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BEST APPROXIMATION IN VECTOR VALUED FUNCTION SPACES

by

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ABSTRACT. Let T be the unit circle, and m be the normalized Lebesgue measure on T. If H is a separable Hilbert space, we let $L^\infty(T,H)$ be the space of essentially bounded functions on T with values in H. Continuous functions with values in H are denoted by $C(T,H)$, and $H^\infty(T,H)$ is the space of bounded holomorphic functions in the unit disk with values in H. The object of this paper is to prove that $(H^\infty+C)(T,H)$ is proximinal in $L^\infty(T,H)$. This generalizes the scalar valued case done by Axler, S. et al. We also prove that $(H^\infty+C)(T,\ell^\infty)|H^\infty(T,\ell^\infty)$ is an M-ideal of $L^\infty(T,\ell^\infty)|H^\infty(T,\ell^\infty)$, and $V(T,\ell^\infty)$ is an M-ideal of $L^\infty(T,\ell^\infty)$ whenever V is an M-ideal of L^∞, where $V(T,\ell^\infty) = \{g \in L^\infty(T,\ell^\infty): \langle g(t),\delta_n\rangle \in V \text{ for all } n\}$.

§0. **Introduction**. Let T be the unit circle and m be the normalized Lebesgue measure on T. The space of p-Bochner integrable functions on T with values in a Banach space X is denoted by $L^p(T,X)$, $1 \leq p < \infty$. If $p = \infty$, then $L^\infty(T,X)$ is the space of essentially bounded functions on T with values in X. $L^p(T,X)$ are Banach spaces with the usual norms:

$$\|f\|_p = \Big(\int_T \|f(t)\|^p dm\Big)^{1/p}, \quad 1 \leq p < \infty$$

$$\|f\|_\infty = \text{ess.}\sup_t \|f(t)\|, \quad p = \infty.$$

We refer to, [3], for the basic structure of these spaces.

The problem of best approximation in L^1 and L^∞ is of much interest. The existence of best approximation in $C(\Omega)$ to functions in $L^\infty(\Omega)$, was proved in [9], where Ω is paracompact, and later, [13, 16], was generalized to the vector valued case, where X is assumed to be uniformly convex.

The existence of best approximation in $H^\infty+C$ to functions in $L^\infty(T)$ was proved in [2], where H^∞ is the space of bounded analytic functions in the unit disk and C is C(T), the space of continuous functions on T.

The object of this paper is to try to generalize the result in [2] to the vector valued case, where X is assumed to be a Hilbert space. Further, if V is a closed subspace of $L^\infty(T)$ and

$$V(T,\ell^\infty) = \{g \in L^\infty(T,\ell^\infty): \langle g(t),\delta_n\rangle \in V \text{ for all } n\}$$

we prove that $V(T,\ell^\infty)$ is proximinal in $L^\infty(T,\ell^\infty)$ whenever V is proximinal in $L^\infty(T)$, noting that ℓ^∞ is not uniformly convex.

In Section 1, we prove some results on $(H^\infty+C)(T,H)$, similar to the scalar valued case. A representation of the dual of $L^\infty(T,H)$ is included. In Section 2, we prove that $(H^\infty+C)(T,H)$ is proximinal in $L^\infty(T,H)$. In Section 3 we give an example, ℓ^∞, where C(T,X) is proximinal in $L^\infty(T,X)$, without X being uniformly convex.

Throughout the paper, if X and Y are Banach spaces, then $X\hat{\otimes}Y$, $X\check{\otimes}Y$ denote the projective and the injective tensor products of X and Y, respectively. L(X,Y) is the space of bounded linear operators from X into Y. The dual of a Banach space X is X^*. The complex numbers are denoted by $\mathbb{C}$.

§1. Vector valued function spaces.

We let $H^p(T)$ be the classical Hardy spaces, $1 \leq p < \infty$, and $H^\infty(T)$ is the space of bounded analytic functions in the unit disk. One can consider $H^p(T)$ as closed subspaces of $L^p(T)$, $1 \leq p \leq \infty$, for which $\int_T f(t)e^{int}dm(t) = 0$ for all $n < 0$; $f \in L^p(T)$, [8].

If X is a Banach space, we define

$$H^p(T,X) = \{f \in L^p(T,X) \mid x^* \circ f \in H^p(T),\ x^* \in X^*\}.$$

This definition saves us the trouble of proving the existence of the radial limit if we were to consider $H^p(T,X)$ as functions on $D = \{z \in \mathbb{C}: |z| < 1\}$.

As in the scalar valued case one can prove:

THEOREM 1.1. $H^p(T,X)$ *is a closed subspace of* $L^p(T,X)$, $1 \leq p \leq \infty$.

Now, we take our Banach space to be a separable Hilbert space H. Consequently every element $f \in H^p(T,H)$ has a representation:

$$f = \sum_{n=1}^{\infty} f_n e_n, \quad f_n \in H^p(T),$$

(e_n) is some orthonormal basis of H. For p = 2, one has

$\|f\| = (\sum^{\infty}\|f_n\|^2)^{\frac{1}{2}}$, [7].

Since H is reflexive, $L^p(T,H)$, $1 < p < \infty$ are reflexive Banach spaces, and $L^p(T,H) = [L^{p'}(T,H)]^*$, $L^1(T,H)^* = L^{\infty}(T,H)$, [3]. From the definition of $H^p(T,H)$ and the fact that the $H^p(T)$, $1 \leqslant p \leqslant \infty$ are w^*-closed in $L^p(T,H)$ the following result follows:

LEMMA 1.2. $H^p(T,H)$ *is* w^**-closed in* $L^p(T,H)$, $1 < p \leqslant \infty$.

Set $A(T,H) = H^{\infty}(T,H) \cap C(T,H)$. If $f \in A(T,H)$, then $f = \sum_{n=1}^{\infty} f_n e_n$, $f_n \in A(T)$, the disk algebra.

If Y is a subspace of the Banach space X, then for $x \in X$, $d(x,Y) = \inf\{\|x-y\| : y \in Y\}$. The proof of the following result is the same as for the scalar valued case and will be omitted, [6].

THEOREM 1.2. *Let* $f \in C(T,H)$. *Then* $d(f,H^{\infty}(T,H)) = d(f,A(T,H))$.

The subspace $H^{\infty}+C \subset L^{\infty}$ was proved to be a closed subspace, [6]. This result is true for the vector valued case:

THEOREM 1.3. $H^{\infty}(T,H) + C(T,H)$ *is a closed subspace of* $L^{\infty}(T,H)$.

THEOREM 1.4. $[L^{\infty}(T,H)]^*$ *is isometrically isomorphic to the space of finitely additive vector measures which vanish on* m*-null sets, equipped with the total variation norm.*

Proof. Since one can integrate a vector valued function against finitely additive vector measures, [4], the proof is similar to the scalar valued case, [5], and will be omitted. Q.E.D.

§2. Best approximation in $L^{\infty}(T,H)$.

Let X be a Banach space and Y be a closed subspace of X. For $x \in X$, an element $y \in Y$ is called a *best approximant of* x *in* Y if

$$\|x-y\| = d(x,Y) = \inf\{\|x-z\|,\ z \in Y\}.$$

If every element $x \in X$ has a best approximant in Y, then Y is called *proximinal* subspace of X. It is an interesting problem to determine whether a given closed subspace of Banach space is proximinal or not. In [2], Axler et al. proved that $H^{\infty}+C$ is proximinal in $L^{\infty}(T)$. Luecking, [12], gave a different proof for the same result, using the idea of M-ideals. We refer to the references in [2] for further results on best approximation. The object of this section is to prove that $(H^{\infty}+C)(T,H)$ is proximinal in $L^{\infty}(T,H)$, for every separable Hilbert space H.

A closed subspace Y of X is called an L-*summand* of X if there exists a subspace Y' of X such that $X = Y \oplus Y'$ and if $x = y + y' \in Y \oplus Y'$ then $\|x\| = \|y\|+\|y'\|$. Y is called an M-*ideal* of X if $Y^{\perp}$ is an L-summand of X^{*}, where $Y^{\perp} = \{\phi \in X^{*} : \phi(Y) = 0\}$.

THEOREM 2.1. $(H^{\infty}+C)(T,H)|H^{\infty}(T,H)$ *is an* M-*ideal of* $L^{\infty}(T,H)|H^{\infty}(T,H)$.

Proof. We identify $(L^{\infty}(T,H)|H^{\infty}(T,H))^{*}$ with $H^{\infty}(T,H)^{\perp} \subseteq L^{\infty}(T,H)^{*}$, and $((H^{\infty}+C)(T,H)|H^{\infty}(T,H))^{\perp}$ with $(H^{\infty}+C)(T,H)^{\perp}$

$\subseteq L^{\infty}(T,H)^*$, [6]. Let $F \in H^{\infty}(T,H)^{\perp}$. Being an element of $L^{\infty}(T,H)^*$, it follows from Theorem 1.8 that $F = \sum_{n=1}^{\infty} \mu_n e_n$ for some sequence (μ_n), $\mu_n \in L^{\infty}(T)^*$, and a (fixed) orthonormal basis (e_n) of H, where

$$\langle F(E),x\rangle = \sum_{n=1}^{\infty} \mu_n(E)\langle e_n,x\rangle$$

for every Lebesgue measurable set $E \subset T$, and $x \in H$. If X is the maximal ideal space of $L^{\infty}(T)$, then $L^{\infty}(T) \simeq C(X)$, the space of continuous function on X. Hence $L^{\infty}(T)^* \simeq M(X)$, the space of regular Borel bounded measures on X, [6]. Consequently, every element μ_n in the representation of F can be considered as a countably additive measure on X. Set

$$F_N = \sum_{n=1}^{N} \mu_n e_n.$$

Clearly $\langle F_N(E),x\rangle \to \langle F(E),x\rangle$ for each $\hat{m}$-measurable set E in X and each $x \in H$, where $\hat{m}$ is the lifting of the Lebesgue measure m on T to X, [6]. It follows from Grothendieck's theorem, [10], that

$$\langle F_N(g),x\rangle \to \langle F(g),x\rangle$$

for ecery $g \in L^{\infty}(T) \simeq C(X)$ and $x \in H$. Thus we can consider F as a countably additive vector measure on X. Let $F_a+F_s = F$ be the Lebesgue decomposition of F with respect to $\hat{m}$, [3], where F_a is $\hat{m}$-continuous and xF_s is singular with respect to m for all $x \in H$. One can easily show that if $F_a = \sum_1^{\infty} e_n$ and $F_s = \sum_1^{\infty} w_n e_n$, then ν_n is $\hat{m}$-continuous and w_n is orthogonal to $\hat{m}$ for all n. It follows from Pettis theorem, [3], that each ν_n is absolutely continuous with respecto to n. Since (e_n) is an orthonormal basis for H, it follows that $\mu_n = \nu_n + w_n$

is the Lebesgue decomposition of μ_n with respect to $\hat{m}$. Since $F \in H^{\infty}(T,H)^{\perp}$, it follows that $\mu_n \in H^{\infty\perp}$ for all n. Consequentely by the abstract Riesz theorem, [6], $w_n \in H^{\infty\perp}$ and $\nu_n \in H^{\infty\perp}$. Define the following map:

$$P:H^{\infty}(T,H)^{\perp} \to (H^{\infty}+C)(T,H)^{\perp}, \quad P(F) = F_s.$$

It follows from the above argument and theorem 2.4 of [12], that $F_s \in (H^{\infty}+C)(T,H)^{\perp}$. Further, since $\|F\| = \|F_a\|+\|F_s\|$, [3], one has P bounded, where $\|F\|$ is the total variation of F. To complete the proof of the theorem, we have to show that P is onto. Let $\phi \in (H^{\infty}+C)(T,H)^{\perp}$, and $\phi = \sum_{n=1}^{\infty}\mu_n e_n$ for some $\mu_n \in L^{\infty}(T)^{*}$. Then each $\mu_n \in (H^{\infty}+C)^{\perp}$, and by Theorem 2.4 of [12], μ_n is orthogonal to $\hat{m}$. Consequentely ϕ is singular to $\hat{m}$. Hence $P(\phi) = \phi$, and P is onto. Q.E.D.

THEOREM 2.2. $(H^{\infty}+C)(T,H)$ *is proximinal in* $L^{\infty}(T,H)$.

Proof. Since an M-ideal of a Banach space X is proximinal in X, [1], it follows that $(H^{\infty}+C)(T,H)/H^{\infty}(T,H)$ is proximinal in $L^{\infty}(T,H)/H^{\infty}(T,H)$. Lemma 1.2. together with a compactness argument imply that $H^{\infty}(T,H)$ is proximinal in $L^{\infty}(T,H)$. Thus for $f \in L^{\infty}(T,H)$, there exists $g \in (H^{\infty}+C)(T,H)$ such that

$$d(f,(H^{\infty}+C)(T,H)) = d(f-g,H^{\infty}(T,H)) = \|f-g-g_o\|$$

for some $g_o \in H^{\infty}(T,H)$. Thus $g+g_o \in (H^{\infty}+C)(T,H)$ is a best approximant of f. Q.E.D.

PROBLEM. Is Theorem 2.2 true if H is replaced by arbitrary Banach space? If not, what are those Banach spaces for which the result is true?

§3. **Proximinality in $L^\infty(T,\ell^\infty)$.** Let ℓ^∞ be the Banach space of bounded sequences, so that if $f \in \ell^\infty$, then $\|f\|_\infty = \sup_n |f(n)| < \infty$.

THEOREM 3.1. *Let* V *be a proximinal subspace of* $L^\infty(T)$. *Then* $V(T,\ell^\infty)$ *is proximinal in* $L^\infty(T,\ell^\infty)$.

Proof. Let $f \in L^\infty(T,\ell^\infty)$. Since $L^\infty(T,\ell^\infty) = (L^1(T,\ell^1))^*$ $= (\ell^1(N,L^1))^* = \ell^\infty(N,L^\infty)$, it follows that

$$\|f\| = \|\sum_{n=1}^{\infty} f_n \delta_n\| = \sup_t \sup_n |f_n(t)| = \sup_n \sup_t |f_n(t)|$$

$$= \sup_n \|f_n\|_\infty .$$

Here, $\delta_n(j) = 1$ if $n = j$, $\delta_n(j) = 0$ otherwise. Consider the function $\tilde{f} = \sum_{n=1}^{\infty} \tilde{f}_n \delta_n$, where $\|f_n - \tilde{f}_n\| = d(f_n, V)$. If $g = \sum_{n=1}^{\infty} g_n \delta_n \in V(T,\ell^\infty)$ then

$$\|f-\tilde{f}\| = \sup_n \|f_n - f_n\| \leqslant \sup_n \|f_n - g_n\| = \|f-g\|.$$

Hence $\tilde{f}$ is a best approximant of f in $V(T,\ell^\infty)$. Q.E.D.

COROLLARY 3.2. *The spaces* $C(T,\ell^\infty)$ *and* $(H^\infty+C)(T,\ell^\infty)$ *are proximinal in* $L^\infty(T,\ell^\infty)$.

Proof. Follows from Theorem 3.1 and the fact that $H^\infty+C$ and $C(T)$ are proximinal in $L^\infty(T)$. Q.E.D.

THEOREM 3.3. *If* V *is an* M-*ideal of* $L^\infty(T)$, *then* $V(T,\ell^\infty)$ *is an* M-*ideal of* $L^\infty(T,\ell^\infty)$.

Proof. Let $B(f^i,r_i)$ be any three open balls in $L^\infty(T,\ell^\infty)$ with centers f^i and radii r_i such that $\bigcap_{i=1}^{3} B(f^i,r_i)$ $\neq \phi$ and $V(T,\ell^\infty) \cap B(f^i,r_i) \neq \phi$ for $i = 1,2,3$. Let

$g^i \in V(T,\ell^\infty)$ such that $g^i \in B(f^i,r_i)$. If $g^i = \sum_{n=1}^{\infty} g_n^i \delta_n$ and $f^i = \sum_{n=1}^{\infty} f_n^i \delta_n$, then $\sup \| f_n^i - g_n^i \| < r_i$. Hence $g_n^i \in B(f_n^i, r_i) \subset L^\infty(T)$. By the same argument, we have $\bigcap_{i=1}^{3} B(f_n^i, r_i) \neq \phi$ for all n. It follows that $V \cap (\bigcap_{i=1}^{3} B(f_n^i, r_i) \neq \phi$, for all n, [1].

Let $g_n \in V \cap (\bigcap_{i=1}^{3} B(f_n^i, r_i))$. The function $g = \sum_{n=1}^{\infty} g_n \delta_n \in V(T,\ell^\infty)$. Further

$$\| g - f^i \| = \sup_n \| g_n - f_n^i \| < r_i.$$

Hence $g \in V(T,\ell^\infty) \cap (\bigcap_{i=1}^{3} B(f^i, r_i))$. It follows, [1], that $V(T,\ell^\infty)$ is an M-ideal of $L^\infty(T,\ell^\infty)$. Q.E.D.

Using the same argument of Theorem 3.3 one can prove:

THEOREM 3.4. $(H^\infty + C)(T,\ell^\infty) | H^\infty(T,\ell^\infty)$ *is an* M-*ideal of* $L^\infty(T,\ell^\infty) | H^\infty(T,\ell^\infty)$.

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