Revista Colombiana de Matemáticas
Vol. XXI (1987), págs. 5-12

ON THE CONVEX SUM OF CERTAIN UNIVALENT FUNCTIONS AND THE IDENTITY FUNCTION

by

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Abstract. If $g(z)$ is analytic, univalent, and convex in the unit disk $E = \{|z| < 1\}$, $g(0) = 0$, $g'(0) = 1$, we extend some known results about the classification of $H_\lambda(z) = \lambda z + (1-\lambda)g(z)$ when $\lambda \geq 0$. In particular, it is proved H_λ is starlike in E for each $\lambda \in [0,1]$ when $g''(0) = 0$.

§1. Introduction. Let S be the class of analytic univalent functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the unit disk $E = \{z:|z| < 1\}$. Let K, S^*, and C denote the subclasses of S whose members are respectively convex, starlike, and close-to-convex in E. The conjecture [4] that $\frac{1}{2}(f+g) \in S^*$ whenever f, g are in K was proved false by MacGregor [5] and by Trimble [9]. If, however, f and g are in K and $f''(0) = g''(0) = 0$, then Hallenbeck and Ruscheweyh [3] used a result of Styer and Wright [8] to prove $\frac{1}{2}(f+g) \in S^*$.

Trimble [9] considered convex combinations $H_\lambda(z) = (1-\lambda)z + \lambda g(z)$, where $0 < \lambda < 1$ and $g \in K$, and showed H_λ is always in S^* provided $\lambda \geq 2/3$ but H_λ need not be starlike for $0 < \lambda < 2/3$. Later, Chichra and Singh [1] found subclasses of S such that $H_\lambda \in S^*$ for all $0 < \lambda < 1$ whenever g is in the subclass. In this paper, we extend a number of these results.

THEOREM 1. *If* $g \in K$, *then* $H_\lambda(z) = (1-\lambda)z + \lambda g(z)$ *is in* C *for* $0 \leqslant \lambda \leqslant 4/3$ *and for each* $\lambda \notin [0,4/3]$ *there is a* $g \in K$ *such that* $H_\lambda \notin S$. *Furthermore,* H_λ *is starlike of order* $\sigma = (1-3|1-\lambda|)/(2-2|1-\lambda|)$ *provided* $\lambda \in [2/3,4/3]$, *and for each* $\lambda \notin [2/3,4/3]$ *there is a* $g \in K$ *such that* $H_\lambda \notin S^*$.

This theorem extends the result in [9] to the interval $-\infty < \lambda < \infty$ and it improves the known starlike order of the family $\{H_\lambda = (1-\lambda)z + \lambda g : 0 < \lambda < 1,\ g \in K\}$ to σ, which we prove is best possible.

THEOREM 2. *If* $g \in K$ *and* $0 < a \leqslant \sqrt{13\sqrt{13} - 46}$, *then* $H_\lambda(z) = (1-\lambda)z + \lambda g(az)/a$ *is in* S^* *for all* $\lambda \in [0,1]$. *The upper bound on* a *is best possible.*

If f and g in K, then MacGregor [5] has proved that $(1-\lambda)f(bz)/b + \lambda g(bz)/b$ is in S^* for $0 < b \leqslant 1/\sqrt{2}$. This upper bound is best possible.

For $\alpha > -1$, set $g_\alpha(z) = z + (\alpha+1)\sum_{n=2}^{\infty} z^n/(\alpha+n)$. In [1] it is proved that for $g \in K$ the function $H_\lambda(z) = (1-\lambda)z + \lambda(g_1 * g)(z)$, where $g_1 * g$ is the Hadamard product, is in S^* when $0 < \lambda < 1$. More generally, we prove

THEOREM 3. *If* $g \in K$, *then* $H_\lambda(z) = (1-\lambda)z + \lambda(g_\alpha * g)(z)$ *is in* S^* *when* $0 < \lambda < 1$ *and* $-1 < \alpha \leqslant 1$.

We suspect not all H_λ are in S^* for $\alpha > 1$ although we have been unable to prove this result. We do prove that $H_\lambda \notin S^*$ for α sufficiently large when $0 < \lambda < 2/3$.

Styer and Wright [8] proved, for f and g odd functions in K, $(1-\lambda)f + \lambda g$ need not be in S^* when $0 < \lambda < 1$, $\lambda \neq 1/2$. In our case, however, we have

THEOREM 4. *If* $g \in K$ *and* $g''(0) = 0$, *then* $H_\lambda(z) = (1-\lambda)z$

$+ \lambda g(z)$ *is in* S^* *for each* $\lambda \in [0,1]$.

The authors interest in these problems was stimulated by Chapter 5 in A.W. Goodman's recent volumes [2] on univalent functions. The first volume serves as a references for the definitions of the classes discussed in this paper.

§2. Proof of Theorem 1. Consider the function $h_\lambda(z) = (1-\lambda)z + \lambda z/(1-z) = z[1+(\lambda-1)z]/(1-z)$. Since $h_\lambda(0) = h_\lambda(1/(1-\lambda)) = 0$, we have $h_\lambda \notin S$ whenever $|1-\lambda| > 1$. Furthermore, $h_\lambda'(z) = 0$ when $z = 1 - \sqrt{\frac{\lambda}{\lambda-1}}$ so $h_\lambda \notin S$ if $\lambda > 4/3$. Since for $z \in E$

$$\frac{z\, h_\lambda'(z)}{h_\lambda(z)} = \frac{1}{1-z} + \frac{(\lambda-1)z}{1+(\lambda-1)z}, \tag{1}$$

we have

$$\operatorname{Re} \frac{zh_\lambda'(z)}{h_\lambda(z)} \geq \frac{1}{2} - \frac{|\lambda-1|}{1-|\lambda-1|} = \frac{1-3|\lambda-1|}{2(1-|\lambda-1|)} = \sigma \geq 0,$$

provided $|\lambda-1| \leq 1/3$. Thus, $h_\lambda \in S^*$ for all (complex) λ in $|\lambda-1| \leq 1/3$.

This order of starlikeness σ cannot be improved. Indeed if $-\pi < \arg(\lambda-1) < \pi$, set $\alpha = -\arg(\lambda-1)$ and $z = -e^{i\alpha}$ to obtain

$$\frac{z\, h_\lambda'(z)}{h_\lambda(z)} = \frac{1+e^{-i\alpha}}{2+2\cos\alpha} - \frac{|\lambda-1|}{1-|\lambda-1|}.$$

The real part of this expression is σ. For $2/3 \leq \lambda \leq 1$, set $z = e^{i\theta}$, $\theta \neq 0$. Then $\operatorname{Re}\{e^{i\theta}h_\lambda'(e^{i\theta})/h_\lambda(e^{i\theta})\}$ tends to σ as $\theta \to 0$.

For $0 \leq \lambda \leq 1$, we have

$$\operatorname{Re}\{(1-z)h_\lambda'(z)\} \geq \lambda \operatorname{Re}\{\frac{1}{1-z}\} + (1-\lambda)\operatorname{Re}(1-z) \geq \lambda/2,$$

and hence $h_\lambda \in C$. Since $S^* \subset C$, we also have $h_\lambda \in C$ when $|\lambda-1| \leq 1/3$.

Since the Hadamard product of $g \in K$ and a function in C or in S^* is respectively in C or in S^* [7], the general result of Theorem 1 is a consequence of our special case. In fact, we have slightly more than is stated, namely, $(1-\lambda)z + \lambda g(z)$ is in S^* for all $g \in K$ if and only if λ is a complex number in $|\lambda-1| \leqslant 1/3$.

§3. **Proof of Theorem 2.** We begin by finding the radius of starlikeness of the family $h_\lambda(z) = (1-\lambda)z + \lambda z/(1-z)$ where $0 < \lambda < 2/3$. This is equivalent to finding $r = \min r_\lambda$, $0 < \lambda < 2/3$, where r_λ is the modulus of the smallest zero of $\mathrm{Re}\{z\, h_\lambda'(z)/h_\lambda(z)\}$. For a fixed λ, this real part is zero at $z = re^{i\theta}$ if by (1)

$$V = V(r,\mu,t) = 1 + \mu^2 r^2(2+r^2) - (1+3\mu)(1+\mu r^2)rt + 4\mu r^2 t^2 = 0 ,$$

where $\mu = 1-\lambda\ (-1/3 < \mu < 1)$ and $t = \cos\theta$. By completing the square,

$$V \geqslant 1+\mu^2 r^2(2+r^2) - (1+3\mu)^2(1+\mu r^2)^2/16\mu \qquad (2)$$

and equality holds if $t = t(r,\mu) = (1+3\mu)(1+\mu r^2)/8\mu r$. Since, for fixed μ, t is a decreasing function of r, we have $t(r,\mu) > t(1,\mu) = (1+3\mu)(1+\mu)/8\mu$ and the latter is less than unity for $1/3 < \mu < 1$. Thus, for each μ in $(1/3,1)$ there is a range of r, $r_0 < r < 1$, such that $t = t(r,\mu) < 1$. In this range, equality holds in (2) and the right hand side of (2) is

$$V(r,\mu,t(r,\mu)) = (1-\mu)\{(4+s-(4-s)\mu r^2)(4-s-(4+s)\mu r^2)\}/32,$$

where $s = \sqrt{2(1-\mu)/\mu} = \sqrt{2\lambda/(1-\lambda)}$. Since $\mu < 1$ and $r < 1$, the last expression is zero if and only if $\mu r^2 = (4-s)/(4+s)$, that is,

$$r^2 = r_\lambda^2 = \frac{1}{1-\lambda}\,\frac{4 - \sqrt{2\lambda/(1-\lambda)}}{4 + \sqrt{2\lambda/(1-\lambda)}}$$

The minimum with respect to λ of r_λ^2 occurs when $\lambda = (5-\sqrt{13})/9$

and $r^2 = a^2 = 13\sqrt{13}-46$ for this value of λ.

We have proved that $h_\lambda(az)/a$ is starlike for $0 \leqslant \lambda \leqslant 1$. The Hadamard product of this function with a $g \in K$ is therefore in S^* [7]. This product is $H_\lambda(z) = (1-\lambda)z + \lambda g(az)/a$ and the first part of Theorem 2 is proved. The function h_λ with $\lambda = (5-\sqrt{13})/9$ is used to prove the sharpness part of the theorem.

For a fixed $\lambda \in (0,2/3)$ we also have that the Hadamard product of $g \in K$ and $h_\lambda(r_\lambda z)/r_\lambda$, where r_λ is determined from (3), is starlike in E. Thus, $(1-\lambda)z + \lambda g(r_\lambda z)/r_\lambda$ is in S^* for all $g \in K$.

§4. **Proof of Theorem 3.** There is an elementary sufficient condition for $H_\lambda = (1-\lambda)z + \lambda f(z)$ to be in S^* for $0 \leqslant \lambda \leqslant 1$ when $f \in S^*$.

LEMMA. *If* $f \in S^*$ *and* $\mathrm{Re}\{f'+f/z\} \geqslant 0$ *for* $z \in E$, *then* $H_\lambda \in S^*$.

For, if $\mu = \lambda/(1-\lambda)$, $0 \leqslant \lambda \leqslant 1$, we have

$$\mathrm{Re}\,\frac{zH_\lambda'}{H_\lambda} = \mathrm{Re}\,\frac{1+\mu f'}{1+\mu f/z} = \frac{(1+\mu\mathrm{Re}(f'+f/z) + \mu^2|f/z|^2\cdot\mathrm{Re}(zf'/f)}{|1+\mu f/z|^2} > 0$$

for all $z \in E$ and μ, $0 \leqslant \mu < \infty$, when $f \in S^*$ and $\mathrm{Re}\{f'+f/z\} \geqslant 0$ in E.

It is known [6] that for $\alpha > -1$ the function

$$g_\alpha(z) = z + \sum_{n=2}^{\infty} \frac{\alpha+1}{\alpha+n} z^n = (\alpha+1)\int_0^1 \frac{z}{1-zt}\, t^\alpha dt$$

is starlike of order 1/2 (and in fact convex when $\alpha \geqslant 0$) in E. Furthermore, in E

$$\mathrm{Re}\,\frac{g_\alpha(z)}{z} = (\alpha+1)\int_0^1 \mathrm{Re}\{\frac{1}{1-zt}\}\, t^\alpha dt \geqslant \frac{1}{2}.$$

Since $z\,g_\alpha' + \alpha g_\alpha = (\alpha+1)z/(1-z)$, we have

$$\operatorname{Re}\{g' + \frac{g_\alpha}{z}\} = (\alpha+1)\operatorname{Re}\frac{1}{1-z} + (1-\alpha)\operatorname{Re}\frac{g_\alpha}{z} \geq \frac{1}{2}$$

when $\alpha \leq 1$. By the lemma, we conclude $h_\lambda = (1-\lambda)z + \lambda g_\alpha$ is in S^* for $\lambda \in [0,1]$ whenever $\alpha \in [-1,1]$. Theorem 3 now follows since, for such λ and α, $H_\lambda = h_\lambda * g$ is in S^* for all $g \in K$ [7].

Since $g_\alpha(z) \to z/(1-z)$ as $\alpha \to \infty$ (uniformly on compact subsets of E), we have for fixed λ that $h_\lambda(z,\alpha) = (1-\lambda)z + \lambda g_\alpha(z) \to (1-\lambda)z + \lambda z/(1-z)$ as $\alpha \to \infty$. Since in §2 we prove the limit function is starlike if and only if $|\lambda-1| \leq 1/3$, the functions $h_\lambda(z,\alpha)$ are not starlike for all $\alpha > -1$ when $|\lambda-1| > 1/3$.

§5. Proof of Theorem 4. Let $K_0 = \{g \in K : g''(0) = 0\}$. It is known [3] that $\{w : |w| < \pi/4\} \subset g(E)$ and $|Im(g/z)| \leq \pi/4$ for $z \in E$ when $g \in K_0$.

Suppose $g \in K_0$ and g is analytic in $|z| \leq 1$. A tangent vector to the convex analytic curve $w = g(e^{i\theta})$, $-\pi \leq \theta \leq \pi$ at $\theta = 0$ is $g(1) + ig'(1)$. Thus, $g(1) + g'(1)$ is an outward normal to this curve. Since $\operatorname{Re}(g/z) \geq \frac{1}{2}$ in E [2,Vol.I, p.135] we have $\operatorname{Re} g(1) \geq \frac{1}{2}$, $|Im\, g(1)| \leq \pi/4$. Let T_1 and T_2 be the points of $|w| = \pi/4$ such that $T_1 - g(1)$ and $T_2 - g(1)$ are tangent to this circle. It follows that T_1, T_2 are in the right half plane $\operatorname{Re} w \geq 0$ and $\arg g'(1)$ is between $\arg T_1$ and $\arg T_2$. This implies $\operatorname{Re} g'(1) \geq 0$. Since, for $0 < |z_0| < 1$, we have $g(z_0 z)/z_0 \in K_0$ whenever $g \in K_0$, we have shown that $\operatorname{Re} g'(z) \geq 0$ for $z \in E$ whenever $g \in K_0$. By the lemma, it follows that $H_\lambda(z) = (1-\lambda)z + \lambda g(z)$ is in S^* for all $0 \leq \lambda \leq 1$.

REFERENCES

[1] Chichra, P. and Singh, R., *Convex sum of univalent functions*, J. Austral. Math. Soc., 14 (1972), 503-507. MR 47 #7014.

[2] Goodman, A.W., *Univalent Functions*, I and II, Mariner Publishing Co., Inc., Tampa, Florida, 1983.
[3] Hallenbeck, D.J. and Ruscheweyh, S., *Subordination by convex functions*, Proc. Amer. Math. Soc. 52 (1975), 191-195. MR 51 #10603.
[4] Hayman, W.K., *Research problems in function theory*, Anthlone Press [University of London], 1967. MR 36 #359.
[5] MacGregor, T.H., *The univalence of a linear combination of convex mappings*, London Math. Soc., 44 (1969), 210-212. MR **38** # 4665.
[6] Ruscheweyh, S., *New criteria for univalent functions*, Proc. Amer. Math. Soc., 49 (1975), 109-115. MR 51 #3418.
[7] Ruscheweyh, S. and Sheil-Small, T., *Hadamard product of schlicht functions and the Polya-Schoenberg conjecture*, Comment. Math. Helv. 48 (1973), 119-135. MR 48 #6393.
[8] Styer, D. and Wright, D., *On the valence of the sum of two convex functions*, Proc. Amer. Math. Soc., 37 (1973), 511-516. MR 47 #2049.
[9] Trimble, S., *The convex sum of convex functions*, Math. Z., 109 (1969), 112-114. MR 39 #7085.

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(Recibido en junio de 1986).