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# ON THE RELATION BETWEEN CONNECTION AND SPRAYS

by

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A well-known (and simple) result of linear algebra asserts that any bilinear symmetric form $\mathbb{R}^n \times \mathbb{R}^n \overset{\delta}{\to} \mathbb{R}^n$ is completely determined by its values on the diagonal: letting $\sigma(t) = \delta(t,t)$ be the quadratic form associated to $\delta$, we may recover the original form $\delta$ simply as the polar form of $\sigma$, i.e. $\delta(t_1,t_2) = \frac{1}{2}(\sigma(t_1+t_2)-\sigma(t_1)-\sigma(t_2))$.

This result has been "lifted" to differential geometry in Ambrose, Palais, Singer (1960), by showing the existence of a natural bijection between symmetric affine connections (which play the role of bilinear symmetric forms) and sprays (which play the role of quadratic forms) on a smooth finite-dimensional manifold. To state this result in a precise way, let us recall some of the standard terminology and notation.

Let $T(M) \xrightarrow{\pi_M} M$ be the *tangent bundle* of M. It has a canonical vector bundle structure over M with fiber $\mathbb{R}^m$, where $m = \dim(M)$. The *second tangent space* $T^2(M) = T(T(M))$ has two vector bundle structures over $T(M)$, given by $T(\pi_M)$ and $\pi_{T(M)}$ respectively. Similarly, the fibered product $T(M) \underset{M}{\times} T(M)$ again has two different vector bundle structures over $T(M)$, given by the canonical projections $p_1$ and $p_2$.

$$\begin{array}{ccc} T(M)\underset{M}{\times}T(M) & \xrightarrow{p_2} & T(M) \\ \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle \pi_M} \\ T(M) & \xrightarrow{\pi_M} & M \end{array}$$

An *affine connection* on M is a map

$$T(M) \underset{M}{\times} T(M) \xrightarrow{\nabla} T^2(M)$$

of vector bundles over T(M) with respect to *both* structures (i.e. $\nabla$ is a smooth map such that $\pi_{TM} \circ \nabla = p_1$, $T(\pi_M) \circ \nabla = p_2$, and $\nabla$ is linear with respect to both structures). Writing $\Sigma : T^2(M) \to T^2(M)$ for the symmetry (or twist) map, we call $\nabla$ *symmetric* if for all $(t_1, t_2) \in T(M) \underset{M}{\times} T(M)$, $\nabla(t_1, t_2) = \Sigma(\nabla(t_2, t_1))$.

The importance of connections in differential geometry, mechanics, relativity theory, etc. is well-established, and it is unnecessary to elaborate this here. Is should be pointed out, however, that for many purposes (such as covariant differentiation, geodesics,...) the simpler notion of a *spray* (french: gerbe) suffices. Furthermore, sprays appear quite naturally in these domains, for instance in the theory of dinamical systems in situations where kinetic energy and field of forces are homogeneous of the same degree (cf. Godbillon (1969)). A *spray* on a manifold M is a smooth map

$$T(M) \xrightarrow{\sigma} T^2(M)$$

which is symmetric section of $T(\pi_M)$ (i.e. $\Sigma \circ \sigma = \sigma$, and $T(\pi_M) \circ \sigma = \mathrm{id}$, or equivalently $\pi_{T(M)} \circ \sigma = \mathrm{id}$), which is homogeneous in the sense that

$$\sigma(\alpha \cdot t) = \alpha \cdot (\alpha \Theta \sigma(t)) \qquad (\alpha \in \mathbb{R},\ t \in T(M))$$

where on the right-hand side, $\cdot$ refers to the $\pi_{T(M)}$-vector bundle structure and $\Theta$ to the $T(\pi_M)$-one (cf. e.g. Lang(1972)).

Notice that a spray is a particular case os a second-order differential equation (i.e. of a symmetric vector field $T(M) \to T(T(M))$, which suggests the interest of this notion in mechanics.

We can now formulate the result of Ambrose, Palais, and Singer.

**THEOREM.** (Ambrose, Palais, Singer (1960)). *Let M be a smooth finite dimensional manifold. There is a natural bijection between symmetric affine connections*

$$T(M) \underset{M}{\times} T(M) \xrightarrow{\nabla} T^2(M)$$

*and sprays* $T(M) \xrightarrow{\sigma} T^2(M)$ *given by* $\sigma(t) = \nabla(t,t)$.

The aim of this paper is to generalize this theorem beyond the case of smooth finite-dimensional manifolds, for example to manifolds with singularities (such as arbitrary fibered products of manifolds), and spaces of smooth functions. Let us explicitly formulate the latter case here. Let M and N be smooth manifolds, and let $C^\infty(M,N)$ be the space of smooth functions from M to N. The natural way of defining the tangent space is as $TC^\infty(M,N) =_{def} C^\infty(M,TN)$, with the canonical projection $\pi_{C^\infty(M,N)} = (\pi_N)_*: C^\infty(M,TN) \to C^\infty(M,N)$ induced by $\pi_N$ via composition. This gives a "vector bundle" over $C^\infty(M,N)$. One may then define (symmetric) affine connections and sprays on $C^\infty(M,N)$ as smooth maps (in the appropriate sense to be explained in section 3): $C^\infty(M,TN) \underset{C^\infty(M,N)}{\times} C^\infty(M,YN) \xrightarrow{\nabla} C^\infty(M,T^2N)$, respectively $C^\infty(M,TN) \to C^\infty(M,T^2N)$. In section 3, we will explain this is detail, and prove

**THEOREM.** ("Ambrose-Palais-Singer for function spaces") *Let M and N be smooth manifolds. There is a (natural) bijection between symmetric connections* $\nabla$ *on* $C^\infty(M,n)$ *and sprays* $\sigma$ *on* $C^\infty(M,N)$, *given by* $\sigma(f) = \nabla(f,f)$ *for* $f \in C^\infty(M,TN)$.

As will be pointed out, this theorem is more general than the preceding one, not every (symmetric) affine conection on $C^\infty(M,N)$ comes from one on N via composition.

The original proof of Ambrose, Palais, Singer (1960) proceeds by locally integration the spray (considered as a second-order differential equation), and the proof cannot simply be generalized to the case of spaces $C^\infty(M,N)$ of smooth functions. Instead, we will proceed by adapting the functorial approach to algebraic schemes (as in SGA 3, Demazure (1970), Demazure & Gabriel (1970)) to differential geometry, much in the spirit of Ehresmann and Weil. We hasten to point out, however, that the reader is not assumed to be familiar with the theory of schemes (as developed in the references just mentioned). But for those readers to whom this material is known, we mention here that our method also gives the following result.

**THEOREM.** ("Ambrose-Palais-Singer in algebraic geometry") *Let* X *be a scheme (over a field* k*). There is a natural bijection between symmetric affine connections* $T(X) \underset{X}{\times} T(X) \xrightarrow{\nabla} T^2(X) = T(T(X))$ *and sprays* $T(X) \xrightarrow{\sigma} T^2(X)$, *given by* $\sigma = \nabla \circ \delta$ *where* $T(X) \xrightarrow{\delta} T(X) \underset{X}{\times} T(X)$ *is the diagonal.*

Furthermore, our method yields a corresponding result for schemes over an arbitrary base scheme, as well as one for spaces Hom(X,Y) of morphisms from one scheme to another.

## §1. Microlinear spaces.

As said in the introduction, algebraic schemes over a field k (or over a fixed base scheme S) can be considered as set-valued functors on a category of k-algebras. We will here follow a similar road, but starting from a more general setting so as to include both algebraic schemes and $C^\infty$-manifolds.

**1.1 /A-functors.** Let k be a field, and let /A be a subcategory of the category of k-algebras. For $A,B \in \mathbb{A}$, $\mathrm{Hom}_{\mathbb{A}}(A,B)$ denotes the set of morphisms in $\mathbb{A}$, or $\mathbb{A}$-*homomorphisms*, from A to B. $\mathbb{A}$ need not be a full subcategory, i.e. $\mathrm{Hom}_{\mathbb{A}}(A,B)$ may be a proper subset of the set $\mathrm{Hom}_k(A,B)$ of all k-algebra maps $A \to B$. We will assume that $\mathbb{A}$ has the following propertie:

(A1) $\mathbb{A}$ has binary coproducts.

For $A,B \in \mathbb{A}$, we write $A \otimes_{\mathbb{A}} B$ for their coproduct in the category $\mathbb{A}$. This need *not* be the same as the coproduct $A \otimes_k B$ in the category of all k-algebras. We will impose some more conditions on $\mathbb{A}$ as we proceed. But for concreteness, the reader may just think of $\mathbb{A}$ as being the category of all finitely generated k-algebras. (Later on, we will consider examples where $\mathbb{A}$ consists of rings of smooth function on manifolds, see section 3).

An $\mathbb{A}$-*functor* is a set-valued functor $\mathbb{A} \overset{X}{\to}$ Sets. A *morphism of* $\mathbb{A}$-*functors* is of course a natural transformation. So if $X,Y:\mathbb{A} \to$ Sets are two $\mathbb{A}$-functors, a morphism $\tau: X \to Y$ consists of a collection of functions $\tau_A : X(A) \to Y(A)$, A running over the objects of $\mathbb{A}$, such that for every $\mathbb{A}$-homomorphism $A \overset{h}{\to} B$ we have $\tau_B \circ X(h) = Y(h) \circ \tau_A$. If X and Y are $\mathbb{A}$-functors, we write $\mathbb{A}(X,Y)$ for the set of morphisms from X to Y.

An important class of $\mathbb{A}$-functors are the *representable* ones, i.e. $\mathbb{A}$-functors of the form

$$\bar{A} = \mathrm{Hom}_{\mathbb{A}}(A,-) : \mathbb{A} \to \text{Sets}.$$

Note that $\mathbb{A}(\bar{A},X) \simeq X(A)$, $\mathbb{A}(\bar{A},\bar{B}) \simeq \mathrm{Hom}_{\mathbb{A}}(B,A)$.

Another $\mathbb{A}$-functor which plays a central role is the underlying set functor, which we denote by R,

$$R:\mathbb{A} \to \text{Sets}, \quad R(A) = |A| = \text{the underlying set of } A.$$

Since each $A \in \mathbb{A}$ is a k-algebra, it is clear that R is a k-algebra object in the category of $\mathbb{A}$-functors, with "pointwise" operation (i.e. for example addition $+: R\times R \to R$ is the morphism given by taking $+_A : R(A)\times R(A) \to R(A)$ to be the addition of A. $R\times R$ is the product, see 1.3(a) below). R need not be representable, but if $\mathbb{A}$ has a free object on one generator, this object will represent R. So $R = \overline{k[x]}$ in the case where $\mathbb{A}$ is the category of initely generated k-algebras. ($\mathbb{A}$ may have a free object on one generator different from $k[x]$, as in the case of rings of smooth functions considered in section 3).

For the particular case where $\mathbb{A}$ consists of all finitely generated k-algebras, every scheme $(X, \mathcal{O}_X)$ of finite type corresponds to an $\mathbb{A}$-functor $s(X): \mathbb{A} \to Sets$ defined by

$$s(X)(A) = Hom_k(\Gamma\mathcal{O}_X, A) = \text{morphisms } Spec(A) \to X.$$

So $s(Spec\ A) = \bar{A}$. This gives a full and faithful embedding of the schemes (of finite type) into the category of $\mathbb{A}$-functors. (The finite type assumption can be eliminated by choosing for $\mathbb{A}$ a suitable category of "models"). All this is extensively discussed in e.g. Demazure & Gabriel (1970).

**1.2 Weil algebras.** These are k-algebras of the form

$$W = k[x_1, \ldots, x_n]/I$$

where $I \supset M^{p+1}$ for some natural number p, and $M = (x_1, \ldots, x_n)$ is the maximal ideal. We will assume

> (A2) $\mathbb{A}$ contains all the Weil algebras, and all the k-algebra maps of Weil algebras; and for each Weil algebra W and each $A \in \mathbb{A}$, $A \underset{\mathbb{A}}{\otimes} W \simeq A \underset{k}{\otimes} W$.

So coproducts in $\mathbb{A}$ are the same as coproducts of k-algebras if one of the factors is a Weil algebra. Weil algebras of the form $k[x_1, \ldots x_n]/M^{p+1}$ will occur very often in this paper, and it is useful to introduce some notation: we will write

$$J_p(n) = k[x_1, \ldots, x_n]/M^{p+1},$$

and delete n when n = 1, i.e. $J_p = k[x]/(x^{p+1})$.

## 1.3 Some standard constructions of A-functors.

(a) If X, Y are $\mathbb{A}$-functors, the *product* X×Y is simply the $\mathbb{A}$-functor defined by taking "pointwise" the product of Sets:

$$X\times Y(A) = X(A)\times Y(A).$$

Similarly, if $X \xrightarrow{\sigma} Z \xleftarrow{\tau} Y$ are morphisms of $\mathbb{A}$-functors, the *fibered product* $X \underset{Z}{\times} Y$ is the $\mathbb{A}$-functor given by

$$X \underset{Z}{\times} Y(A) = X(A) \underset{Z(A)}{\times} Y(A) = \{(x,y) \in X(A)\times Y(A) \mid \sigma_A(x) = \tau_A(y)\}$$

More generally, the inverse limit $\varprojlim X_i$ of any system $(X_i)_i$ of $\mathbb{A}$-functors can be constructed in this way:
$(\varprojlim X_i)(A) = \varprojlim X_i(A)$.

(b) If X and Y are $\mathbb{A}$-functors, the *function space* $Y^X$ of $\mathbb{A}$-morphisms $X \to Y$ is the $\mathbb{A}$-functor defined by

$$Y^X(A) = \mathbb{A}(\bar{A}\times X, Y).$$

For any $\mathbb{A}$-functor Z, there is a canonical isomorphism

$$(Y^X)^Z \simeq Y^{(X\times Z)}.$$

In particular, evaluating this isomorphism at $A = k \in \mathbb{A}$ (by (A2)) we get

$$\mathbb{A}(Z, Y^X) \simeq \mathbb{A}(Z\times X, Y). \qquad (1)$$

(c) Let X be an $\mathbb{A}$-functor. The *tangent bundle* of X is defined to be the function space $X^{\bar{J}_1}$,

$$T(X) = X^{\bar{J}_1} = X^{\overline{k[\varepsilon]}} \qquad (2)$$

where $k[\varepsilon]$ is the ring of dual numbers. So for $A \in \mathbb{A}$, $T(X)(A) = X(A[\varepsilon])$ where $A[\varepsilon] = A \otimes_k k[\varepsilon] \in \mathbb{A}$ by (A2). There is a canonical *base-point projection*

$$T(X) \xrightarrow{\pi_X} X$$

induced by the A-algebra map $A[\varepsilon] \xrightarrow{0_A} A$, $0_A(\varepsilon) = 0$, i.e. $(\pi_X)_A = X(0_A): T(X)(A) = X(A[\varepsilon]) \to X(A)$, for $A \in \mathbb{A}$; and a zero-section

$$X \xrightarrow{0_X} T(X)$$

induced by the A-algebra map $A \xhookrightarrow{i_A} A[\varepsilon]$, i.e.

$$(0_X)_A = X(i_A): X(A) \to X(A[\varepsilon]).$$

This construction is obviously functorial: a morphism $X \overset{\sigma}{\to} Y$ of $\mathbb{A}$-functors induces a morphism

$$T(X) \xrightarrow{d\sigma} T(Y), \quad (d\sigma)_A = \sigma_{A[\varepsilon]}$$

with $\pi_Y \circ d\sigma = \sigma \circ \pi_X$, and $d\sigma \circ 0_X = 0_Y \circ \sigma$. Furthermore, T commutes with fibered products:

$$T(X \underset{Y}{\times} Z) \simeq T(X) \underset{T(Z)}{\times} T(Y)$$

(and more generally with all inverse limits).

**(d)** As a generalization of the tangent bundle of an $\mathbb{A}$-functor, we can define the *jet bundle* (a la Ehresmann) of an $\mathbb{A}$-functor, simply by replacing $J_1$ by $J_p(n)$. For an $\mathbb{A}$-functor X, we let

$$J_p^n(X) = X^{\overline{J_p(n)}},$$

so $J_p^n(X)(A) = X(A \underset{k}{\otimes} J_p(n))$, and as in (c) before there is a canonical base point projection $J_p^n(X) \xrightarrow{\pi_X} X$ induced by $A \underset{k}{\otimes} J_p(n) \xrightarrow{0_A} A$, and a zero-section $X \xrightarrow{0_X} J_p^n(X)$ induced by $A \xrightarrow{i_A} A \underset{k}{\otimes} J_p(n)$, and all this is again functorial in X.

More generally, for any Weil algebra W, the canonical k-algebra projection $W \to k$ induces a morphism

$$X^{\bar{W}} \xrightarrow{\pi_X} X^{\bar{k}} = X$$

for any $\mathbb{A}$-functor X. called the $\bar{W}$-*prolongation* of X. (For the case of manifolds, these prolongations were defined in Weil (1953), see 3.2 below).

**1.4 Inverse limits in $\mathbb{A}$.** Finite inverse limits need not exist in $\mathbb{A}$. However, if $(A_i)_i$ is a finite diagram of algebras in $\mathbb{A}$, we can compute the inverse limit in the category of k-algebras (this is just the inverse limit of the underling sets), $A = \varprojlim A_i$. We will assume that *if* A is an object of $\mathbb{A}$, then it is the inverse limit of the $A_i$ in the category $\mathbb{A}$. In technical terms (cf. MacLane (1971)),

(A3) The inclusion functor $\mathbb{A} \hookrightarrow$ k-algebras reflects finite inverse limits.

**1.5 Microlinear spaces.** The category of $\mathbb{A}$-functors is really too large to do geometry in. Accordingly, we will now restrict our attention to the class of microlinear spaces. Intuitively, these are $\mathbb{A}$-functors X which behave as if they had local coordinates, at least with respect to maps of the form $\bar{W} \to X$, where W is a Weil algebra:

**DEFINITION.** A *microlinear space* (or more explicity, microlinear $\mathbb{A}$-space) is an $\mathbb{A}$-functor X such that for any finite inverse limit diagram of Weil algebras $\{W \xrightarrow{\pi_i} X_i\}_{i\in I}$, $W = \varprojlim_I W_i$, the induced diagram of function spaces $\{X^{\bar{W}} \to X^{\bar{W}_i}\}_{i\in I}$ is an inverse limit of $\mathbb{A}$-functors: $X^{\bar{W}} = \varprojlim_I X^{\bar{W}_i}$.

For example, the diagram

$$\begin{array}{ccc} J_1(2) & \xrightarrow{p_1} & J_1 \\ \downarrow{\scriptstyle p_2} & & \downarrow \\ J_1 & \longrightarrow & k \end{array} \qquad (1)$$

is an inverse limit (a fibered product) of Weil algebras (where $J_1 = k[\varepsilon]$, $J_1[2] = k[\varepsilon_1,\varepsilon_2]$, with $\varepsilon^2 = \varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_1\varepsilon_2 = 0$, and $p_1(\varepsilon_1) = \varepsilon = p_2(\varepsilon_2)$, $p_1(\varepsilon_2) = 0 = p_2(\varepsilon_1)$). So if X is microlinear, we have a fibered product of function spaces corresponding to (1), i.e.

$$T(X) \underset{X}{\times} T(X) = X^{\overline{J_1(2)}}, \qquad (2)$$

which is essentially what is needed to define addition of tangent vectors as a morphism $T(X) \underset{X}{\times} T(X) \overset{+}{\to} T(X)$. (2) is a special case of "condition (E)" considered in Demazure (1970); see also Mumford (1966), p.26. Any microlinear space satisfies condition (E), but microlinearity is strictly stronger. Some more inverse limits of Weil algebras like (1) above are listed in the next section, see 2.1.

There is a plentiful supply of microlinear spaces:

**1.6. PROPOSITION.** (a) *Let* $(X_j)_{j\in J}$ *be an arbitrary diagram of microlinear spaces. Then* $\varprojlim_J X_j$ *is again microlinear.*
(b) *If* X *is a microlinear space, then so is* $X^Y$ *for any* $\mathbb{A}$-*functor* Y.
(c) *The underlying set functor* R *is a microlinear space.*
(d) *For every* $B \in \mathbb{A}$, *the representable* $\mathbb{A}$-*functor* $\bar{B}$ *is a microlinear space.*

*Proof.* (a) and (b) are simple general properties of inverse limits. For (c), let $\{W \xrightarrow{\pi_i} W_i\}$ be a finite diagram of Weil algebras, $W = \varprojlim W_i$. Then for each $A \in \mathbb{A}$, $\{A \otimes_k W \to A \otimes_k W_i\}_{i\in I}$ is also an inverse limit of k-algebras, and hence an inverse limit in $\mathbb{A}$, by (A2) and (A3). But $R^{\bar{W}}(A) = A \otimes_k W = A \otimes_{\mathbb{A}} W$, and similarly for $R^{W_i}(A)$, so (c) follows. The proof of (d) makes a similar use of (A2) and (A3), since for any Weil algebra W, $\bar{B}^{\bar{W}} \cong \mathrm{Hom}_{\mathbb{A}}(B, W \otimes -)$. ▲

It is easy to see that in the case of $\mathbb{A}$ = all finitely generated k-algebras, the $\mathbb{A}$-functor s(X) corresponding to a scheme X is a microlinear space (cf. the example al the end of 1.1).

**1.7 R-module structure.** (a) Let $E \xrightarrow{p} X$ be a morphism of microlinear spaces. An *R-module structure on* $E \xrightarrow{p} X$ is given by morphisms $E \times_X E \xrightarrow{+} E$, $R\times E \xrightarrow{\cdot} E$, and $X \xrightarrow{0} E$ *over* X which make the appropriate diagrams (expressing that these maps constitute an R-module structure on the fibres) commute. Equivalently, for each algebra $A \in \mathbb{A}$ we have maps $+_A: E(A) \times_{X(A)} E(A) \to E(A)$, $\cdot_A: A\times E(A) \to E(A)$ and $0_A: X(A) \to E(A)$ over X(A) which give the fibers of $p_A: E(A) \to X(A)$ an A-module structure, and this structure is natural in A (e.g. for · this means that for any $\mathbb{A}$-homomorphism $A \xrightarrow{f} B$ the square

$$\begin{array}{ccc} A\times E(A) & \xrightarrow{\cdot A} & E(A) \\ \downarrow{\scriptstyle f\times E(f)} & & \downarrow{\scriptstyle E(f)} \\ B\times E(B) & \xrightarrow{\cdot B} & E(B) \end{array}$$

commutes, etc.).

If $E \xrightarrow{p} X$ and $F \xrightarrow{q} X$ both have such an R-module structure, a map $F \xrightarrow{\phi} E$ over X (i.e. $p\phi = q$) is R-*linear* if for each $A \in \mathbb{A}$, $\phi_A : F(A) \to E(A)$ induces A-linear maps on the fibres: $q_A^{-1}(x) \xrightarrow{\phi A, x} p_A^{-1}(x)$, for $x \in X(A)$.

(b) If $E \xrightarrow{p} X$ has an R-module structure, and $Y \xrightarrow{f} X$ is a morphism of microlinear spaces, we obtain an obvious R-module structure on $E \times_X Y \to Y$, called the *induced* R-*module structure*. If $E \xrightarrow{p} X$ and $F \xrightarrow{q} Y$ both have an R-module structure, an R-*linear* map from p to q is a pair $(\phi, f)$ with $F \xrightarrow{\phi} E$, $Y \xrightarrow{f} X$, $p\phi = fq$, such that the map $F \xrightarrow{(\phi,q)} E \times_X Y$ is an R-linear map from $F \xrightarrow{q} Y$ to $E \times_X Y \to Y$ equipped with the induced structure.

(c) **PROPOSITION.** *If* X *is microlinear, then* $TX \xrightarrow{\pi X} X$ *has a canonical* R-*module structure, which is natural in* X, *in the sense that for* $Y \xrightarrow{f} X$, *the pair* $(Y \xrightarrow{f} X,\ TY \xrightarrow{df} TX)$ *is an* R-*linear map from* $\pi_Y$ *to* $\pi_X$.

*Proof.* This result can already be found in Demazure (1970), with "condition (E)" instead of microlinearity, but for the convenience of the reader, we outline the proof. The structure is defined as follows. The zero section $X \xrightarrow{0} TX$ has already been described: $X(A) \xrightarrow{0_A} X(A[\varepsilon]) = T(X)(A)$ is simply $X(i_A)$ with $A \xrightarrow{i_A} A[\varepsilon]$ being the canonical A-algebra map. Multiplication $\cdot_A : A\times T(X)(A) \to T(X)(A)$ is the function

$$A\times X(A[\varepsilon]) \to X(A[\varepsilon]), \quad (a,x) \mapsto X(\mu_a)(x),$$

where $\mu_a : A[\varepsilon] \to A[\varepsilon]$ is the A-algebra map given by $\mu_a(\varepsilon) = a\varepsilon$.

It is for the definition of $+_A$ that we use microlinearity: if X is microlinar, the fibered product (1) in 1.5 gives for each $A \in \mathbb{A}$ a fibered product of Sets

$$\begin{array}{ccc} X(A \otimes J_1(2)) & \xrightarrow{X(A \otimes p_1)} & X(A \otimes J_1) \\ \Big\downarrow {\scriptstyle X(A \otimes p_2)} & & \Big\downarrow {\scriptstyle (\pi_X)_A} \\ X(A \otimes J_1) & \xrightarrow{(\pi_X)_A} & X(A) \end{array}$$

let $\delta : J_1(2) = k[\varepsilon_1, \varepsilon_2] \to k[\varepsilon] = J_1$ be the k-algebra-map given by $\delta(\varepsilon_i) = \varepsilon$. Then $+_A : X(A[\varepsilon]) \underset{X(A)}{\times} X(A[\varepsilon]) \to X(A[\varepsilon])$ is defined by

$$+_A(t_1, t_2) = X(A \otimes \delta)(\langle t_1, t_2 \rangle),$$

where $\langle t_1, t_2 \rangle \in X(A \otimes J_1(2))$ is the unique element with $X(A \otimes p_i)(\langle t_1, t_2 \rangle) = t_i$. It is routine to check that this indeed defines an R-module structure which is natural in X, as stated in the proposition (the fact that (f,df) is R-linear is an easy consequence of naturality).

(d) **REMARK.** If $E \overset{p}{\to} X$ is equipped with an R-module structure, then we obtain an R-module structure on the morphism $E^Y \xrightarrow{p^Y} X^Y$ for each $\mathbb{A}$-functor Y in the obvious way. (For example, addition $+ : E^Y \times_{X^Y} E^Y \to E^Y$ is the morphism $E^Y \times_{X^Y} E^Y \cong (E \times_X E)^Y \xrightarrow{(+)^Y} E^Y$, where + is the addition of the given R-module structure on p.

**1.8 Vector bundles. (a)** Let $E \overset{p}{\to} X$ be a morphism of microlinear spaces. The *fiber tangent bundle* (or "vertical tangent bundle") is defined as the fibered product $T_X(E) = X \times_{T(X)} T(E)$

$$\begin{array}{ccc} T_X(E) & \longrightarrow & T(E) \\ \Big\downarrow {\scriptstyle \pi_p} & & \Big\downarrow {\scriptstyle dp} \\ X & \xrightarrow{0} & T(X) \end{array}$$

So $T_X(E)(A) = \{t \in E(A[\varepsilon]) \mid dp_A(t) = 0\}$ (recall that $dp_A(t) = 0$ can be unwound as $p_{A[\varepsilon]}(t) = X(i_A \circ 0_A)(p_{A[\varepsilon]}(t))$, with $A \xrightarrow{i_A} A[\varepsilon]$ and $A[\varepsilon] \xrightarrow{0_A} A$ as before).

**(b)** If $E \overset{p}{\to} X$ as in (a) is equipped with an R-module structure, this gives an associated R-module structure on $T(E) \xrightarrow{dp} T(X)$ by remark 1.7(d) (since $T(X) = X^{J_1}$), and hence we obtain an induced R-module structure on $T_X(E) \xrightarrow{\pi p} X$ by 1.7 (b). Explicitly, the R-module structure on $T_X(E) \xrightarrow{\pi p} X$ can be described as follows:

**(i)** the zero section $0: T(X) \to T_X(E) \subset T(E)$ is the composite $X \overset{0}{\to} E \xrightarrow{0_E} T(E)$.

**(ii)** the multiplication $R \times T_X(E) \to T_X(E)$ is given by the components

$$\cdot_A : A \times T_X(E)(A) \to T_X(E)(A) \subset E(A[\varepsilon]), \quad (a,t) \mapsto a \cdot_{A[\varepsilon]} t ,$$

where on the right-hand side, $\cdot_{A[\varepsilon]}$ refers to the multiplication of p, $R \times E \to E$, evaluated at $A[\varepsilon] \in \mathbb{A}$.

**(iii)** Similarly, the addition $T_X(E) \times T_X(E) \overset{+}{\to} T_X(E)$ is given by the components

$$T_X(E)(A) \underset{X(A)}{\times} T_X(E)(A) \xrightarrow{+_A} T_X(E)(A), \quad (t_1, t_2) \mapsto t_1 +_{A[\varepsilon]} t_2 ,$$

where again, on the right-hand side $+_{A[\varepsilon]}$ refers to the addition $E \underset{X}{\times} E$ of p evaluated at $A[\varepsilon] \in \mathbb{A}$ (this makes sense since $T_X(E)(A) \underset{X(A)}{\times} T_X(E)(A) \subset (E \underset{X}{\times} E)(A[\varepsilon])$).

**(c) DEFINITION.** Let $E \overset{p}{\to} X$ be a morphism of microlinear spaces equipped with an R-module structure, as above. $E \overset{p}{\to} X$ is called a *vector bundle* if the canonical map $E \underset{X}{\times} E \overset{\alpha}{\to} T_X(E)$ is an isomorphism, where $\alpha$ is defined by components

$$\alpha_A : E(A) \underset{X(A)}{\times} E(A) \to T_X(E)(A), \quad \alpha_A(e_1, e_2) = E(i_A)(e_1) + \varepsilon \cdot E(i_A)(e_2),$$

with + and $\cdot$ referring to the R-module structure on $E \overset{p}{\to} X$, i.e. the $A[\varepsilon]$-module structure on the fibers of $E(A[\varepsilon]) \to X(A|\varepsilon|)$, and $A \xrightarrow{i_A} A[\varepsilon]$ as before. Note that $\alpha$ is bilinear (i.e. each $\alpha_A$ is A-bilinear).

**1.9 LEMMA.** *Let* $E \overset{p}{\to} X$ *be a vector bundle as in* (c) *above.*

(i) *For any morphism of microlinear spaces* $Y \to X$, *the induced structure* (1.7(b)) *on* $E \times_X Y \to Y$ *is again a vector bundle.*

(ii) *For any* $\mathbb{A}$-*functor* Y, *the corresponding structure on* $E^Y \to X^Y$ (1.7(d)) *is again a vector bundle.*

*Proof.* Straightforward verification.

**1.10 PROPOSITION.** *Let* X *be a microlinear space. Then* $T(X) \xrightarrow{\pi_X} X$ *equipped with the canonical* R-*bundle structure of* 1.7(c) *is a vector bundle.*

*Proof.* Write $A \otimes J_1 \otimes J_1 = A[\varepsilon_1, \varepsilon_2]$, and $q_i$ for the $A[\varepsilon_i]$-algebra map $q_i = 0_{A[\varepsilon_i]}: A[\varepsilon_1, \varepsilon_2] \to A[\varepsilon_i]$. So for the case where $E = T(X)$, the fiber tangent bundle of 1.8(a) is the $\mathbb{A}$-functor

$$T_X T(X)(A) = \{s \in X(A[\varepsilon_1,\varepsilon_2]) \mid X(q_2)(s) = 0 \in T(X)(A) = X(A[\varepsilon_2])\}, \quad (1)$$

and the map $\alpha$ has components

$$\alpha_A : X(A[\varepsilon_1]) \times_{X(A)} X(A[\varepsilon_1]) \to X(A[\varepsilon_1, \varepsilon_2])$$

$$\alpha_A(t_1, t_2) = X(i_{A[\varepsilon_1]})(t_1) + \varepsilon_2 \cdot X(i_{A[\varepsilon_1]})(t_2),$$

where $i_{A[\varepsilon_1]}: A[\varepsilon_1] \hookrightarrow A[\varepsilon_1, \varepsilon_2]$, and + and · refer to the $A[\varepsilon_2]$-module structure of $T(X)(A[\varepsilon_2]) = X(A[\varepsilon_1, \varepsilon_2])$. Now given $s \in T_X T(X)(A)$ as in (1) above, we claim that there are unique $t_1, t_2 \in T(X)(A)$ with $(\pi_X)_A(t_1) = (\pi_X)_A(t_2)$ such that $s = \alpha_A(t_1, t_2)$. Clearly for $t_1$ we have to take

$$t_1 = X(q_1)(s) \in X(A[\varepsilon_1]).$$

To define $t_2$, we use microlinearity of X. Observe that

$$k[\varepsilon_1] \xrightarrow{m} k[\varepsilon_1] \otimes k[\varepsilon_2] \underset{r_2}{\overset{r1}{\rightrightarrows}} k[\varepsilon_1] \otimes k[\varepsilon_2] \quad (2)$$

is an inverse limit of Weil algebras, where $m(\varepsilon_1) = \varepsilon_1 \cdot \varepsilon_2$, and

$r_i(\varepsilon_i) = \varepsilon_i$, $r_1(\varepsilon_2) = 0 = r_2(\varepsilon_1)$. Hence

$$X(A[\varepsilon_1]) \xrightarrow{X(A\otimes m)} X(A[\varepsilon_1,\varepsilon_2]) \underset{X(A\otimes r_2)}{\overset{X(A\otimes r_1)}{\rightrightarrows}} X(A[\varepsilon_1,\varepsilon_2])$$

is an equalizer, and since clearly $X(A\otimes r_1)(s-\alpha(t_1,0)) = X(A\otimes r_2)(s-\alpha(t_1,0))$ (where $s-\alpha(t_1,0)$ is given by the $A[\varepsilon_2]$-module structure of $T(X)(A[\varepsilon_1]) = X(A[\varepsilon_1,\varepsilon_2])$), there is a unique $t_2 \in T(X)(A)$ with $X(A\otimes m)(t_2)+\alpha(t_1,0) = s$. But by definition of $\alpha$, we have $\alpha_A(t_1,t_2) = X(A\otimes m)(t_2)+\alpha(t_1,0)$, thus proving the proposition. ▲

Working with vector bundles, rather than just R-module structures, greatly simplifies the computations in the next section, by the following result.

**1.11 PROPOSITION.** *Let* $E \overset{p}{\to} X$ *and* $F \overset{q}{\to} X$ *be vector bundles over* X, *and let* $E \overset{f}{\to} F$ *be a morphism with* $qf = p$. *Then* f *is* R-*linear iff* f *is* R-*homogeneous.*

*Proof.* Suppose f is R-homogeneous, i.e. for each $A \in \mathbb{A}$ we have $f_A(a\cdot e) = a\cdot f_A(e)$ for all $a \in \mathbb{A}$, $e \in E(A)$. To see that f must be additive, take $A \in \mathbb{A}$ and $e_1, e_2 \in E(A)$, and let $u = f_{A\otimes J_1(2)}(\varepsilon_1\cdot E(i)(e_1)+\varepsilon_2\cdot E(i)(e_2))$, $v = \varepsilon_1\cdot f_{A\otimes J_1(2)}(E(i)(e_1))+\varepsilon_2\cdot f_{A\otimes J_1(2)}(E(i)(e_2))$, where $A \overset{i}{\hookrightarrow} A\otimes J_1(2) = A[\varepsilon_1,\varepsilon_2]$. By (1) of 1.5, we have a fibered product

$$(F(A\otimes p_1), F(A\otimes p_2)) : F(A[\varepsilon_1,\varepsilon_2]) \xrightarrow{\sim} F(A[\varepsilon_1]) \underset{F(A)}{\times} F(A[\varepsilon_2]).$$

But by homogeneity, $F(A\otimes p_i)(u) = F(A\otimes p_i)(v)$ for $i = 1,2$, and therefore $u = v$. Thus also $F(\mu)(u) = F(\mu)(v)$ where $\mu : A[\varepsilon_1,\varepsilon_2] \to A[\varepsilon_1,\varepsilon_2]$, $\mu(\varepsilon_1) = \mu(\varepsilon_2) = \varepsilon_2$. But by homogeneity of f, with $\alpha$ as in 1.8(c), $F(\mu)(u) = \alpha(0, F(i_A)f_A(e_1+e_2))$, (where $A \overset{i_A}{\hookrightarrow} A[\varepsilon_1]$) while by definition $F(\mu)(v) = \alpha(0, F(i_A)(f_A(e_1)+f_A(e_2))$. Since $\alpha$ is an isomorphism (1.8(c)), we conclude that $f_A(e_1,e_2) = f_A(e_1)+f_A(e_2)$.

## §2. Connections and sprays on microlinear spaces.

The object of this section is to prove a generalization of the theorem of Ambrose, Palais, and Singer (1960) on the correspondence between affine connections and sprays, which applies to any microlinear space over a given category $\mathbb{A}$ of k-algebras satisfying the axioms (A1)-(A3) of the previous section. Before going into this, however, we list some inverse limits of Weil algebras that we will need in the course of the proof. (Some of these inverse limits also appear in Koch (1983)). At the end of this section, the special case of connections and sprays on algebraic schemes will be spelled out.

**2.1 Some inverse limits of Weil algebras.** Microlinearity will be applied to the following inverse limits of Weil algebras. (i) and (ii) have already occurred in the previous section. The proofs that these diagrams are indeed inverse limits are completely straightforward, and omitted.

(i)
$$\begin{array}{ccc} J_1(2) & \xrightarrow{p_1} & J_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow \\ J_1 & \longrightarrow & k \end{array}$$

If we write $J_1(2) = k[\eta_1,\eta_2]$, $J_1 = k[\varepsilon]$, then $p_i(\eta_i) = \varepsilon$, $p_1(\eta_2) = 0 = p_2(\eta_1)$ (this is diagram (1) in 1.5).

(ii) $J_1 \xrightarrow{m} J_1 \otimes J_1 \overset{r_1}{\underset{r_2}{\rightrightarrows}} J_1 \otimes J_1$

writing $J_1 = k[\varepsilon]$, $J_1 \otimes J_1 = k[\varepsilon_1,\varepsilon_2]$, $m(\varepsilon) = \varepsilon_1 \cdot \varepsilon_2$, $r_i(\varepsilon_i) = \varepsilon_i$, $r_1(\varepsilon_2) = 0 = r_2(\varepsilon_1)$ (this is (2) in the proof of 1.10).

(iii) $J_2 \xrightarrow{s} J_1 \otimes J_1 \overset{\tau}{\underset{id}{\rightrightarrows}} J_1 \otimes J_1$,

writing $J_2 = k[\delta]$, $J_1 \otimes J_1$ as in (ii), $s(\eta) = \varepsilon_1 + \varepsilon_2$ ; $\tau$ is the twist-map $\tau(\varepsilon_1) = \varepsilon_2$, $\tau(\varepsilon_2) = \varepsilon_1$.

(iv) $J_2 \xrightarrow{s} J_1 \otimes J_1 \underset{q_2}{\overset{q_1}{\rightrightarrows}} J_1$.

with s as in (iii), $q_1$ and $q_2$ the projections, $q_i(\varepsilon_i) = \varepsilon$, $q_1(\varepsilon_2) = 0 = q_2(\varepsilon_1)$.

(v) $J_2(2) \xrightarrow{m} J_2 \otimes J_2(2) \underset{m\otimes J_2}{\overset{J_2\otimes m}{\rightrightarrows}} J_2 \otimes J_2(2) \otimes J_2$,

writing $J_2(2) = k[\delta_1,\delta_2]$, $J_2$ as in (iv), $m(\delta_i) = \delta\cdot\delta_i$.

(vi) $J_1 \xrightarrow{m} J_1 \otimes J_1 \underset{m\otimes J_1}{\overset{J_1\otimes m}{\rightrightarrows}} J_1 \otimes J_1 \otimes J_1$,

notation as in (ii).

(vii) $J_1(2) \xrightarrow{m} J_1 \otimes J_1(2) \underset{m\otimes J_1}{\overset{J_1\otimes m}{\rightrightarrows}} J_1 \otimes J_1(2) \otimes J_1$,

$J_1(2)$, $J_1$ as in (i), $m(\eta_i) = \varepsilon\cdot\eta_i$.

(viii) $J_2(2) \xrightarrow{s} J_1(2) \otimes J_1(2) \underset{id}{\overset{\tau}{\rightrightarrows}} J_1(2) \otimes J_1(2)$,

writing $J_2(2)$ as in (v), and $J_1(2) = k[\lambda_1,\lambda_2] \simeq k[\mu_1,\mu_2]$, $J_1(2) \otimes J_1(2) = k[\lambda_1,\lambda_2,\mu_1,\mu_2]$, s is given by $s(\delta_i) = \lambda_i+\mu_i$; $\tau$ is the twist-map.

**2.2 Affine connections.** (a) Let X be a microlinear space. The iterated tangent bundle T(TX) has two vector bundle structures over TX. Namely the usual tangent bundle $T(TX) \xrightarrow{\pi_{TX}} T(X)$ of proposition 1.10, and the vector bundle structure given by 1.9(ii), i.e. obtained from the tangent bundle structure $T(X) \xrightarrow{\pi_X} X$ by taking functions spaces:

$$T(TX) \xrightarrow{T(\pi_X)} T(X) = T(X)^{\bar{J}_1} \xrightarrow{(\pi_X)^{\bar{J}_1}} X^{\bar{J}_1}.$$

(b) $T(X) \underset{X}{\times} T(X)$ also has two vector bundle structures over T(X), both induced from the tangent bundle structure (1.9(i)): $T(X) \underset{X}{\times} T(X) \xrightarrow{\pi_1} T(X)$ can be given the induced structure obtained by taking the fibered product along $T(X) \xrightarrow{\pi_2} X$, and

$T(X) \underset{X}{\times} T(X) \xrightarrow{\pi_2} T(X)$ can be given the induced structure obtained from $T(X) \xrightarrow{\pi_1} X$.

**(c)** An *affine connection on* X is a map

$$T(X) \underset{X}{\times} T(X) \xrightarrow{\nabla} T(T(X))$$

which is a linear (or homogeneous, cf. 1.11) map of vector bundles for *both* structures over T(X):

$$(1)\quad \begin{array}{ccc} T(X)\underset{X}{\times}T(X) & \xrightarrow{\nabla} & T(TX) \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_{TX}} & \\ & T(X) & \end{array} \qquad (2)\quad \begin{array}{ccc} T(X)\underset{X}{\times}T(X) & \xrightarrow{\nabla} & T(TX) \\ & {\scriptstyle \pi_2}\searrow \quad \swarrow{\scriptstyle T\pi_X} & \\ & T(X) & \end{array}$$

**(d)** Let $T(X) \underset{X}{\times} T(X) \xrightarrow{\Sigma} T(X) \underset{X}{\times} T(X)$ be the twist-map, $\Sigma = (\pi_2, \pi_1)$. There is a similar symmetry map on T(TX), which we also call $\Sigma$, $T(TX) \xrightarrow{\Sigma} T(TX)$. Writing $T(TX) = (X^{J_1})^{J_1} = X^{\overline{J_1 \otimes J_1}}$, $\Sigma$ is simply induced by the symmetry map $J_1 \otimes J_1 \xrightarrow{\tau} J_1 \otimes J_1$ (occuring in 2.1(iii)). An affine connection $\nabla$ on X is called *symmetric* (or torsion free) if $\nabla \circ \Sigma = \Sigma \circ \nabla$.

**(e)** Let us rewrite this is terms of $\mathbb{A}$-functors. $T(X) \underset{X}{\times} T(X) = X(A \otimes J_1(2))$ (by microlinearity, cf. 1.5(2)), and $T(TX)(A) = X(A \otimes J_1 \otimes J_1)$. $(\pi_{TX})_A : X(A \otimes J_1 \otimes J_1) \to X(A \otimes J_1)$ is the map $X(A \otimes q_1)$, with $q_1$ as in 2.1(iv); and $(T\pi_X)_A$ is $X(A \otimes J_1 \otimes J_1) \xrightarrow{X(A \otimes q2)} X(A \otimes J_1)$. So a symmetric affine connection $\nabla$ has components

$$\nabla_A : X(A \otimes J_1(2)) \to X(A \otimes J_1 \otimes J_1)$$

inducing A-linear maps on the fibers corresponding to (1) and (2) in (c) above, and satisfying moreover $X(A \otimes \tau) \circ \nabla_A = \nabla_A \circ X(A \otimes \tilde{\tau})$ with $J_1 \otimes J_1 \xrightarrow{\tau} J_1 \otimes J_1$, $J_1(2) \xrightarrow{\tilde{\tau}} J_1(2)$ the obvious twist-maps.

**2.3 Sprays.** Let X be a microlinear space. The second-order tangent space $T_2(X)$ of X is the microlinear space

$$T_2(X) = X^{\bar{J}_2} .$$

The canonical projection $T_2(X) \xrightarrow{p_X} T(X)$ has components $(p_X)_A : T_2(X)(A) = X(A \otimes J_2) \to X(A \otimes J_1)$, $(p_X)_A = X(A \otimes u)$, where $J_2 \xrightarrow{u} J_1$ is the quotient map ($u(\delta) = \varepsilon$ in the notation of 2.1). There is an obvious multiplicative action on the fibers of $\pi_X \circ p_X : T_2(X) \to X$, $R \times T_2(X) \xrightarrow{\cdot} T_2(X)$, with components $A \times X(A \otimes J_2) \xrightarrow{\cdot_A} X(A \otimes J_2)$, $\cdot_A(a,t) = X(\mu_a)(t)$, where (writing $J_2 = k[\delta]$ as in 2.1) $A \otimes J_2 \xrightarrow{\mu_a} A \otimes J_2$ is the A-algebra map given by $\mu_a(\delta) = a \cdot \delta$.

(b) A *spray on* X is a homogeneous section of $p_X$, i.e. a morphism

$$T(X) \xrightarrow{\sigma} T_2(X)$$

with $p_X \circ \sigma = id$, which preserves the multiplicative action (in other words, each component $\sigma_A : X(A \otimes J_1) \to X(A \otimes J_2)$ is A-homogeneous, $\sigma_A(a \cdot t) = a \cdot \sigma_A(t)$).

(c) Since X is microlinear, a spray on X can equivalently be defined as a homogeneous map

$$TX \xrightarrow{\sigma} T(TX)$$

which is symmetric, i.e. $\Sigma \circ \sigma = \sigma$, and commutes with the two projections: $\pi_{TX} \circ \sigma = id = T\pi_X \circ \sigma$ (this way of defining a spray is perhaps slightly more common in the literature). The equivalence follows immediately from the fact that, by 2.1(iii), the following diagram is a fibered product, where $\Delta$ is the diagonal

$$\begin{array}{ccc} T_2(X) & \longrightarrow & T(TX) \\ \downarrow & & \downarrow \Delta \\ T(TX) & \xrightarrow{(\Sigma, id)} & T(TX) \times T(TX) \end{array} .$$

**2.4 THEOREM.** *Let X be a microlinear space. There is a natural one-to-one correspondence between symmetric affine connections $\nabla$ on X and sprays $\sigma$ on X, given by the formula*

$$X(A\otimes s)(\sigma_A(t)) = \nabla_A(t,t),$$

*where* $t \in T(X)(A)$, *and* $J_2 \overset{s}{\to} J_1 \otimes J_1$ *as in* 2.1(iii). ($\sigma$ *is called the geodesic spray associated with* $\nabla$).

*Proof.* We first describe the correspondence.

(a) *From connections to sprays:* Given $\nabla$, we define $\sigma$ as follows. For $t \in T(X)(A) = X(A\otimes J_1)$, we note that by definition (with $q_i$ as in 2.1(iv)) $X(A\otimes q_1)(\nabla_A\langle t,t\rangle) = \nabla_A(\pi_1\langle t,t\rangle) = \nabla_A(t) = \nabla_A(\pi_2\langle t,t\rangle) = X(A\otimes q_2)(\nabla_A\langle t,t\rangle)$. But by 2.1(iv),

$$X(A\otimes J_2) \xrightarrow{X(A\otimes s)} X(A\otimes J_1\otimes J_1) \underset{X(A\otimes q_2)}{\overset{X(A\otimes q_1)}{\rightrightarrows}} X(A\otimes J_1)$$

is an equalizer. So there is a unique $u \in X(A\otimes J_2)$ with $X(A\otimes s)(u) = \nabla_A\langle t,t\rangle$. We define $\sigma_A(t) = u$ for this u. One easily checks that $\sigma$ satisfies the conditions of 2.2(b). (Note that we did not use the symmetry of $\nabla$ to define this spray $\sigma$).

(b) *From sprays to connections:* Here, we use the equalizer 2.1(v) of Weil algebras. Given a spray $\sigma$ and tangent vectors $t_1,t_2 \in T(X)(A)$ with $(\pi_X)_A(t_1) = (\pi_X)_A(t_2)$, consider $\delta_1\cdot X(j_A)(t_1)+\delta_2\cdot X(j_A)(t_2) \in T(X)(A\otimes J_2(2)) = T(X)(A[\delta_1,\delta_2])$, where $J_2(2) = k[\delta_1,\delta_2]$ as in 2.1(v), and $j_A : A \hookrightarrow A\otimes J_2(2)$. Write $v = \sigma_{A\otimes J_2(2)}(\delta_1\cdot X(j_A)(t_1)+\delta_2\cdot X(j_A)(t_2)) \in T_2(X)(A\otimes J_2(2)) = X(A\otimes J_2\otimes J_2(2))$. Then by homogeneity of $\sigma$, $X(A\otimes J_2\otimes m)(v) = X(A\otimes m\otimes J_2)(v) \in X(A\otimes J_2\otimes J_2(2)\otimes J_2)$, so by microlinearity of X and 2.1(v) we obtain a unique $r \in X(A\otimes J_2(2))$ with $X(A\otimes m)(r) = v$. We let $\nabla_A(t_1,t_2) = X(A\otimes q)(r)$ for this r, where $J_2(2) \overset{q}{\to} J_1\otimes J_1$ is the canonical quotient map ($q(\delta_i) = \varepsilon_i$ in the notation of 2.1(v), 2.1(ii)).

We have to check that $\nabla$ is indeed a symmetric affine connection on X. Naturality of $\nabla_A$ in A follows easily from the uniqueness of r as given by microlinearity. $\nabla$ commutes with the projections to T(X): we check the case of diagram (1) in 2.2(c), that of (2) being similar. So we need to check that for $t_1,t_2 \in T(X)(A)$ as above,

$$X(A \otimes q_1)\nabla_A(t_1,t_2) = t_1 \in X(A \otimes J_1) \tag{1}$$

with $q_1$ as in 2.1(iv). But by equalizer 2.1(vi),(1) follows from

$$X(A\otimes m\circ A\otimes q_1)(\nabla_A(t_1,t_2)) = X(A\otimes m)(t_1) \in X(A\otimes J_1\otimes J_1), \tag{2}$$

and (2) is just a consequence of the homogeneity of $\sigma$:

$$X(A\times m\circ q_1)(\nabla_A(t_1,t_2)) = (p_X)_{A[\varepsilon_2]}\sigma_{A[\varepsilon_2]}(\varepsilon_2\cdot(T(X)(i_A(t_1))$$

(by definition of $\nabla$, where $i_A : A \to A[\varepsilon_2]$ and $A\otimes J_1\otimes J_1 = A[\varepsilon_1,\varepsilon_2]$)

$$= \varepsilon_2\cdot T(X)(i_A)(p_X)_A(\sigma_A(t_1))$$

(by naturality of $p_X$ and $\sigma$, and homogeneity of $\sigma$ )

$$= \varepsilon_2\cdot T(X)(i_A(t_1)$$

$$= X(A\otimes m)(t_1).$$

The homogeneity (and hence linearity, by 1.11) of $\nabla$ in each variable, and the symmetry of $\nabla$ as defined from the given $\sigma$ are obvious.

(c) *These operations are inverse to each other:* First, given a spray $\sigma$, (b) and (a) give a new spray $\tilde{\sigma}$ determined by the equality

$$X(A\otimes s)(\tilde{\sigma}_A(t)) = \nabla_A(t,t) \tag{3}$$

(where s an in 2.1(iv)), where $\nabla_A$ in (3) is the connection defined from $\sigma$ as in (b). To show that $\sigma$ itself also satisfies equation (3), it is clearly sufficient to show

$$X(A\otimes \tilde{s})(\sigma_A(t)) = r \in X(A\otimes J_2(2)), \tag{4}$$

where r is as in (b) above for the particular case that $t_1 = t_2 = t$, and $J_2 \overset{\tilde{s}}{\to} J_2(2)$ is given by $\tilde{s}(\delta) = \delta_1+\delta_2$ notation as in 2.1(v)), so $q\circ\tilde{s} = s$ for q as in (b) and s as in 2.1(iii)). By definition of r, we thus have to verify the equation

$$X(A\otimes(m\circ\tilde{s}))(\sigma_A(t)) = \sigma_{A\otimes J_2(2)}(X(A\otimes(m\circ s))(t)) \in X(A\otimes J_2 \otimes J_2(2)). \qquad (5)$$

But (5) is immediate from the homogeneity of $\sigma$.

The other way round is more complicated. Given a symmetric affine connection $\nabla$, (a) and (b) give a new symmetric affine connection $\tilde{\nabla}$ which is by its definition completely determined by the equation

$$X(u\circ j)\tilde{\nabla}_A(t_1,t_2) \qquad (6)$$

$$= X(\bar{q})\big(\nabla_{A[\lambda_1,\lambda_2]}(\lambda_1 T(X)(i)(t_1)+\lambda_2 T(X)(i)(t_2),\lambda_1 T(X)(i)(t_1)+\lambda_2 T(X)(i)(t_2))\big)$$

where $t_1,t_2 \in T(X)(A)$ is a pair of tangent vectors with $(\pi_X)_A(t_1) = (\pi_X)_A(t_2)$, so $\nabla_A(t_1,t_2) \in X(A\otimes J_1 \otimes J_1) = X(A[\varepsilon_1,\varepsilon_2]$, j is the inclusion $A\otimes J_1\otimes J_1 \hookrightarrow A\otimes J_1\otimes J_1\otimes J_2(2)$, i.e. $A[\varepsilon_1,\varepsilon_2] \xrightarrow{j} A[\lambda_1,\lambda_2,\varepsilon_1,\varepsilon_2]$ (where $\lambda_1^{\alpha_1}\cdot\lambda_2^{\alpha_2} = \varepsilon_i^2 = 0$ for $i = 1,2$, $\alpha_1+\alpha_2 = 3$), and u is the A-algebra map $A[\lambda_1,\lambda_2,\varepsilon_1,\varepsilon_2] \xrightarrow{u} A[\bar{\lambda}_1,\bar{\lambda}_2,\bar{\varepsilon}_1,\bar{\varepsilon}_2]$ given by $u(\lambda_i) = \lambda_i$, $u(\varepsilon_i) = (\bar{\varepsilon}_1+\bar{\varepsilon}_2)\cdot\bar{\lambda}_i$, and $A[\bar{\lambda}_1,\bar{\lambda}_2,\bar{\varepsilon}_1,\bar{\varepsilon}_2]$ is defined by the relations $\bar{\lambda}_1^{\alpha_1}\bar{\lambda}_2^{\alpha_2} = \bar{\varepsilon}_i^2 = ((\bar{\varepsilon}_1+\bar{\varepsilon}_2)\cdot\bar{\lambda}_i)^2 = 0$ $(i = 1,2;\ \alpha_1+\alpha_2 = 3)$, while finally $A \xrightarrow{i} A[\lambda_1,\lambda_2]$ is the embedding, and $A[\lambda_1,\lambda_2,\varepsilon_1,\varepsilon_2] \xrightarrow{\bar{q}} A[\bar{\lambda}_1,\bar{\lambda}_2,\bar{\varepsilon}_1,\bar{\varepsilon}_2]$ is the quotient map.

We need to show $\tilde{\nabla} = \nabla$. So fix $t_1,t_2 \in T(X)(A)$ as above, and let

$$S = \nabla_{A[\underline{\lambda},\underline{\mu}]}(\lambda_1 t_1+\lambda_2 t_2,\mu_1 t_1+\mu_2 t_2), \qquad (7)$$

where $A[\underline{\lambda},\underline{\mu}] = A\otimes J_1(2)\otimes J_1(2) = A[\lambda_1,\lambda_2,\mu_1,\mu_2]$, and the occurences of $t_i$ in (5) stand for $X(A \to A[\underline{\lambda},\underline{\mu}])(t_i)$. (Below, we will continue suppressing such embeddings and just write $t_i$). Similarly, we define

$$\tilde{S} = \tilde{\nabla}_{A[\underline{\lambda},\underline{\mu}]}(\lambda_1 t_1+\lambda_2 t_2,\mu_1 t_1+\mu_2 t_2). \qquad (\tilde{7})$$

So S and $\tilde{S}$ are elements of $X(A\otimes J_1(2)\otimes J_1(2)\otimes J_1\otimes J_1) = X(A[\underline{\lambda},\underline{\mu},\varepsilon_1,\varepsilon_2])$. By symmetry of $\nabla$ we have

$$X(A\otimes\tau\otimes\tau)(S) = S \qquad (8)$$

where $\tau$ denotes the twist-map of 2.1(viii) resp. 2.1(iii), i.e. $(A\otimes\tau\otimes\tau)(\lambda_i) = \mu_i$, $(A\otimes\tau\otimes\tau)(\varepsilon_1) = \varepsilon_2$, $(A\otimes\tau\otimes\tau)(\mu_i) = \lambda_i$, $(A\otimes\tau\otimes\tau)(\varepsilon_2) = \varepsilon_1$. And by homogeneity of $\nabla$ for each of the two R-module structures we have

$$X(m_1)(S) = X(\bar{m}_1)(S) \in X(A[\underline{\lambda},\underline{\mu},\varepsilon_1,\varepsilon_2,\alpha]) \tag{9}$$

$$X(m_2)(S) = X(\bar{m}_2)(S) \in X(A[\underline{\lambda},\underline{\mu},\varepsilon_1,\varepsilon_2,\alpha]), \tag{10}$$

where $A[\underline{\lambda},\underline{\mu},\varepsilon_1,\varepsilon_2,\alpha] = A\otimes J_1(2)\otimes J_1\otimes J_1\otimes J_1$, i.e. $\alpha^2 = 0$, and the $m_i,\bar{m}_i$ are the A-algebra maps given by the two multiplications in the i-th argument, i.e. $m_1(\lambda_i) = \alpha\lambda_i$, $m_i(\varepsilon_1) = \varepsilon_1$, $m_i(\mu_i) = \mu_i = \bar{m}_1(\mu_i)$, $m_1(\varepsilon_2) = \bar{m}_1(\varepsilon_2) = \varepsilon_2$, $\bar{m}_1(\lambda_i) = \lambda_i$, $\bar{m}_1(\varepsilon_1) = \alpha\varepsilon_1$; $m_2(\lambda_i) = \lambda_i = \bar{m}_2(\lambda_i)$, $m_2(\varepsilon_1) = \varepsilon_1 = \bar{m}_2(\varepsilon_1)$, $m_2(\mu_i) = \alpha\mu_i$, $m_2(\varepsilon_2) = \varepsilon_2$, $\bar{m}_2(\mu_i) = \mu_i$, $\bar{m}_2(\varepsilon_2) = \alpha\varepsilon_2$.

There are corresponding equations $(\tilde{8}),(\tilde{9}),(\tilde{10})$ for $\tilde{S}$. By the equalizer 2.1(vii), or rather, 2.1(vii) tensored with itself, we conclude that there is a *unique* $T \in X(A\otimes J_1(2)\otimes J_1(2)) = X(A[\underline{\lambda},\underline{\mu}])$ with

$$X(A\otimes m\otimes m)(T) = S \tag{11}$$

with m as in 2.1(vii), i.e. $A\otimes m\otimes m: A[\underline{\lambda},\underline{\mu}] \to A[\underline{\lambda},\underline{\mu},\varepsilon_1,\varepsilon_2]$ is given by $\lambda_i \mapsto \varepsilon_1\cdot\lambda_i$, $\mu_i \mapsto \varepsilon_2\cdot\mu_i$. But then by (8) above and 2.1(viii), there is a unique $U \in X(A\otimes J_2(2))$ with

$$X(A\otimes s)(U) = T, \tag{12}$$

where s is as in 2.1(viii). And analogously, there are unique $\tilde{T}$ and $\tilde{U}$ satisfying $(\tilde{11})$: $X(A\otimes m\otimes m)(\tilde{T}) = \tilde{S}$ and $(\tilde{12})$: $X(A\otimes s)(\tilde{U}) = \tilde{T}$.

We now prove $\tilde{S} = S$, using U and $\tilde{U}$. By the diagrams 2.1(iv) and (v) and microlinearity of X, it sufficies to show that

$$X(u)(S) = X(u)(\tilde{S}) \in X(A[\underline{\lambda},\underline{\mu},\underline{\gamma},\underline{\beta}]), \tag{13}$$

where $\underline{\gamma} = (\gamma_1,\gamma_2)$, $\underline{\beta} = (\beta_1,\beta_2)$ are indeterminates given by the relations $\beta_i^2 = 0 = \gamma_1^{\alpha_1}\cdot\gamma^{\alpha_2} = ((\beta_1+\beta_2)\gamma_i)^2$, for i = 1,2; $\alpha_1+\alpha_2 = 3$, while u is the $A[\underline{\lambda},\underline{\mu}]$-algebra map given by $u(\varepsilon_i) = (\beta_1+\beta_2)\cdot\gamma_i$. But by spelling out (6) and the definition of S, we have

$$X(u)(\tilde{S}) = X(v)(S) \tag{14}$$

where $A[\underline{\lambda},\underline{\mu},\varepsilon_1,\varepsilon_2] \xrightarrow{v} A[\underline{\lambda},\underline{\mu},\underline{\gamma},\underline{\beta}]$ is the A-algebra map given by $v(\lambda_i) = \gamma_1\lambda_i+\gamma_2\mu_i$, $v(\mu_i) = \gamma_1\lambda_i+\gamma_2\mu_i$, $v(\varepsilon_i) = \varepsilon_i$. And by definition of T and U,

$$X(v)(S) = X(w)(U) \tag{15}$$

where $w:A[\gamma_1,\gamma_2] \to A[\underline{\lambda},\underline{\mu},\underline{\gamma},\underline{\beta}]$ is given by $w(\gamma_i) = (\beta_1+\beta_2)(\gamma_1\lambda_i+\gamma_2\mu_i)$, with $A[\gamma_1,\gamma_2] = A\otimes J_2(2)$ as before. Writing w for the composite

$$A[\gamma_1,\gamma_2] \xrightarrow{w_1} A[\underline{\lambda},\underline{\mu}] \xrightarrow{w_2} A[\underline{\lambda},\underline{\mu},\underline{\gamma},\underline{\beta}]$$

$w_1(\gamma_i) = \lambda_i+\mu_i$, $w_2(\lambda_i) = (\beta_1+\beta_2)\gamma_1\lambda_i$, $w_2(\mu_i) = (\beta_1+\beta_2)\gamma_2\mu_i$, we find that

$$\begin{aligned} X(w)(U) &= X(w_2)X(w_1)(U) \\ &= X(w_2)(T) \\ &= X(u)(S), \end{aligned}$$

the last identity by definition of T, cf. (11), and the fact that $u\circ(A\otimes m\otimes m) = w_2$. This proves $S = \tilde{S}$.

To conclude the proof, define maps

$$F,\tilde{F}:A^2\otimes A^2 \to X(A\otimes J_1\otimes J_1)$$

by $F(\underline{a}\otimes\underline{b}) = \nabla_A(a_1t_1+a_2t_2,b_1t_1+b_2t_2)$, $\tilde{F}(\underline{a}\otimes\underline{b}) = \tilde{\nabla}_A(a_1t_1+a_2t_2,b_1t_1+b_2t_2)$. Write $A\otimes J_2(2)\otimes J_2(2)\otimes J_1\otimes J_1 = A[\underline{\lambda},\underline{\mu},\varepsilon_1,\varepsilon_2]$ as before, and $A\otimes J_1\otimes J_1\otimes J_1\otimes J_1 = A[\bar{\varepsilon}_1,\bar{\varepsilon}_2,\varepsilon_1,\varepsilon_2]$ and define

$$A[\underline{\lambda},\underline{\mu},\varepsilon_1,\varepsilon_2] \xrightarrow{h_{\underline{a},\underline{b}}} A[\bar{\varepsilon}_1,\bar{\varepsilon}_2,\varepsilon_1,\varepsilon_2]$$

by $h_{\underline{a},\underline{b}}(\lambda_i) = \bar{\varepsilon}_1 \cdot a_i$, $h_{\underline{a},\underline{b}}(u_i) = \bar{\varepsilon}_2 \cdot b_i$. Moreover, let $A[\varepsilon_1,\varepsilon_2] \xrightarrow{j} A[\bar{\varepsilon}_1,\bar{\varepsilon}_2,\varepsilon_1,\varepsilon_2]$ be the map $j = A \otimes m \otimes m$, i.e. $j(\varepsilon_i) = \varepsilon_i \cdot \bar{\varepsilon}_i$. By microlinearity of X and the limit 2.1(iv) of Weil algebras, X(j) is a 1-1 function, and by bilinearity of $\nabla_A$ resp. $\tilde{\nabla}_A$ we have $X(h_{\underline{a},\underline{b}})(S) = X(j)(F(\underline{a}\otimes\underline{b}))$, resp. $X(h_{\underline{a},\underline{b}})(\tilde{S}) = X(j)(\tilde{F}(\underline{a}\otimes\underline{b}))$. Hence since $S = \tilde{S}$ and X(j) is 1-1, $F(\underline{a}\otimes\underline{b}) = \tilde{F}(\underline{a}\otimes\underline{b})$. Putting $\underline{a} = (1,0)$, $\underline{b} = (0,1)$, we conclude that $\nabla_A(t_1,t_2) = \tilde{\nabla}_A(t_1,t_2)$.

This completes the proof of Theorem 2.4.

**2.5 Application to schemes.** Let $\mathbb{A}$ be the category of $\mathbb{A}$-algebras (or rather, to avoid size problems, an appropriate small subcategory of "models"). As discussed extensively in e.g. Demazure (1970), Demazure & Gabriel (1970), a scheme X over k can be considered as an $\mathbb{A}$-functor s(X), $s(X)(A) = \mathrm{Hom}_k(\Gamma\mathcal{O}_X,A)$, and this provides a full and faithful embedding of the category of schemes into the category of $\mathbb{A}$-functors. It is easy to see that s(X) is microlinear for every scheme X. Moreover, s preserves tangent bundles and other prolongations (cf. 1.3(c),(d)). More precisely, $s(TX) \simeq T(sX)$, where on the right-hand side T(sX) is the $\mathbb{A}$-functor defined in 1.3(c), and on the left-hand side X is the tangent bundle constructed as a scheme over X in the usual way, from the sheaf $\Omega^1_{X/k}$ of $\mathcal{O}_X$-modules on X. Similarly, one can define the scheme $T_2(X)$, e.g. as the fibered product of schemes as at the end of 2.3 above, so that $s(T_2(X)) \simeq T_2(sX)$, etc. etc. In this way, we obtain the following result as a special case of theorem 2.4.

**COROLLARY.** (Ambrose-Palais-Singer theorem for schemes) *Let* X *be a scheme over* k. *There is a 1-1 correspondence between symmetric affine connections* $T(X) \times_X T(X) \xrightarrow{\nabla} T(TX)$ *on* X *and sprays* $T(X) \xrightarrow{\sigma} T_2(X)$ *on* X *(or alternatively,* $T(X) \xrightarrow{\sigma} T(TX)$, *cf.* 2.3(c)).

Note, however, that theorem 2.3 gives much more for the case where $\mathbb{A}$ is the category of all k-algebras. For example, one obtains an Ambrose-Palais-Singer theorem for "schemes" $Y^X$ of morphisms from one scheme X to another Y (cf. 1.6(b)). Moreover, a completely analogous argument as the proof of 2.4 will give a result like the corollary just stated, for *schemes over a fixed base scheme* S. (Classically, one constructs the tangent bundle of $X \to S$ from $\Omega^1_{X/S}$. As $\mathbb{A}$-functors, this corresponds to taking the vertical tangent bundle, i.e. the fibered product

$$\begin{array}{ccc} T_S(X) & \longrightarrow & TX \\ \downarrow & & \downarrow \\ S & \xrightarrow{0_S} & TS \end{array}$$

One can now copy the proof of 2.4, replacing T(X), $(X) \underset{X}{\times} T(X) \overset{\nabla}{\to} T(TX)$, etc. by $T_S(X)$, $T_S(X) \underset{X}{\times} T_S(X) \overset{\nabla}{\to} T_S T_S(X)$, etc. throughout).

We will not elaborate these "algebraic" instances of 2.4 further, but turn instead to the context of differential geometry.

## §3. Applications to differential geometry.

In this section, we will describe a category A of algebras appropriate for differential geometry, and show how the classical result of Ambrose, Palais, Singer (1960) is a special case of theorem 2.4. We will also explicity formulate some more general versions of theorem 2.4 in this context.

**3.1 $C^\infty$-rings and $C^\infty$-functors.** We consider $\mathbb{R}$-algebras of the form $C^\infty(\mathbb{R}^n)/I$, where $C^\infty(\mathbb{R}^n)$ is the ring of $C^\infty$-functions on $\mathbb{R}^n$, and I is an ideal. A $C^\infty$*-homomorphism* $C^\infty(\mathbb{R}^n)/I \overset{\phi}{\to} C^\infty(\mathbb{R}^m)/J$ of two such rings is an $\mathbb{R}$-algebra homomorphism induced by a smooth function $\mathbb{R}^m \overset{\phi}{\to} \mathbb{R}^n$ via composition. In other words, a $C^\infty$-homomorphism $C^\infty(\mathbb{R}^n)/I \to C^\infty(\mathbb{R}^m)/J$ is just an equivalence class of $C^\infty$-functions $\mathbb{R}^m \overset{\phi}{\to} \mathbb{R}^n$ with the property that

$f \in I \Rightarrow f \circ \phi \in J$, two such functions $\phi$ and $\bar{\phi}$ being equivalent if for each $i = 1, \ldots, n$, $\pi_i \circ \phi = \pi_i \circ \bar{\phi}$ mod J. Let $C^\infty$ be the category whose objects are $\mathbb{R}$-algebras (isomorphic to ones) of the form $C^\infty(\mathbb{R}^n)/I$ ("$C^\infty$-*rings*"), and whose morphisms are the $C^\infty$-homomorphisms. For $A, B \in C^\infty$, $\mathrm{Hom}_{C^\infty}(A,B)$ is the set of $C^\infty$-homomorphisms $A \to B$. A $C^\infty$-*functor* is a functor $C^\infty \to$ Sets (an $\mathbb{A}$-functor where $\mathbb{A} = C^\infty$). As in section 1, $C^\infty(X,Y)$ denotes the set of morphisms from a $C^\infty$-functor X to another one Y.

Axiom (A1) is satisfied for $C^\infty$-rings. Coproducts of $C^\infty$-rings are given by

$$C^\infty(\mathbb{R}^n)/I \underset{\infty}{\otimes} C^\infty(\mathbb{R}^m)/J \simeq C^\infty(\mathbb{R}^n \times \mathbb{R}^m)/(I,J). \qquad (1)$$

Notice that the underlying set functor R of 1.2 is representable: $R \simeq \overline{C^\infty(\mathbb{R})}$. Clearly $\mathbb{R}[x_1, \ldots, x_n]/I \simeq C^\infty(\mathbb{R}^n)/I$ when $I \supset m^{p+1}$ for some $p > 0$, so $C^\infty$ contains all the Weil algebras, in fact as a full subcategory. Moreover, (1) and a simple Taylor series argument gives $A \underset{\infty}{\otimes} W \simeq A \underset{\mathbb{R}}{\otimes} W$ for any $C^\infty$-ring A and Weil algebra W. So axiom (A2) is satisfied. (A3) is easily verified directly.

**3.2 Manifolds and prolongations.** Let M be a (paracompact) $C^\infty$-manifold. The ring $C^\infty(M)$ of $C^\infty$-functions on M is an object of $C^\infty$, since

$$C^\infty(M) \simeq C^\infty(\mathbb{R}^n)/m_M^o \qquad (1)$$

where $M \hookrightarrow \mathbb{R}^n$ is identified with a closed subspace of $\mathbb{R}^n$ (Whitney embedding theorem), and $m_M^o$ is the ideal of functions which vanish on M. So M gives rise to a $C^\infty$-functor

$$s(M) = \mathrm{Hom}_{C^\infty}(C^\infty(M), -) : C^\infty \to \text{Sets} \qquad (2)$$

Note that s(M) is a *microlinear space*, by 1.6(d).

Regarding $C^\infty$-manifolds as $C^\infty$-functors does not change the class of morphisms: if M and N are manifolds, then a morphism $s(M) \to s(N)$ of $C^\infty$-functors corresponds (by the formula $A(\bar{A},\bar{B}) \simeq \mathrm{Hom}_A(B,A)$ of 1.1) to a $C^\infty$-homomorphism

$C^\infty(N) \to C^\infty(M)$, which corresponds in turn to a smooth map $N \to M$,

$$C^\infty(sM,sN) \simeq \mathrm{Hom}_{C^\infty}(C^\infty(N),C^\infty(M)) \simeq C^\infty(M,N). \qquad (3)$$

So s is a full embedding of the category of manifolds into the category of $C^\infty$-functors.

Fibered products of manifolds need not exist. But if $N_1 \overset{f}{\to} M \overset{g}{\to} N_2$ are smooth maps of manifolds which are transversal (i.e. $(f,g) \pitchfork \nabla_M \subseteq M\times M$) then $N_1 \underset{M}{\times} N_2$ is again a manifold (e.g. Golubitsky & Guillemin (1973), p.9), and it is easy to see that in this case $s(N_1 \underset{M}{\times} N_2) \simeq s(N_1) \underset{s(M)}{\times} s(N_2)$, i.e. s preserves transversal fibered products (see e.g. Kock (1981)).

The embedding s preserves tangent spaces,

$$s(TM) \simeq T(sM) \qquad (4)$$

where TM is the usual tangent bundle of M and T(sM) is the tangent space as constructed in 1.3(c). And more generally for any Weil algebra W

$$s(M)^{\bar{W}} \simeq s({}^{W}M) \qquad (5)$$

where $s(M)^{\bar{W}} \simeq s({}^{W}M)$ is the function space as in 1.3(d) and ${}^{W}M$ is the prolongation of M by W (see Weil (1953), for the case of jet-bundles, this is due to Ehresmann). The isomorphisms (4) and (5) follow immediately from the definitions. Moreover, it is not difficult to check that s maps the usual vector bundle structure on $TM \to M$ to the vector bundle structure defined in 1.7(c) (note that $s(\mathbb{R}) = R$).

**3.3 Connections and sprays on manifolds.** Recall that, classically, an affine connection $\nabla$ on a manifold M is a smooth map $T(M) \underset{M}{\times} T(M) \overset{\nabla}{\to} T(T(M))$ which is linear with respecto to both vector bundle structures over T(M). Since s preserves transversal fibered products and tangent bundles and s is full and faithful (3.2), it is clear that such an affine connection on M is the same as an affine connection on s(M)

$s(M)$ as a $C^\infty$-functor in the sense of 2.2.

A spray on M is usually defined either as a homogeneous section $T(M) \xrightarrow{\sigma} T_2(M)$ of the projection $T_2(M) \to T(M)$, $T_2(M)$ being the manifold of second-order tangent vectors (i.e. the jet-bundle $J^2(\mathbb{R},M)_0$ in the notation of Golubitsky & Guillemin (1973)), or alternatively as a section $T(M) \xrightarrow{\sigma} T(T(M))$ of $T(TM) \xrightarrow{T(\pi_M)} T(M)$ which is symmetric ($\Sigma\circ\sigma = \sigma$ where $T(TM) \xrightarrow{\Sigma} T(TM)$ is the twist-map) and homogeneous. These two definitions are equivalent, and clearly come down to the same thing as a spray on $s(M)$ as defined in the general context of $\mathbb{A}$-functors in 2.3. By the embedding s, we thus obtain as a special instance of theorem 2.4 the known case for differentiable manifolds:

COROLLARY. (cf. Ambrose, Palais, Singer (1960)). *Let* M *be a manifold. There is a natural bijection between symmetric (or torsion-free) affine connections* $TM \times_M TM \xrightarrow{\nabla} TTM$ *on* M *and sprays* $TM \xrightarrow{\sigma} T_2M$.

Let us immediately note that theorem 2.4 gives much more for the case of $C^\infty$-functors. For example, since inverse limits of manifolds are still microlinear spaces when regarded as $C^\infty$-functors, the corollary just stated applies equally well to *manifolds with singularities*.

We will now formulate an other instance of theorem 2.4 in the context of differential topology, for spaces of smooth functions. However, it is convenient to work with a proper subcategory of the category $C^\infty$ of $C^\infty$-rings described in 3.1 above, and we first turn to the description of this subcategory, called $\mathbb{W}$.

**3.4 The category $\mathbb{W}$ of $C^\infty$-rings.** $\mathbb{W}$ is the full subcategory of $C^\infty$ whose objects are the $C^\infty$-rings (isomorphic to ones) of the form

$$C^\infty(M) \otimes_\infty W \qquad (1)$$

where M is a (paracompact) manifold and W is a Weil algebra. We remark that if $W = C^\infty(\mathbb{R}^k)/I$ (with $x^\alpha \in I$ for every multi-index $\alpha = (\alpha_1,\ldots,\alpha_k)$ with $|\alpha| \geq p+1$ say), the coproduct in (1) can also be described as

$$C^\infty(M) \underset{\infty}{\otimes} C^\infty(\mathbb{R}^k)/I \cong C^\infty(M\times\mathbb{R}^k)/(I), \tag{2}$$

where (I) refers to the ideal generated by the compositions $M\times\mathbb{R}^k \overset{\pi}{\to} \mathbb{R}^k \overset{f}{\to} \mathbb{R}$ with $f \in I$. This follows from the coproduct formula (1) in 3.1 by writing $C^\infty(M) = C^\infty(\mathbb{R}^n)/m_M^o$ and using a simple Taylor series argument.

From the corresponding facts for the category $C^\infty$, it is clear that $\mathbb{W}$ also satisfies (A1)-(A3). Moreover, the embedding s of manifolds into $\mathbb{W}$-functors given by

$$s(M) = \mathrm{Hom}_{\mathbb{W}}(C^\infty(M),-)$$

has the same properties as for the case of $C^\infty$: s is full and faithful, preserves tangent bundles and prolongations, and transversal fibered products; all this is just as in 3.2.

For later use, we prove here the following lemma.

**LEMMA.** *Let* $W = C^\infty(\mathbb{R}^n)/I$ *be a Weil algebra, and let* X *and* Y *be manifolds. Then any morphism of* $\mathbb{W}$*-functors* $\bar{W}\times s(X) \to s(Y)$ *is induced by a smooth function* $\mathbb{R}^n\times X \to Y$. *(In other words,* $s(Y)^{s(X)}(q)$ *is surjective, where* $q \in \mathrm{Hom}_{\mathbb{W}}(C^\infty(\mathbb{R}^n), C^\infty(\mathbb{R}^n)/I)$ *is the quotient map).*

*Proof*. We identity Y with a closed submanifold of $\mathbb{R}^e$. By 3.4(2) and 3.2, a morphism $\bar{W}\times s(X) \to s(Y)$ is represented by a smooth function $f(t,x):\mathbb{R}^n\times X \to \mathbb{R}^e$. The problem is that f does not need to map into Y, although by definition (see 3.1) f maps $o\times X$ into Y.

Let U be a tubular neighbourhood of Y, so $Y \subset U \subset \mathbb{R}^e$ and there is a smooth retraction $U \overset{r}{\to} Y$. Let $V = f^{-1}(U)$. Moreover, fix for $a > 0$ a smooth diffeomorphism $\phi_a(t):\mathbb{R}^n \to B(o,a) = \{t \in \mathbb{R}^n \mid |t| < a\}$, which is the identity near 0, and depends smoothly on the parameter $a > 0$. Now let $a(x):X \to \mathbb{R}_{>0}$ be a smooth function with $|t| < a(x) \Rightarrow (t,x) \in V$, and let

$$g:\mathbb{R}^n\times X \to Y, \quad g(t,x) = rf(\phi_{a(x)}(t),x)$$

We claim that $\pi_i\circ f = \pi_i\circ g \bmod (I)$ $(i = 1,\ldots,e)$, with $(I) \subseteq C^\infty(\mathbb{R}^n\times X)$, i.e. g represents the same morphism $\bar{W}\times s(X) \to s(Y)$ as f does (cf. 3.4(2)). Since I is finitely generated (remember that W is a Weil algebra, so I is just an ideal in $\mathbb{R}[x_1,\ldots,x_n]$), a simple partition of unity argument shows that it sufficies to prove that

$$\pi_i\circ f - \pi_i\circ g \in (I) \subseteq C^\infty(W) \quad (i = 1,\ldots,e) \tag{1}$$

where V' is an open neighbourhood of $o\times X$ in $\mathbb{R}^n\times X$. Now take V' such that $\phi_{a(x)}(t) = t$ whenever $(t,x) \in V'$. Since $\pi_i\circ r - \pi_i:\mathbb{R}^r \to \mathbb{R}$ vanishes on Y and f defines a morphism $\bar{W}\times s(X) \to s(Y)$, we have by definition that $\pi_i rf - \pi_i \in (I) \subseteq C^\infty(\mathbb{R}^n\times X)$. Hence (1) follows by definition of g and choice of V'.

**3.5 Spaces of smooth functions. (a)** For manifolds X and Y, $C^\infty(X,Y)$ denotes the spaces of smooth functions from X to Y. If Z and P are two other manifolds, a function $C^\infty(X,Y)\to C^\infty(P,Z)$ is called *smooth* if F can be extended to a natural transformation (a morphism of $\mathbb{W}$-functors) $s(Y)^{s(X)} \xrightarrow{\tau} s(Z)^{s(P)}$, i.e. $F = \tau_{\mathbb{R}}$. Notice that it follows from naturality of $\tau$ and the preceding lemma that if such a $\tau$ exists, it is necessarily unique.

**(b)** Recall that a function $C^\infty(X,Y) \xrightarrow{F} C^\infty(P,Z)$ is called *path-smooth* if for every smooth $\mathbb{R}\times X \xrightarrow{\alpha} Y$, the function $\mathbb{R}\times P \xrightarrow{F(\alpha)} Z$ defined by

$$F(\alpha)(t,p) = F(\alpha(t,-))(p)$$

is smooth. It follows from Boman's theorem (Boman (1967)) that $\mathbb{R}$ can be replaced by any manifold; that is to say, if F is path-smooth and T is a manifold, then for any smooth $T\times X \xrightarrow{\alpha} Y$, $F(\alpha):T\times P \to Z$ is again smooth. The following proposition shows that in some cases, including the important cases of the space of functions $C^\infty(M,\mathbb{R})$ and the space of paths $C^\infty([0,1],M)$ of a manifold M, smooth maps of function spaces are the same as path-smooth maps. (For the case $C^\infty([0,1],M)$ of paths, we remark that the proof of (ii) of the following proposition also applies to manifolds with boundary). Part (ii) of the proposition (observed independently by A. Kock) is not the most general formulation possible; we conjecture that the converse of (i) holds for all manifolds of finite type.

PROPOSITION. *Let* X, Y, Z, *and* P *be manifolds.*

(i) *Every smooth map* $C^\infty(X,Y) \to C^\infty(P,Z)$ *is path-smooth.*

(ii) *In case* X *is compact or* $Y = \mathbb{R}^d$, *the converse of* (i) *also holds.*

*Proof.* (i) If $F = \tau_{\mathbb{R}}$ as in (a) above, then for $\mathbb{R}\times X \overset{\alpha}{\to} Y$, $F(\alpha) = \tau_{C^\infty(\mathbb{R})}(\alpha)$ where $\alpha$ is considered as an element of $s(Y)^{s(X)}(C^\infty(\mathbb{R}))$, so (i) is clear.

(ii) We only prove the case where X is compact; the case $Y = \mathbb{R}^d$ is similar, but much easier. So suppose we are given a path-smooth function $C^\infty(X,Y) \overset{F}{\to} C^\infty(P,Z)$. We will define a morphism $\tau: s(Y)^{s(X)} \to s(Z)^{s(P)}$ of $\mathbb{W}$-functors with $\tau_{\mathbb{R}} = F$. Let $C^\infty(M) \underset{\infty}{\otimes} W$ be an object of $\mathbb{W}$, with $W = C^\infty(\mathbb{R}^n)/I$ say, and let $f \in s(Y)^{s(X)}(C^\infty(M) \underset{\infty}{\otimes} W)$, i.e. f is a morphism $\bar{W}\times s(M)\times s(X) \to s(Y)$ of $\mathbb{W}$-functors. By the lemma in 3.4, we may assume f is represented by a $C^\infty$-function $f(t,m,x): \mathbb{R}^n\times M\times X \to Y$. Let $\tau_{C^\infty(M)\underset{\infty}{\otimes} W}(f): \bar{W}\times s(M)\times s(P) \to s(Z)$ be represented by the function $F(f(t,m,-))(p): \mathbb{R}^n\times M\times P \to Z$.

We claim that $\tau_{C^\infty(M)\underset{\infty}{\otimes} W}$ is well-defined. Indeed, suppose f and $g: \mathbb{R}^n\times M\times X \to Y \subseteq \mathbb{R}^e$ represent the same $\mathbb{W}$-morphism, i.e. $f_i(t,m,x) - g_i(t,m,x) \in (I) \subseteq C^\infty(\mathbb{R}^n\times M\times X)$ for all $i = 1, \ldots, e$. Write

$$f_i(t,m,x) - g_i(t,m,x) = \sum_{\alpha=1}^{k_i} A^i_\alpha(t,m,x)\phi^i_\alpha(t) \tag{1}$$

with $\phi^i_\alpha \in I$, and define

$$A: \mathbb{R}^n\times M\times X\times \mathbb{R}^{k_1}\times\ldots\times\mathbb{R}^{k_e} \to \mathbb{R}^e$$

$$A_i(t,m,x,s^1,\ldots,s^e) = g_i(t,m,x) + \sum_{\alpha=1}^{k_i} A^i_\alpha(t,m,x)s^i_\alpha \quad (i = 1,\ldots,e)$$

(where $s^i = (s^i_1,\ldots,s^i_{k_i})$). Then $A(t,m,x,o,\ldots,o) \in Y$, so by compactness of X we find for each $m_o \in M$ a neighbourhood $U_{m_o}$ and an $\varepsilon > 0$ with

$$A(t,m,x,s^1,\ldots,s^e) \in U$$

whenever $m \in U_{m_o}$ and $|t|$, $|s^i| < \varepsilon$, where U is a tubular neighbourhood of Y as in 3.4, with retraction r.

Fix $m_o \in M$, and suppose Z is a closed submanifold of $\mathbb{R}^d$. Then for $j = 1,\ldots,d$ we can write for $m = U_{m_o}$ and $|t|$, $|s^i| < \varepsilon$:

$$F(rA(t,m,-,s^1,\ldots,s^e))(p)_j = F(rA(t,m,-,o,\ldots,o))(p)_j + \sum_{\alpha,i} B^i_{\alpha j}(m,y,s^1,\ldots,s^e)(p)s^j_\alpha \,, \tag{2}$$

by the fact that F is path-smooth. So if $\delta > 0$ is so small that $|t| < \delta \Rightarrow |\phi^i(t)| = |(\phi^i_1(t),\ldots,\phi^i_{k_i}(t))| < \varepsilon$ then substituting $\phi^i(t)$ for $s^i$ in (2) gives

$$F(f(t,m,-))(p)_j - F(g(t,m,-))(p)_j \in (I) \subset C^\infty(B(0,\delta)\times U_{m_o}\times P) \tag{3}$$

Since this holds for each $m_o \in M$ and I is a finitely generated ideal with 0 as only zero, a simple partition of unity argument gives

$$F(f(t,m,-))(p)_j - F(g(t,m,-))(p)_j \in (I) \subset C^\infty(\mathbb{R}^n\times M\times P), \tag{4}$$

showing that $\tau_{C^\infty(M)\otimes W}(f)$ is well-defined. (This argument is similar to the proof of theorem 8 in Bruno (1985)).

Once we know that $\tau$ is well-defined, it is easily checked that $\tau$ is a natural transformation, and we leave this to the reader.

(c) **REMARK.** Using Boman's theorem, it follows from (i) of the above proposition that a function $C^\infty(X,Y) \overset{F}{\to} C^\infty(P,Z)$ is *smooth* iff it induces in a *natural* way *path-smooth* functions ${}^W F: C^\infty(X, {}^W Y) \to C^\infty(P, {}^W Z)$ for each Weil algebra W (${}^W Y$ denotes the prolongation, cf. 3.2).

**3.6 Connections and sprays on spaces of smooth functions.** Let M and N be manifolds. An affine connection on $C^\infty(M,N)$ is a smooth map

$$C^\infty(M,TN \underset{N}{\times} TN) \xrightarrow{\nabla} C^\infty(M,T^2N)$$

which commutes with the two projections into $C^\infty(M,TN)$ (i.e. $T(\pi_M)\circ\nabla(f) = \pi_2\circ f$, $\pi_{TM}\circ\nabla(f) = \pi_1\circ f$, for $M \xrightarrow{f} TN \underset{N}{\times} TN$), and is linear for both vector space structures on the fibres.

Similar, a spray on $C^\infty(M,N)$ is a smooth section $C^\infty(M,TN) \xrightarrow{\varsigma} C^\infty(M,T_2N)$ of the map $C^\infty(M,T_2N) \to C^\infty(M,TN)$ induced by the projection $T_2N \to TN$, and which satisfies the obvious homogeneity condition. (Alternatively, a spray can be defined as a map $C^\infty(M,TN) \xrightarrow{\varsigma} C^\infty(M,T^2N)$, cf. 2.3).

As a special case of theorem 2.3 we obtain

**COROLLARY.** *Let* M *and* N *be manifolds. There is a natural bijection between symmetric connections* $C^\infty(M,TN \underset{N}{\times} TN) \xrightarrow{\nabla} C^\infty(M,T^2N)$ *on* $C^\infty(M,N)$ *and sprays* $C^\infty(M,TN) \xrightarrow{\varsigma} C^\infty(M,T_2N)$ *on* $C^\infty(M,N)$.

**3.7 REMARK.** The corollary above is really more general than the classical theorem of Ambrose, Palais, Singer (1960) (cf. 3.3 above), since not every (symmetric) connection on $C^\infty(M,N)$ comes from one on N by composition. To take a simple example, let $M = \mathbb{R} = N$ and define a symmetric connection $\nabla$ on $C^\infty(\mathbb{R},\mathbb{R})$ as follows. Let $f \in C^\infty(\mathbb{R},\mathbb{R})$ and $X,Y \in T_f(C^\infty(\mathbb{R},\mathbb{R}))$. We can write $X(t) = (f(t),g(t))$, $Y(t) = (f(t),h(t))$, and let $\nabla_f(X,Y)(t) = (f(t),g(t),h(t),g(t)\cdot h(t)\cdot f'(t))$. Then $\nabla_f(X,Y)(t)$ depends not only on $f(t)$ but also on $f'(t)$, and hence $\nabla_f$ cannot come from a connection on $\mathbb{R}$.

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(*) note added in proof: after hearing from our results, M. Bunge informed us that the treatment of Bunge-Sawyer can be generalized. Their new version is almost as general as ours.

author gives a categorical proof of the 1-1 correspondence between symmetric connections and sprays for the case of (essentially) smooth manifolds of finite dimension (see also Kock-Lavendhomme (1984)). All we did was to show that this correspondence extends to arbitrary microlinear spaces (otherwise known as infinitesimally linear objects, cf. Bergeron (1980)), so as to include e.g. the case of function spaces, manifolds with singularities, and algebraic schemes. We wish to point out here that there are essentially two ways of presenting the proof of theorem 2.4, and we encourage the reader to compare these approaches. One way is in the spirit of SGA3 and Demazure & Gabriel (1971), as we have chosen here. The other is to use "functorial semantics" (Lawvere (1963)), or equivalently, give a presentation in the context of "synthetic differential geometry" (Kock (1983) and Kock-Lavendhome (1984) are written in this spirit, and a synthetic proof of the general case will appear in our forthcoming monograph Moerdijk & Reyes (198?)).

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## REFERENCES

Ambrose, W., Palais, R.S., Singer, I.M., *Sprays*. An.Ac. Bras. de Ciencias 32, 1960.

Bergeron, F., "Object infinitésimalement linéaire dans un modèle adapté de G.D.S"., in: Rapport de Recherches 80-12, Univ. de Montréal, 1980.

Boman, J., *Differentiability of a function and of its compositions with functions of one variable*. Math. Scand. 20, 1967.

Bruno, O., "Vector fields on $R^R$ in well-adapted models of synthetic differential geometry". Manuscript, 1985.

Bunge, M. Sawyer, P., *On connections, geodesics and sprays in synthetic differential geometry*, in: *Category Theoretic Methods in Geometry*, Aarhus Var. Publ. Series 35, 1983.

Demazure, M., *Fibrés tangents et algèbres de Lie*. Exposé II in SGA3 (see below).

Demazure, M. Gabriel, P., *Groupes algébriques*. Tome 1, Masson/North-Holland, 1970 (partly translated as *Introduction to Algebraic Geometry and Algebraic Groups*. North-Holland, 1980).

Godbillon, C., *Géometrie différentielle et Mécanique Analytique*. Hermann, 1969.

Golubitsky, M. Guillermin, V., *Stable Mappings and their Singularities*. Springer, 1973.

Kock, A., *Synthetic Differential Geometry*. Cambridge UP, 1981.

Kock, A., *Remarks on connections and sprays*. in: *Category Theoretic Methods in Geometry*. Aarhus Var. Publ. Series 35, 1983.

Kock, A. Lavendhomme, A., *Strong infinitesimal linearity, with applications to strong difference and affine connections*. Cahiers de Top. et Géom. Diff. 25 (1984), 311-324.

Lang, S., *Differential Manifolds*. Addison-Wesley, 1972.

Lawvere, F.W., *Functorial semantics of algebraic theories*. Proc. Nat. Acad. Sci. U.S.A. 50, 1963.

MacLane, S., *Categories for the Working Mathematician*. Springer, 1971.

Moerdijk, I. Reyes, G.E., *Models for smooth Infinitesimal Analysis*. (Monograph, in preparation, 198?).

Mumford, D., *Lectures on curves on an algebraic surface*. Ann. Math. Studies 59, Princeton, 1966.

SGA3 - *Séminaire de Géometrie Algébrique*, dir. par M. Demazure et A. Grothendieck. Springer LNM 151, 1970.

Weil, A., *Théorie de points proches sur les variétés différentiables*, in: Colloque de Géométrie Différentielle, C.N.R.S., 1953.

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