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# ANOTHER UPPER BOUND FOR THE DOMINATION NUMBER OF A GRAPH

by

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**ABSTRACT.** If $\delta$ and $\Delta$ are the minimum and maximum degrees of a simple graph G of size n, then, for its domination number $\beta(G)$, we show that $\beta(G) \leqslant \lfloor (n-\Delta-1)(n-\delta-2)/(n-1) \rfloor + 2$.

**Introduction.** Graphs, considered here, are *finite* and *simple* (without loops or multiple edges), and [1,2] are followed for terminology and notation. Let G = (V,E) be an *undirected graph* with V the set of *vertices* and E the set of *edges*. A graph is said to be *complete*, if every two vertices of the graph are joined by an edge. We shall denote by $K_n$ the complete graph on n vertices. The *complement* $G^c$ of G is the graph with vertex set V, two vertices being adjacent in $G^c$ if and only if they are not adjacent in G. For any vertex v of G, the *neighbour set* of v is the set of all vertices adjacent to v; this set is denoted by N(v). A vertex is said to be an *isolated* vertex, if its neighbour set is empty. A set of vertices in a graph is said to be a *dominating set,* if every vertex not in the set is adjacent to one or more vertices in the sets. A *minimal dominating set* is a dominating set such that no proper subset of it is also a dominating set. The *domination number* $\beta(G)$ of G is the size of the smallest minimal dominating set. A well known upper bound for $\beta(G)$ is due to V.G. Vizing [1,4] and it is as follows:

$$\beta(G) < n+1-\sqrt{1+2m}\ ,$$

where $n = |V|$ and $m = |E|$. But, if $\beta(G) > 2$, this bound can be attained only for graphs having at least an isolated vertex. In [3], we have suggested an upper bound for $\beta(G)$, which can be attained for graphs with no isolated vertices and having $\beta(G) > 2$. More exactly, we have proved that for a simple graph $G = (V,E)$ without isolated vertices and for which $\beta(G) > 2$, we have $\beta(G) < \lceil (n+1-\delta)/2 \rceil$, where $\delta = \min N(v)$, and $\lceil x \rceil$ denotes the smallest integer greater than or equal to the real number x. Our aim, in this note, is to suggest another upper bound for $\beta(G)$, when $\beta(G) > 2$.

**The main result.** In the sequel, we shall denote $\Delta = \max_{v \in V} |N(v)|$. For any real number x, we use $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x.

**LEMMA.** $\beta(G^c) \leqslant \lfloor \delta(\Delta-1)/(n-1) \rfloor + 2$.

*Proof.* Obviously, if G contains at least an isolated vertex, then $\delta = 0$, $\beta(G^c) = 1$ and the theorem is proved.

So, suppose that G does not contain isolated vertices.

Let $v \in V$, such that $|N(v)| = \delta$, and $W = V-(N(v) \cup \{v\})$.

If W is empty, then, by the choice of v, we must have $|N(u)| \geq \delta$ for each $u \in N(v)$, i.e., $G = K_n$. Thus, $\delta = \Delta = n-1$, and every vertex of $G^c$ is isolated. Consequently, $\beta(G^c) = n$, and the theorem is proved.

Let then $|W| \geq 1$. Obviously, the following holds:

(1) $\qquad (v,w) \notin E$, for each $w \in W$.

Let $u \in N(v)$ and $D = N(v) \cap N(u)$. Therefore, we have

(2) $\qquad (u,t) \notin E$, for each $t \in N(v) - D$.

From (1) and (2), it follows that $D \cup \{v\} \cup \{u\}$ is a dominating set of $G^c$, i.e.,

(3) $\qquad \beta(G^c) \leq e + |D|$.

On the other hand, we have $W \cap N(u) = N(u)-(D \cup \{v\})$, i.e.,

(4) $\qquad |W \cap N(u)| \leq \Delta - |D| - 1$.

Hence, from (3) and (4), we obtain

(5) $\qquad |W \cap N(u)| \leq \Delta + 1 - \beta(G^c)$, for each $u \in N(v)$.

Let $w \in W$ and $\tilde{D} = N(v) \cap N(w)$. Obviously, we have

(6) $\qquad (u,w) \notin E$, for each $u \in N(v) - \tilde{D}$

and

(7) $\qquad (v,t) \notin E$, for each $t \in N(w) - \tilde{D}$.

From (1), (6) and (7), it follows that $D \cup \{v\} \cup \{w\}$ is a dominating set of $G^c$, i.e.,

(8) $\beta(G^c)-2 \leqslant |N(v) \cap N(w)|$, for each $w \in W$.

From (8), it follows that

(9) $$|W|[\beta(G^c)-2] \leqslant \sum_{w \in W} |N(v) \cap N(w)|.$$

But,

(10) $$\sum_{w \in W} |N(v) \cap N(w)| = \sum_{u \in N(v)} |W \cap N(u)|.$$

Hence, from (9), (10) and (5), we obtain

$$|W|[\beta(G^c)-2] \leqslant [\Delta+1-\beta(G^c)]|N(v)|, \quad \text{or}$$

(11) $$(n-\delta-1)[\beta(G^c)-2] \leqslant \delta[\Delta+1-\beta(G^c)].$$

From (11), by an elementary calculus, we obtain $\beta(G^c) \leqslant \delta(\Delta-1)/(n-1)+2$, i.e., $\beta(G^c) \leqslant \lfloor \delta(\Delta-1)/(n-1) \rfloor +2$. Q.E.D.

**THEOREM.** $\beta(G) \leqslant \lfloor (n-\Delta-1)(n-\delta-2)/(n-1) \rfloor +2$.

*Proof.* If $\delta$ and $\Delta$ are the minimum and maximum degrees of vertices in G, then $\delta(G^c) = n-\Delta-1$ and $(G^c) = n-\delta-1$ are the corresponding degrees in $G^c$. Thus, the theorem follows by lemma, since $(G^c)^c = G$. Q.E.D.

**EXAMPLE.** Let us consider the graph $G = (V,E)$, where:

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

and

$$E = \{e_1, e_2, e_3, e_4\},$$

such that: $e_1 = (v_2,v_3)$, $e_2 = (v_2,v_4)$, $e_3 = (v_2,v_5)$, $e_4 = (v_2,v_6)$. It is easy to see that $\beta(G) = 2$. For this graph, our upper bound gives the correct value, whereas Vizing's bound is larger.

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