2 *Let* $\sigma \in L^{1,1}$ *and suppose there is a model of* σ *in the* κ *interpretation where* κ *is regular and* $\kappa > \omega$. *Then there is a model of* σ *in the* ω *interpretation.*

The expressive strength of $L^{m,n}$ makes a generalization of this theorem desirable. For example, the sentence $\neg Q^{m,1}\bar{u}v[f\bar{u} \neq v]$ asserts that f is not closed on a κ powered subset of the universe. It follows that in a finite functional type one can express the property of a Jónsson algebra. A strengthening of the theorem above would yield results such as: if there is a Jónsson algebra in $\mathrm{Mod}_\kappa\Sigma$ then there is one in $\mathrm{Mod}_\lambda\Sigma$.

7. OPEN PROBLEMS.

This list of problems is by no means comprehensive, instead it represents the author's particular interests. In many of these problems only relative consistency results can be hoped for.

Is $L^{<\omega}$ $<\kappa$-compact in the κ interpretation when the cofinality of $\kappa \geqslant \omega_1$?

At the moment we do not know if L^2 is countably compact in the $\beth_{\omega_1}$ interpretation or in the first strongly inaccessible interpretation.

In the cases where compactness is known, completeness is also, at least in the sense that the validities are recursively enumerable. Positive answers to any of the above should yield completeness results also.

Let Val_κ be the set of validities of $L^{<\omega}$ in the κ interpretation. Let κ and κ' be successor cardinals and let λ and λ' be of cofinality strictly between ω and κ. Let μ and μ' be inaccessible but not weakly compact, ν and ν' weakly compact. We suspect that $\mathrm{Val}_\kappa = \mathrm{Val}_{\kappa'} \subset \mathrm{Val}_\lambda = \mathrm{Val}_{\lambda'} \subset \mathrm{Val}_\mu = \mathrm{Val}_{\mu'} \subset \mathrm{Val}_\nu = \mathrm{Val}_{\nu'}$ (It is easy to see that $\mathrm{Val}_\kappa \not\supseteq \mathrm{Val}_\lambda \neq \mathrm{Val}_\mu \not\supseteq \mathrm{Val}_\nu$).

A purely set theoretic combinatorial statement equivalent to the countable compactness of $L^{<\omega}$ might be an interesting new axiom for set theory.

We have mentioned that L^{n+1} is more expressive than L^n (even up to relativised reducts). Can this be sharpened in the following way? Let $M = \langle A, R^M, \ldots \rangle$ where R^M is a symmetric n+1-ary relation and the cardinality of A is $\kappa > \omega$. Is there some N equivalent to M with respect to the language L^n such that $N \models Q^{n+1}\bar{x}R\bar{x} \vee Q^{n+1}\bar{x} \neg R\bar{x}$?

Regarding the $L^{m,n}$ languages, there are two obvious questions. In view of Theorem 6.1 it is natural to investigate the relative expressive power of $L^{1,1}$, $L^{1,2}$, $L^{2,1}$, and $L^{2,2}$.

Theorem 6.2 raises the following questions. For what $m,n \in \omega,\kappa,\lambda$ will satisfiability in the κ interpretation of $\sigma \in L^{m,n}$ imply satisfiability of σ in the λ interpretation? In particular, we do not know if satisfiability of $\sigma \in L^{1,1}$ in the κ interpretation, κ uncountable, regular and $> \lambda$ implies satisfiability of $\sigma \in L^{1,1}$ in the ω_2 interpretation. Nor do we know if satisfiability of $\sigma \in L^{1,2}$ or $L^{2,1}$, or $L^{2,2}$ in the κ interpretation, κ uncountable and regular implies the satisfiability of σ in the ω_1 interpretation.

Theorem 5.2 presents an R.E. extention P^2 of Peano's arithmetic in which one can prove the combinatorial principles of Harrington and Paris which are independent of Peano's arithmetic. At the moment there is no 'natural' sentence independent of L^2 that is known. In particular, it is not known if Kruskal's theorem [K] is decidable in P^2.

For each $n \geq 1$ is there a (natural) structure whose L^n theory is decidable but whose L^{n+1} theory is not?

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