ALMOST-HOMEOMORPHISMS AND ALMOST-TOPOLOGICAL PROPERTIES

by

Filippo CAMMAROTO(*)(**)

ABSTRACT. A function is said to be an almost-homeomorphism if it is a bijective almost continuous function (see [25]) with an almost continuous inverse. We characterize such functions in several ways and obtain the relationship between almost-homeomorphisms and semi-homeomorphisms (see [8]). We study those properties which are preserved under this class of functions -the almost topological properties - and characterize them as the semi-regular properties (see [3]). We also introduce the concept of an almost topological class and study the relationship between this classes and the topological, semi-topological, and p-topological classes.

(*) This result was presented to the XII Meeting of the U. M.I. held at the University Perugia in September 1983.
(**) This research was supported by a grant from the C.N.R. (G.N.S.A.G.A.) and the M.P.I. through "Fondi 40%".

57
Introduction. Many authors, among them Biswas [1], Crossley and Hildebrand [8], Fomin [10], Neubrum [17], and Piotrowski [24], have introduced and studied weak forms of homeomorphisms. In section 1 of this paper, using the definition of an almost-continuous function of Singal [25], we introduce the concept of an almost-homeomorphism. In section 2 we obtain some necessary conditions for a function to be an almost-homeomorphism. In section 3 we obtain the relationship between almost-homeomorphisms and semi-homeomorphisms; we introduce the notion of an almost topological property and give some examples of topological properties which are also almost topological properties. We characterize the almost topological properties as the semi-regular properties (see [3]). In section 4 we introduce the concept of an almost topological class and we study its relationship with the concepts of topological class, semi-topological class and, $\rho$-topological class.

In this work the $\delta$-continuous and $\delta$-open functions of [4] play an important role.

The author is very much indebted to the referee for his helpful remarks in connection with the revision of this paper.

80. Preliminaries. Our notation is standard. Spaces will always mean topological spaces in which no separation axioms are assumed, unless explicitly stated. The closure (resp. $\delta$-interior) of a subset $A$ of a space $(X, \tau)$ will be denoted by $\bar{A}^\tau$ (resp. $\overset{\circ}{A}^\tau$) or simply $\bar{A}$ (resp. $\overset{\circ}{A}$). The $\delta$-closure (resp. $\delta$-interior) [4] of $A$ in $X$ will be denoted by $\overline{A}$ (resp. $\overset{\circ}{A}$). If
τ and σ are two topologies in a given set X, τ is said to be finer than σ if σ ⊆ τ, and the relationship is expressed as τ ≤ σ. The empty set will be denoted by ∅. A subset A of X is called regularly open, (resp. regularly closed) if it is the interior (resp. closure) of its own closure (resp. interior) [4]. Let (X,τ) be a topological space. By τ* we denoted the topology (δ-topology, prop. 1.1 of [4]) which has as a base the family of regularly open sets of (X,τ). This topology is called the semiregularization of τ, and every element of τ* is said δ-open set [4]. A subset A of X is said to be semi-open [13] if there exists an open set U of X such that U ⊆ A ⊆ U. The family of all semi-open sets of X will be denoted by SO(X).

DEFINITION 0.1, [4],[21]. A function f:(X,τ) → (Y,σ) is said to be δ-continuous if the inverse image of each δ-open subset of Y is a δ-open subset of X, that is, if τ* ≤ f⁻¹(σ*).

DEFINITION 0.2, [4]. A function f:(X,τ) → (Y,σ) is said to be δ-open (resp. δ-closed) if the image of each δ-open (resp. δ-closed) subset of X is a δ-open (resp. δ-closed) subset of Y, i.e. if σ* ≤ f(τ*).

LEMMA 0.1. Let (X,τ) be a topological space, then for every subset A of X we have: A = Aτ* and A = Aτ*.

DEFINITION 0.3, [25]. A function f:(X,τ) → (Y,σ) is said to be almost-continuous if the inverse image of every regularly open subset of Y is an open subset of X. Also f is called almost-open (resp. almost-closed) is the image
of every regularly open (resp. regularly-closed) subset of
X is an open (resp. closed) subset of Y.

§ 1. Almost-homeomorphisms.

DEFINITION 1.1 A bijective function \( f: (X, \tau) \to (Y, \sigma) \) is said to be an \textit{almost-homeomorphism} if \( f \) and \( f^{-1} \) are almost-continuous functions.

PROPOSITION 1.1. If \( f: (X, \tau) \to (Y, \sigma) \) is a homeomorphism then \( f \) is an almost-homeomorphism.

Proof. Since every continuous function is almost continuous ([25], remark 2.1), this is obvious.

Remark 1. The converse of Prop. 1.1 is not true, as the following example shows.

Example 1. Let \( X = \{a, b, c\} \), and let \( \sigma = \{\emptyset, \{a\}, \{b, c\}, \{c\}, \{a, c\}, X\} \) and \( \tau = \{\emptyset, \{a\}, \{b, c\}, X\} \) be two topologies on \( X \). It is obvious that \( \sigma \leq \tau \). Let \( f: (X, \sigma) \to (X, \tau) \) be the identity function on \( X \). It follows that \( f \) is not a homeomorphism because \( f^{-1} \) is not a continuous function. But \( f \) is an almost-homeomorphism.

PROPOSITION 1.2. Let \( f: (X, \tau) \to (Y, \sigma) \) be a function. The following conditions are equivalent:

1) the function \( f \) is almost-open,
2) the image of each \( \delta \)-open subset of \( X \) is an open subset of \( Y \), i.e. \( \sigma \leq f(\tau^\delta) \),
3) for every \( A \subseteq X \), \( f(\emptyset) \subseteq f^\emptyset(A) \).

Proof. The implication (1) \( \Rightarrow \) (2) is obvious since any \( \delta \)-open subset of \( X \) is the union of regularly open subsets of \( X \). (2) \( \Rightarrow \) (3) Let \( y \in f(\emptyset) \). Then there exists \( x \in \delta A \) such that \( f(x) = y \). Since \( x \in \delta A \), there exists a
neighbourhood $U$ of $x$ such that $\overline{U} \subseteq A$, $f(\overline{U}) \subseteq f(A)$. By (2), $f(\overline{U})$ is an open subset of $Y$ such that $y = f(x) \in f(\overline{U})$ and hence $f(x) = y \in f(\overline{A})$. (3) $\Rightarrow$ (1) Let $A$ be a regularly open subset of $X$, then $A = A^{\circ}$, hence $f(A) = f(\overline{A})$; by (3) we obtain $f(A) \subseteq f(\overline{A})$. Since $f(\overline{A}) \subseteq f(A)$, we obtain also $f(A) = f(A)$ and hence $f(A)$ is open in $Y$. ▲

The proof of the following proposition is omitted since it is similar to that of proposition 1.2.

**PROPOSITION 1.3.** Let $f:(X,\tau) \rightarrow (Y,\sigma)$ be a function. Then the following conditions are equivalent:

1) the function $f$ is almost-closed,
2) the image of each $\delta$-closed subset of $X$ is a closed subset of $Y$,
3) for every $A \subseteq X$, $\overline{f(A)} \subseteq f(\overline{A})$. ▲

**Remark 2.** It is obvious that if $f$ is a bijective function then $f$ is almost-open iff $f$ is almost-closed.

**COROLLARY 1.3.** A function $f:(X,\tau) \rightarrow (Y,\sigma)$ is $\delta$-open (resp. $\delta$-closed) iff for each $A \subseteq X$, $f(\overline{A}) \subseteq f(A)$ (resp. $f(A) \subseteq f(\overline{A})$).

**Proof.** If $f$ is $\delta$-open, by Prop. 2.4 of [4], $f:(X,\tau^*) \rightarrow (Y,\sigma^*)$ is an open function; then by theorem 11.3 of [9], $f(\overline{A}^{\tau^*}) \subseteq f(A^{\tau^*})$ for any $A \subseteq X$. Hence by lemma 0.1 $f(A)$ $f(A)$. In similar way we prove the corollary when $f$ is a closed function.

**Remark 3.** It is immediate that if $f$ is a bijective function then $f$ is $\delta$-open iff $f$ is $\delta$-closed.
The following results are direct generalizations of well-known facts on homeomorphisms.

**THEOREM 1.1** Let \((X, \tau) \rightarrow (Y, \sigma)\) be a bijective function. Then the following conditions are equivalent:
1) the function \(f\) is an almost-homeomorphism,
2) the function \(f\) is almost-continuous and almost-open (resp. almost-closed),
3) the function \(f\) is \(\delta\)-continuous and \(\delta\)-open, (resp. \(\delta\)-closed),
4) the function \(f\) and \(f^{-1}\) are \(\delta\)-continuous functions,
5) the subset \(f(A)\) is regularly open (resp. closed) in \(Y\) if and only if \(A\) is regularly open in \(X\) (resp. regularly closed),
6) \(f(\tau^*) = \sigma^*\), (resp. \(f(\tau_C^*) = \sigma_C^*)\),
7) the subset \(f^{-1}(B)\) is regularly open in \(X\) if and only if \(B\) is regularly open in \(Y\) (resp. regularly closed). △

**THEOREM 1.2** Let \(f:(X, \tau) \rightarrow (Y, \sigma)\) be a bijective function. Then the following conditions are equivalent:
1) the function \(f\) is an almost-homeomorphism,
2) for every regularly open subset \(A\) of \(X\) and for every regularly open subset \(f(B)\) of \(Y\) we have \(f(A) = \overline{f(A)}\) and \(f(B) = \overline{f(B)}\),
3) for every regularly closed subset \(A\) of \(X\) and for every regularly closed subset \(f(B)\) of \(Y\) we have \(f(A) = f(A)\) and \(f(B) = f(B)\),
4) for every subset \(A\) of \(X\) we have \(f(A) = \overline{f(A)}\),
5) for every subset \(B\) of \(X\) we have \(f(B) = f(B)\).

Proof. We prove only the implications (1) \(\Rightarrow\) (2), (3) \(\Rightarrow\) (1) and (4) \(\Rightarrow\) (5). The implications (2) \(\Rightarrow\) (3), (1) \(\Rightarrow\) (3), (1) \(\Leftrightarrow\) (4) and (5) \(\Rightarrow\) (4) are straightforward and are
left to the reader. We prove first \( (1) \Rightarrow (2) \): let \( A \subseteq X \) be a regularly open subset of \( X \). Then \( f(A) \) and \( f(\overline{A}) \) are open and closed subsets of \( Y \), respectively (by Th. 1.1 (2)), hence \( \overline{f(A)} \subseteq f(\overline{A}) \). Since \( A = f^{-1}(f(A)) \) and \( \overline{f(A)} \) is a regularly closed subset of \( Y \), \( f^{-1}(\overline{f(A)}) \) is a closed subset of \( X \) and hence \( \overline{A} \subseteq f^{-1}(\overline{f(A)}) \) and \( f(\overline{A}) \subseteq f(A) \). We have then \( f(\overline{A}) = \overline{f(A)} \). Let \( f(B) \) be a regularly open subset of \( Y \). Then by theorem 1.1 \( f^{-1}(f(B)) = B \) is a regular open subset of \( X \) and by the proof above we have \( f(B) = \overline{f(B)} \).

\( (3) \Rightarrow (1) \): by (2) of Prop. 1.2, we prove that \( f \) is an almost-continuous and almost-open function. Then, let \( f(B) \subseteq Y \) be a regularly open subset of \( Y \). If \( C = f^{-1}(\overline{f(B)}) \), then \( f(C) = \overline{f(B)} \) is a regularly closed subset of \( Y \), we apply (3) to \( f(C) \) obtaining \( f(C) = f(C) = \overline{f(B)} \), then \( f(C) = f(B) \); hence we have \( B = C \), then \( B \) is open and so \( f \) is almost-continuous.

Let \( A \subseteq X \) be a regularly open subset of \( X \), then \( \overline{A} \) is a regularly closed subset of \( X \). We apply (3) to \( \overline{A} \) and obtain \( f(\overline{A}) = \overline{f(A)} \); hence \( f(A) = f(\overline{A}) \). Therefore, \( f(A) \) is an open subset of \( Y \). This shows that \( f \) is almost-open function.

\( (4) \Rightarrow (5) \): we apply (4) to \( A = X-B \), with \( B \subseteq X \), and obtain \( f(X-B) = f(X-B) \), which gives:

\[
Y - f(X-B) = Y - f(X-B) = Y - (f(X) - f(B)) = Y - (\overline{Y - f(B)}) = \overline{f(B)}.
\]

On the other hand we have:

\[
Y - f(X-B) = f(X) - f(X-B) = f(X - (X-B)) = f(B),
\]

hence, \( f(B) = \overline{f(B)} \). ▲

The following proposition if an immediate consequence of Theorem 1.1.
PROPOSITION 1.4 A function \( f:(X,\tau) \to (Y,\sigma) \) is an almost-homeomorphism iff \( f:(X,\tau^*) \to (Y,\sigma^*) \) is a homeomorphism. \( \Delta \)

§2. Properties of almost-homeomorphisms.

DEFINITION 2.1, [26]. A space \((X,\tau)\) is said to be nearly-compact (resp. almost-compact) if every open cover admits of a finite subfamily, the interior of the closure (resp. closure) of whose members cover the space.

DEFINITION 2.2, [31]. A \( T_2 \) space \((X,\tau)\) is said to be \( T_2 \)-closed if \((X,\tau)\) is closed in every \( T_2 \) extension of it.

DEFINITION 2.3, [5]. A space \((X,\tau)\) is said to be weakly-compact if every regular cover [5] (an open cover \( \{A_i\}_{i \in J} \) such that every \( A_i \) contained a regular closed \( C_i \) verifying \( \bigcup_{i \in J} C_i = X \)) admits of a finite subfamily the closure of whose members cover the space.

PROPOSITION 2.1 Let \( f:(X,\tau) \to (Y,\sigma) \) be an almost continuous function. If \((X,\tau)\) is almost-compact and \( Y \) is a \( T_2 \) space, then \( f \) is an almost-closed function.

Proof. Let \( A \) be a regularly closed subset of \( X \). By theorem 2.6 of [25], \( f\vert_A:A \to Y \) is an almost continuous function. Since \( A \) is almost-compact, \( f(A) \) is \( T_2 \)-closed subset of \( Y \) (Th.3.3 of [26]). Since \( Y \) is a \( T_2 \) space, \( f(A) \) is a closed subset of \( Y \). \( \Delta \)
Combining Prop. 2.1 with Th. 1.1 we obtain the following corollary:

**COROLLARY 2.1** If \( f: (X, \tau) \rightarrow (Y, \sigma) \) is a bijective almost-continuous function from a \( T_2 \)-closed space onto \( T_2 \)-space, then \( f \) is an almost-homeomorphism. ▲

**PROPOSITION 2.2** Let \( f: (X, \tau) \rightarrow (Y, \sigma) \) be an almost-continuous function. If every regularly closed subset of \( X \) is weakly-compact and \( Y \) is a \( T_{2-\frac{1}{2}} \) space (or a completely Hausdorff \([30]\) space), then \( f \) is an almost-closed function.

Proof. Let \( A \) be a regularly closed subset of \( X \). By theorem 2.6 of \([25]\), \( f/A: A \rightarrow Y \) is an almost continuous function. By hypothesis, \( A \) is weakly-compact space and hence by theorem 3.4 of \([6]\) \( f(A) \) is weakly-compact space in \( Y \). Since \( Y \) is a \( T_{2-\frac{1}{2}} \) space, by theorem 2 of \([7]\) \( f(A) \) a closed subset of \( Y \). ▲

An immediate consequence of this fact and theorem 1.1 is the next result.

**COROLLARY 2.2** If \( f: (X, \tau) \rightarrow (Y, \sigma) \) is a bijective almost continuous function \( X \) where every regularly closed subset is weakly-compact, into a \( T_{2-\frac{1}{2}} \) space \( Y \), then it is an almost-homeomorphism. ▲

§3. Almost-homeomorphisms, semi-homeomorphisms, Almost-topological properties and semiregular properties. We recall some definition used in the following.
DEFINITION 3.1, [8]. A function \( f: X \rightarrow Y \) is said to be \textit{irresolute} if, for any semi-open \([8]\) subset \( S \) of \( Y \), \( f^{-1}(S) \) is semi-open in \( X \).

DEFINITION 3.2, [8]. A function \( f: X \rightarrow Y \) is said to be \textit{pre-semi-open} if for any semi-open subset \( S \) of \( X \), \( f(S) \) is semi-open in \( Y \).

DEFINITION 3.3, [8]. Two topological spaces \( X \) and \( Y \) are said to be \textit{semi-homeomorphic} if there exists a bijective function \( f \) such that \( f \) is irresolute and pre-semi-open. Such an \( f \) is called a \textit{semi-homeomorphism}.

THEOREM 3.1 Homeomorphisms are semi-homeomorphisms, and semi-homeomorphisms are almost-homeomorphisms.

Proof. It is shown in theorem 1.9 of [8] that homeomorphism implies semi-homeomorphism. It is shown in prop. 4.2 of [14] that semi-homeomorphism implies almost-homeomorphism. \( \square \)

REMARK 4. It follows from example 1.2 of [8] and example page 251 of [14] that the converses of the implications in theorem 3.1 are not true in general.

Utilizing Th. 1.1 (4) and Prop. 3.2 of [4] we have the following proposition:

PROPOSITION 3.1 Almost-homeomorphism is an equivalence relation between topological spaces.
DEFINITION 3.4 A topological property which is preserved under almost-homeomorphisms is said to be an almost-topological property.

The following results are direct consequences of respective definitions.

PROPOSITION 3.2 If $P$ is an almost-topological property, then $P$ is a semi-topological property [8] and hence $P$ is a topological property. ▲

PROPOSITION 3.3 If $P$ is a topological property but it is not a semi-topological property, then $P$ is not an almost-topological property. ▲

It follows from [8] that $T_0$, $T_1$, $T_3$, $T_4$, $T_5$, regularity, complete normality, normality, first countability, second countability, compactness, Lindelöfness, metrizability, and local connectedness are not semi-topological properties. By prop. 3.3 they are not almost-topological properties either.

We give now some examples of almost-topological properties.

a) Separation properties

PROPOSITION 3.4 Let $(X, \tau)$ be a $T_2$ (resp. $T_{2-\frac{1}{2}}$) topological space. If $f:(X, \tau) \to (Y, \sigma)$ is an almost-open bijection, then $(Y, \sigma)$ is $T_2$ (resp. $T_{2-\frac{1}{2}}$).

Proof. We prove only the $T_{2-\frac{1}{2}}$ case, since the proof
of the $T_2$ case is analogous. Let $y_1 \neq y_2 \in Y$. Then there exist $x_1 \neq x_2 \in X$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since $X$ is $T_2$, there exist two open sets $U$ and $V$ such that $x_1 \in U$, $x_2 \in V$ and $U \cap V = \emptyset$. Then also $\hat{U} \cap \hat{V} = \emptyset$. By hypothesis $f$ is an almost-open function, then $f(\hat{U})$ and $f(\hat{V})$ are subsets of $Y$ such that $y_1 \in f(\hat{U})$, $y_2 \in f(\hat{V})$ and $f(\hat{U}) \cap f(\hat{V}) = \emptyset$. ▲

An immediate consequence of this fact and theorem 1.1 is the next result.

**COROLLARY 3.4** $T_2$ (resp. $T_{2-\frac{1}{3}}$) is an almost-topological property.

**LEMMA 3.1, [11]**. Let $(Y,\sigma)$ be a regular space. If $f:(X,\tau) \rightarrow (Y,\sigma)$ is continuous then $f:(X,\tau^*) \rightarrow (Y,\sigma)$ is continuous. ▲

**PROPOSITION 3.5** Being completely Hausdorff is an almost-topological property.

**Proof**. Let $(X,\tau)$ be a completely Hausdorff space and let $f:(X,\tau) \rightarrow (Y,\sigma)$ be an almost-homeomorphism. We show that $(Y,\sigma)$ is a completely Hausdorff space. Let $y_1 \neq y_2 \in Y$. Then there exist $x_1 \neq x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. By hypothesis there exists a Urysohn function $g:(X,\tau) \rightarrow [0,1]$, [30]. Next, we show that $g \circ f^{-1}:(Y,\sigma) \rightarrow [0,1]$ is a continuous function. For any open subset $A$ of $[0,1]$, $(g \circ f^{-1})(A) = f(g^{-1}(A))$. Since $[0,1]$ is a regular space, by lemma 3.1, $g^{-1}(A) \in \tau^*$. By theorem 1.1, $f$ is almost-open and so by Proposition 1.2, $f(g^{-1}(A))$ is an open subset of $Y$. Therefore, $g \circ f^{-1}$ is continuous. Moreover, since $f$ is bijective, $g \circ f^{-1}$ is a Urysohn function. This
shows that \((Y,\sigma)\) is completely Hausdorff. ▲

**DEFINITION 3.4** [27]. A space \((X,\tau)\) is said to be almost regular if for any regularly closed set \(A\) and any \(x \notin A\) there exist disjoint open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(x \in V\).

**DEFINITION 3.5** [28]. A space \((X,\tau)\) is said to be almost completely regular if for any regularly closed set \(A\) and each point \(x \in X-A\) there is a continuous function \(f:(X,\tau) \to [0,1]\) such that \(f(x) = 1\) and \(f(A) = 0\).

**PROPOSITION 3.6** Almost regularity (almost-complete regularity) is an almost-topological property.

*Proof*. This follows from remark 2, theorem 1.1, and theorem 1 of [20] (prop. 1.4 and Cor. 2.1 of [29]). ▲

b) Connectedness properties.

**DEFINITION 3.6**, [30]. A space \((X,\tau)\) is said to be hyperconnected if every nonempty open set of \(X\) is dense in \(X\).

**PROPOSITION 3.7** Hyperconnectedness is an almost-topological property.

*Proof*. Let \((X,\tau)\) be a hyperconnected space and let \(f:(X,\tau) \to (Y,\sigma)\) be an almost-homeomorphism. We show that \((Y,\tau)\) is hyperconnected. Let \(A \subseteq Y\) be an open set of \(Y\). Then \(A \subseteq \overset{\circ}{A}\) and \(f^{-1}(\overset{\circ}{A})\) is a regularly open set of \(X\) by theorem 1.1 (7). By hypothesis, \(f^{-1}(\overset{\circ}{A}) = X\) and so \(f^{-1}(\overset{\circ}{A}) = X\). We have \(\overset{\circ}{A} = Y\) and hence \(\overset{\circ}{A} = Y\). This shows that \((Y,\tau)\) is hyperconnected. ▲
DEFINITION 3.7 [16]. A space \((X, \tau)\) is said to be almost-locally connected at a point \(p \in X\) if each regularly open neighbourhood of \(p\) contains a connected open neighbourhood of \(p\). \((X, \tau)\) is said to be almost-locally connected if it is almost-locally connected at each of its points.

Utilizing Th. 4 of [15], Cor. 4.7 of [23] and Th.1.1, we have the following proposition.

PROPOSITION 3.8 Connectedness (almost-local connectedness) is an almost-topological property. ▲

c) Covering properties.

PROPOSITION 3.9. Near compactness, almost compactness (weak compactness) are almost-topological properties.

Proof. For near compactness, it is shown in Theorem 3.1 of [26] that the almost continuous and almost-open image of a nearly-compact spaces is nearly-compact, then by theorem 1.1 (2) we have the proposition. For almost compactness, this is a consequence of theorem 3.3 of [26] (theorem 3.4 of [6], and theorem 1.1 (2).) ▲

DEFINITION 3.10 [7]. A \(T_{2\frac{1}{2}}\) space \((X, \tau)\) is said to be \(T_{2\frac{1}{2}}\)-closed if \((X, \tau)\) is closed in every \(T_{2\frac{1}{2}}\) extension of it.

The following proposition is an consequence of Prop. 3.4, Prop. 3.9, and Th. 2 of [7].
PROPOSITION 3.10 $T_2$-closedness (resp. $T_{2-1/2}$-closedness) is an almost-topological property. ▲

DEFINITION 3.11, [3]. A topological property $P$ is called semiregular provided that $(X,\tau)$ has property $P$ if and only if $(X,\tau^*)$ has property $P$.

THEOREM 3.2 A topological property $P$ is an almost-topological property if and only if it is semiregular.

Proof. Let $(X,\tau)$ be a topological space with an almost-topological property $P$. Since the identity function $i:(X,\tau) \to (X,\tau^*)$ is an almost-homeomorphism, $(X,\tau^*)$ has $P$. So if $(X,\tau^*)$ has $P$, $(X,\tau)$ has $P$, because the identity function $i^{-1}:(X,\tau^*) \to (X,\tau)$ is also an almost-homeomorphism. Conversely, we prove that if $P$ is a semiregular property then $P$ is an almost-topological property. Let $(X,\tau)$ be a topological space with a semiregular property $P$ and $f:(X,\tau) \to (Y,\sigma)$ an almost-homeomorphism. Then by prop.1.4, $f:(X,\tau^*) \to (Y,\sigma^*)$ is an homeomorphism and hence $(Y,\sigma^*)$ has $P$. Since $P$ is semiregular, $(Y,\sigma)$ has $P$ and hence $P$ is an almost-topological property. ▲

By theorem 3.2 and results of [12],[22] we have some almost-topological properties.

DEFINITION 3.12, [18],[2]. A space $(X,\tau)$ is said to be weakly locally connected if each component of $(X,\tau)$ is open.

PROPOSITION 3.11, Extremally disconnectedness (resp. weakly local connectedness) is an almost-topological property. ▲
DEFINITION 3.13, [3]. A topological space \((X, \tau)\) is said to be S-closed (locally S-closed [22]) if every semi-open [13] cover has a finite subfamily whose closure covers the space (if each point of \(X\) has an open neighbourhood which is an S-closed subspace of \(X\)).

PROPOSITION 3.13, S-closedness and locally S-closedness are almost-topological properties. ▲

§4. Almost-topological classes. If \(X\) is a set of points, let \(S(X)\) denote the collection of all topological spaces which have \(X\) as their set of points.

DEFINITION 4.1, [3]. Let \((X, \tau)\) and \((X, \sigma)\) be two elements of \(S(X)\), then \((X, \tau)\) and \((X, \sigma)\) are said to be \(\rho\)-equivalent if \(\tau^* = \sigma^*\).

THEOREM 4.1, \(\rho\)-equivalence is an equivalence relation on the collection \(S(X)\). ▲

Thus, the collection \(S(X)\) of topological spaces is partitioned into equivalence classes. Let \(\rho[X, \tau^*]\) denote the equivalence class of topological spaces with the same semi-regularization \(\tau^*\) as \((X, \tau)\).

DEFINITION 4.2 The equivalence classes of \(S(X)\) under the relation of \(\rho\)-equivalence will be called the \(\rho\)-topological classes of \(X\).
We omit the easy proof of the following results.

**THEOREM 4.2** If \( f: (X, \tau) \to (Y, \sigma) \) is a \( \delta \)-continuous (resp. \( \delta \)-open) function and \( (X, \tau_1) \in \rho[X, \tau^*] \) and \( (Y, \sigma_1) \in \rho[Y, \sigma^*] \), then \( f: (X, \tau_1) \to (Y, \sigma_1) \) is \( \delta \)-continuous (resp. \( \delta \)-open). ▲

**THEOREM 4.3** If \( f(X, \tau) \to (Y, \sigma) \) is an almost-continuous (almost-open) function, and \( (X, \tau_1) \in \rho[X, \tau^*] \) \((X, \tau_1) \in \rho[X, \tau^*]\), then \( f: (X, \tau) \to (Y, \sigma_1) \) \((f: (X, \tau_1) \to (Y, \sigma))\) is an almost-continuous (almost-open) function. ▲

**THEOREM 4.4** If \( f(X, \tau) \to (Y, \sigma) \) is an almost-homeomorphism and \( (X, \tau_1) \in \rho[X, \tau^*] \) \((Y, \sigma_1) \in \rho[Y, \sigma^*]\), then \( f: (X, \tau_1) \to (Y, \sigma_1) \) is an almost-homeomorphism. ▲

As was shown in [2], p. 103, \( \rho[X, \tau^*] \) contains a maximal topological space, denoted by \( (X, \tau^*_0) \), in the sense that the topology induced on \( X \) by relation "\( \leq \)" is finer than the topology on any other space in \( \rho[X, \tau^*] \). However, we have the following result.

**COROLLARY 4.4** If \( f(X, \tau) \to (Y, \sigma) \) is an almost-homeomorphism, then \( f: (X, \tau^*_0) \to (Y, \sigma^*_0) \) is an almost-homeomorphism. ▲

**THEOREM 4.5** If \( (X, \tau) \) and \( (X, \sigma) \) are two \( \rho \)-equivalent spaces, then they are almost-homeomorphic. ▲

**Remark 5.** Since an almost-homeomorphism is an equivalence relation (Prop. 3.1), the collection of all
topological spaces is partitioned into equivalence classes. Let $A[X,\tau]$ denote the equivalence class of all topological spaces almost-homeomorphic to $(X,\tau)$. Let $H[X,\tau]$ and $S[X,\tau]$ denote respectively the equivalence classes of all topological spaces homeomorphic to $(X,\tau)$, and those semi-homeomorphic to $(X,\tau)$.

**Definition 4.3** The equivalence class of $(X,\tau)$ under the relation of almost-homeomorphisms will be called the almost-topological class of $X$.

**Proposition 4.1** Let $P$ be a topological property. Then $P$ is an almost-topological property if and only if for some topological space $(X,\tau)$ with $P$ we have that every $(Y,\sigma) \in A[X,\tau]$ has $P$. △

**Remark 6.** In [8], $[X,SO(X,\tau)]$ denote the equivalence class of topological spaces with the same collection of semi-open sets as $(X,\tau)$, i.e. the semi-topological class of $X$.

**Theorem 4.6** Let $(Y,\sigma)$ be a topological space of $A[X,\tau]$ then we have:

$$[Y,SO(Y,\sigma)] \subseteq \rho[Y,\sigma^*] \subseteq A[X,\tau].$$

**Proof.** We prove that $[Y,SO(Y,\sigma)] \subseteq \rho[Y,\sigma^*]$. Let $\sigma_1$ and $\sigma_2$ two topologies on $Y$ such that $SO(Y,\sigma_1) = SO(Y,\sigma_2)$ i.e. $(Y,\sigma_1)$ and $(Y,\sigma_2)$ are elements of $[Y,SO(Y,\sigma)]$. By $\sigma_1^\alpha$ and $\sigma_2^\alpha$ we denote the $\alpha$-topology on $Y$ [19] associate to $\sigma_1$ and $\sigma_2$ respectively. From prop. 1 of [19] we have $\sigma_1^\alpha = \sigma_2^\alpha$ then by prop. 2.4 of [14] $\sigma_1^* = \sigma_2^*$. Hence $(Y,\sigma_1)$ and
We prove that \( p[\sigma_2] \subseteq A[X,\tau] \). Let \((Y,\omega) \in p[\sigma_2] \). Then by theorem 4.5 there exists an almost-homeomorphism \( i:(Y,\omega) \to (Y,\sigma) \). If \( f:(X,\tau) \to (Y,\sigma) \) is the almost-homeomorphism by hypothesis, then \( f^{-1}oi \) is the almost-homeomorphism from \((Y,\omega)\) onto \((X,\tau)\) so \((Y,\omega) \in A[X,\tau] \). △

**COROLLARY 4.6** Let \((X,\tau)\) be a topological space, we have:

1) \( H[X,\tau] \subseteq S[X,\tau] \subseteq A[X,\tau] \),

2) \( A[X,\tau] = \bigcup_{(Y,\sigma) \in A[X,\tau]} p[Y,\tau^*] \).

*Proof.* 1) Obvious.

2) By theorem 4.6 we obtain \( A[X,\tau] \supseteq \bigcup_{(Y,\sigma) \in A[X,\tau]} p[Y,\tau^*] \). The proof of the inverse inclusion is similar to that of theorem 4.6. △

**REFERENCES**


Dipartimento di Matematica
Università di Messina
Via C. Battisti, 90
Messina 98100, Italy.

(Recibido en agosto de 1985; la versión revisada en noviembre de 1985).