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# WHAT IS A MATHEMATICAL THEORY?

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Since Hilbert's and Skolem's work in foundations of mathematics we got used to mathematizing the concept of a theory as a theory formalized in first order logic. This view was very fruitful since it generated model theory and proof theory, but it may have obscured the fact that there are possibilities of other more abstract mathematizations of the concept of a theory which raise other deep and interesting problems. It is the purpose of this lecture to point out two such mathematizations, and the way in which one of them leads to a mathematical concept of finitistic theory.

## 1. τπ-THEORIES.

By a *normal theory* we mean a theory T which is formalized in first order logic with equality and axiomatized by a finite set of axioms or axiom schemata (see [10]) such that $\forall xy\,[x = y]$ is not a theorem of T. By a proof in such a theory we mean a Hilbert style proof from the axioms.

Let $\Sigma$ be a finite alphabet and $\Sigma^*$ the set of all words, i.e. finite sequences of elements of $\Sigma$. For any $\xi \in \Sigma^*$, $|\xi|$ denotes the length of $\xi$.

A $\tau\pi$-*theory* is a set of pairs $T \in \Sigma^*\times\Sigma^*$ such that there exists a polynomials $p(x,y)$ and a Turing machine M such that, for any $(\tau,\pi) \in \Sigma^*\times\Sigma^*$. M can decide in time $\leqslant p(|\tau|,|\pi|)$ if $(\tau,\pi) \in T$.

If $(\tau,\pi) \in T$ then $\tau$ is called a *theorem* of T and $\pi$ is called a *proof* of in T.

Every normal theory defines a $\tau\pi$-theory since the time necessary to check the correctness of a Hilbert style proof in a normal theory can be estimated from above by a polynomial of the length of that proof.

Now, a $\tau\pi$-theory T will be called *amenable* (to *automatization*) iff there exists another polynomial $P_0(x,y)$ and another Turing machine $M_0$ such that, given any word $\tau \in \Sigma^*$ and any positive integer n, $M_0$ can decide in time $\leqslant p_0(|\tau|,n)$ if there exists a $\pi \in \Sigma^*$ with $|\pi| \leqslant n$ such that $(\tau,\pi) \in T$. (Notice that if we replaced the condition $\leqslant p_0(|\tau|,n)$ by the condition $\leqslant p_0(|\tau|,c^n)$ where $c = \text{card}\,\Sigma$, then the concept would trivialize since every $\tau\pi$-theory would be amenable).

It is clear that after Gödel's discovery that all sufficiently strong theories are undecidable, the next question which should have presented itself is the question if the $\tau\pi$-theories (corresponding to normal theories) are amenable. But we had to wait until 1971 (the paper of Cook [1]) for a clear statement of that question. To end this part of my talk let me formulated the following proposition (which is implicit in [1]).

PROPOSITION. *The following three statements are equivalent to each other:*

i) $P \neq NP$;

ii) *There exists a $\tau\pi$-theory which is not amenable;*

iii) *Every $\tau\pi$-theory which is defined by a normal theory is not amenable.*

I think that this proposition constitutes the best way of explaining the great importance of Cook's conjecture $P \neq NP$ for the foundations of mathematics. (Its importance in computer science is also well known [3]).

## 2. INTERPRETABILITY.

Now we want to introduce you to another abstraction which we call the *locall interpretability type*, or, *chapter* of a first order theory.

First a sentence $\sigma$ without functions symbols nor equality is *interpretable* in a theory T if one can substitute the variable of $\sigma$ by n-tuples of variables (for some integer n), and the relation symbols of $\sigma$ by formulas which may have additional free variables (called parameters of the interpretation) such that the existential closure of the resulting formula is a theorem of T. of course, if $\sigma$ and $\sigma'$ are sentences of the same shape, i.e., if they differ only by the names of their relation symbols, then $\sigma$ is interpretable in T iff $\sigma'$ is interpretable in T.

For any first order theory T the *chapter* of T, in symbols $|T|$, is the set of all shapes of sentences interpretable in T. Let $\mathcal{J}$ be the set of all chapters of theories. Thus $\mathcal{J}$ is a family of sets.

It is easy to check that the partial order $<\mathcal{J}, \subseteq>$ constitutes a complete lattice, since the intersection of any set of chapters is again the chapter of some theory.

From the point of view of ordinary informal mathematics the chapter $|T|$ of a theory T is no less interesting than T itself. E.g., $|T|$ does not depend on the choice of the primitive symbols of T, in fact $|T|$ is immune to extensions of T by means of defined symbols, and, $|T|$ reflects very well the mathematical strength of T. Thus a study of the lattice $<\mathcal{J}, \subseteq>$ seems very important. In [5] we have published a preliminary study of this lattice. E.g, $<\mathcal{J},\subseteq>$ is Brouwerian, its zero has one succesor, etc. Now we want to point out some open problems:

**(A)** Does $<\mathfrak{J}, \subseteq>$ have any automorphisms? If it does, are the types of some important theories like PA or ZF fixed points of all automorphisms? (Similar problems for some lattices of equational theories were recently solved by Kezek [4]).

**(B)** We say that a theory T is *connected* iff for all, $a,b \in J$ if $a \vee b = |T|$ then $a = |T|$ or $b = |T|$. P. Pudlák has shown [9] that many interesting theories are connected. Are the theories of real closed fields or of algebraic closed fields connected?

## 3. FINITISM.

A first order theory T will be called *finitistic* iff every finite part of T has finite models. The following proposition follows from Proposition 3(i) of [5].

PROPOSITION. *A theory is finistic iff its type is either zero or the successor of zero in the lattice* $<\mathfrak{J}, \subseteq>$.

It is surprising that there exists finitistic theories (whose type is the successor of zero) with a considerable mathematical content. In fact we have constructed a finitistic recursively axiomatized theory FIN which appears to be as powerful as analysis [6,7,8].

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