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CATEGORIES OF SPACES MAY NOT BE GENERALIZED SPACES AS EXEMPLIFIED BY DIRECTED GRAPHS

by

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It has long been recognized [G1], [L] that even within geometry (that is, even apart from their algebraic/logical role) toposes come in (at least) two varieties: as spaces (possibly generalized, treated via the category of sheaves of discrete sets), or as categories of spaces (analytic [G2], topological [J], combinatorial, etc.). The success of theorems [J'] which approximate toposes by generalized spaces has perhaps obscured the role of the second class of toposes, though some explicit knowledge of it is surely necessary for a reasonable axiomatic understanding of toposes of C^{∞} spaces or of the topos of simplicial sets. Perhaps some of the confusion is due to the lack of a stabilized definition of morphism appropriate to categories of spaces in the way that "geometric morphisms" are appropriate to generalized spaces.

There are certain properties which a topos of spaces often has; a wise selection of these should serve as an axiomatic definition of the subject. While we have not achieved that goal yet, we list some important properties and show that these properties cannot be true for a "generalized space" of the localic or groupoid kind.

We consider a topos $\mathcal{E}$ defined over another topos $\mathcal{S}$.

The latter need not be the category of abstract sets, though it will often be Boolean. In many cases it is instructive to think of S as *derived from* E (rather than the other way around), as Cantor derived "cardinal numbers" (= abstract sets) from "Mengen" (= sets with topological or similar structure, as they arise in geometry and analysis). Indeed S can be viewed as a sheaf topos in E, for an essential topology:

AXIOM 0. $E \to S$ *is local* ; $\Gamma^* \dashv \Gamma_* \dashv \Gamma^!$.

The Γ^* may be considered as the inclusion of *discrete* spaces S into "all" spaces E, whereas the sheaf inclusion $\Gamma^!$ may be considered as the inclusion of *codiscrete* or chaotic spaces into E; that these inclusions have the same domain category S may be summed up in Hegelian fashion by "pure becoming is identical with non-becoming".(*)

Of course, there are some spatial toposes which satisfy axiom 0, although they are extremely special since Γ_* is the fiber-functor for a canonically-defined extremal point of E; for example, the Zariski spectrum of a *local* ring does admit such a point $\Gamma^!$. On the other hand, the topos of G-sets for a groupoid G cannot satisfy axiom 0.

Our further axioms will be stated in terms of a further left adjoint $\Pi = \Gamma_!$ assigning to each space a discrete space of components.

AXIOM 1. $E \to S$ is *essential*, that is $\Gamma_! \dashv \Gamma^*$ exists, but moreover we require that it preserve finite products

$$\Gamma_!(X \times Y) \xrightarrow{\sim} \Gamma_!(X) \times \Gamma_!(Y)$$
$$\Gamma_!(1) \xrightarrow{\sim} 1$$

for all X, Y in E.

(*) That is, completely random motion, as a category in itself, is indistinguishable from immobility, as a category in itself, even though they are of course completely different (except for 0, 1) as subcategories of the category of spaces (= frames for continuous motion).

The axiom is necessary for the naive construction of the homotopic passage from quantity to quality; namely, it insures that (not only Γ_* but also) $\Gamma_!$ is closed functor, thus inducing a second way of associating an $\mathcal{S}$-enriched category to each $\mathcal{E}$-enriched category

$$\mathcal{E}\text{-cat} \xrightarrow{[\ ]} \mathcal{S}\text{-cat}.$$

For example, $\mathcal{E}$ itself as an $\mathcal{E}$-enriched category gives rise to a homotopy category in which

$$[\mathcal{E}](X,Y) = \Gamma_!(Y^X) .$$

This product-preserving property of $\Gamma_!$ is well-known to be false in the group case, where $\Gamma_!(G\times G) = n$, where $n = \#G$, whereas $\Gamma_!(G) = 1$. Again, it *can* hold for some (extremely special) spaces: For a topos $\mathcal{E}$ localic over $\mathcal{S}$, $\Gamma_!$ is left exact if only it preserves products, and hence there is again a canonically defined point, at the opposite extreme; for example, the Zariski spectrum of an integral domain admits a product preserving $\Gamma_!$. If $\mathcal{S}$ is an "exponential variety" in $\mathcal{E}$, then $\Gamma_!(\Gamma^*(A)\times Y) \xrightarrow{\sim} A\times\Gamma_!(Y)$ which is, however, only a fragment of our axiom 1. It is at this point that the constructions of generalized spaces which "cover" a given topos insofar as the "internal logic" is concerned, fail to preserve the structure of a "topos of spaces". (For covering as an "exponential variety" would preserve our axiom 2).

AXIOM 2. $\Gamma_!(\Omega) = 1$, where Ω is the truth-value object in the topos $\mathcal{E}$ of spaces.

Since Ω has the structure of a monoid with zero, in the presence of axiom 1 its being connected (axiom 2) implies its being contractible in that

$$[\mathcal{E}](X,\Omega) = 1$$

for all X in $\mathcal{E}$, and hence that $X \to \Omega^X$ is a natural embedding of every space into a contractible space; moreover, any retract, such as Ω_j for a topology j, (for example the Boolean algebra $\Omega_{\neg\neg}$) is also contractible. Of course axiom 2 cannot be true of a Boolean topos since $\Gamma_!$ preserves any sum such as 1+1.

PROPOSITION. *Axioms 1 and 2 cannot both be true for a localic topos* E *over sets* S.

Proof. Axiom 1 implies that $\Gamma_!$ preserves pullbacks in the localic case. In any case there is pullback diagram

$$\begin{array}{ccc} 2 & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ \Omega & \longrightarrow & \Omega \end{array}$$

in E where 2 = 1+1. Thus applying $\Gamma_!$ we get an impossible pullback diagram

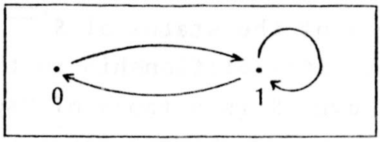
$$\begin{array}{ccc} 2 & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & 1 \end{array}$$

in S. QED.

The above axioms (incomplete though they may be) enable us to make some rather sharp distinctions. For example, there are (at least) two distinct toposes commonly referred to as "the category of directed graphs" and even commonly considered to be more or less of the same value since, for example, the notion of "free category" generated by either kind of graph makes sense. The two are

$$S^{\mathbb{\Delta}_1^{op}} \qquad S^{\cdot \rightrightarrows \cdot}$$

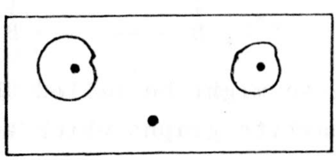
where $\mathbb{\Delta}_1$ is the three-element monoid of all orderpreserving endomaps of the two-element linearly ordered set [1]; splitting the idempotents shows that Γ_* is essentially representable and hence $\Gamma^!$, the notion of codiscrete graph, exists for $S^{\mathbb{\Delta}_1^{op}}$, though not for $S^{\cdot \rightrightarrows \cdot}$. However, the one-dimensional simplicial sets $S^{\Delta_1^{op}}$ and the "irreflexive" graphs $S^{\cdot\rightrightarrows\cdot}$ differ already in regard to axiom 1: the functor $\Gamma_!$ is in either case just the coequalizer of the structural maps, but, as is well-konown, reflexive coequalizers preserve products, whereas irreflexive coequalizers do not. The subobject classifier for $S^{\Delta_1^{op}}$ has five elements



and is obviously connected. A similar statement is true for $S^{\cdot\rightrightarrows\cdot}$ but the foregoing remarks are sufficient to show the following.

PROPOSITION. *The topos* $S^{\Delta_1^{op}}$ *satisfies the axioms* 0, 1, 2 *for a "topos of spaces", whereas the topos* $S^{\cdot\rightrightarrows\cdot}$ *of diagram schemes does not satisfy* 0 *or* 1.

In fact, at least two arguments can be given to show that $S^{\cdot\rightrightarrows\cdot}$ definitely belongs to the other variety of toposes, namely that it is in fact a simple example of a generalized space. For one thing, the category $S^{\cdot\rightrightarrows\cdot}$ of irreflexive graphs is an étendue; in fact, it is locally localic in an illuminating manner: Consider the space



which has three points and five open sets. A sheaf on this space consists of a set E of global sections, two sets V_o, V_1 of sections over the two open points, and two restriction maps $E \to V_o$, $E \to V_1$.

If we consider the two-element group acting on the space by interchanging the two open points, we can take the "quotient" (descent) in the 2-category of toposes by the equivalence relation associated to this action; this has the effect of forcing $V_0 = V_1$, but allowing the two restrictions $E \rightrightarrows V$ to remain different. Conversely, there is an object A in $\mathcal{S}^{\cdot\rightrightarrows\cdot}$ such that $\mathcal{S}^{\cdot\rightrightarrows\cdot}/A$ is (the topos of sheaves on) the three point space, showing explicitly the local homeomorphism of the two toposes.

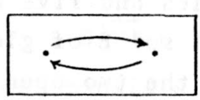
Another aspect of the status of $\mathcal{S}^{\cdot\rightrightarrows\cdot}$ as a generalized space is revealed by its relationship to the category of spaces $\mathcal{S}^{\Delta_1^{op}}$. If $\mathcal{E}$ over $\mathcal{S}$ is a topos of "spaces", then each object B of $\mathcal{E}$ should be capable of serving as a domain of variation in its own right; in particular it should have sense to speak of abstract sets varying over B, giving rise to a topos $\mathcal{S}(B)$ (usually a subcategory of $\mathcal{E}/B$), which should be an example of a generalized space ("should be" since we don't yet have axioms strong enough to capture the special nature of generalized spaces, yet general enough to include the classical petit étale example!). In case $B \in \mathcal{E} = \mathcal{S}^{\Delta_1^{op}}$ is a graph, one reasonable definition of

$$\mathcal{E}/B \supset \mathcal{S}(B)$$

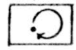
is simply to take all $E \longrightarrow B$ which have discrete fibers in the sense that

$$\begin{array}{ccc} \Gamma^*\Gamma_* E & \longrightarrow & E \\ \downarrow & & \downarrow \\ \Gamma^*\Gamma_* B & \longrightarrow & B \end{array}$$

is a pullback. These might be called "B-partite graphs" generalizing the bipartite graphs which arise as the special case where

B = 

The toposes $\mathcal{S}(B)$ are *all* étendues, and behave with excellen functorial comportment under morphisms $B \longrightarrow B'$ in the topos of spaces $\mathcal{E} = \mathcal{S}^{\Delta_1^{op}}$; thus they seem to embody well one idea of the generalized spaces associated to objects of $\mathbf{E}$.

PROPOSITION. *If* L =  *is the object of* $\mathbf{S}^{\Delta^{op}}$ *obtained by identifying the two points of the representable object* $\Delta[1]$, *then irreflexive graphs may be identified with* L-*partite graphs*:

$$\mathbf{S}^{\bullet \rightrightarrows \bullet} \simeq \mathbf{S}(L).$$

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