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SEMIREFLEXIVE SPACES IN WHICH THE THEOREM ON THE DERIVATIVE-OF-THE-INVERSE FAILS FOR FRECHET DERIVATIVES

by

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Let E be a Hausdorff locally convex topological vector space (LCS). A map $f:E \to E$ is said to be *Fréchet differentiable* at $x_o \in E$ if there is a continuous (or even a sequentially continuous, cf. [2]) linear map $f'[x_o]:E \to E$ such that for any bounded set B in E, the limit

$$\lim_{t \to o} \frac{f(x_o+th) - f(x_o) - tf'[x_o](h)}{t} = 0$$

holds uniformly with respect to $h \in B$.

S. Yamamuro, has obtained ([5], pp.151-153) an attractively simple characterization of those locally convex spaces for which the derivative-of-the-inverse theorem holds, by which we mean the following result:

Suppose the one-to-one and onto map $f:E \to E$ *is Fréchet differentiable at* x_o *in such a way that* $f'[x_o]$ *is a homeomorphism. Then if the inverse map* f^{-1} *is continuous at*

$f(x_o)$, *it is Fréchet differentiable there with derivative* $(f'[x_o])^{-1}$.

His characterizing condition is the following:

For each null sequence of non zero elements $x_n \to 0$ *in* E *there exists a scalar sequence* α_n *for which the sequence* $\alpha_n x_n$ *is bounded but not convergent to* 0.

An LCS with this property is said to be *boundedly levered*. Normed spaces and strict (LB)-spaces are boundedly levered. It is the purpose of this note to show that members of some other important classes of locally convex spaces are not. For basic results and terminology, we refer to [4].

We will be concerned with a class of spaces which contains all strict (LF)-spaces. An LCS E will be called a (GS)-*space*(*) if for each bounded sequence x_n in E, the following is true:

With respect to the induced topology from E, *the closed subspace* F *spanned by* $\{x_n : n = 1, 2, \ldots\}$ *has a weakly separable dual space* F'.

Since the weak topology on E induces the weak topology on F, it follows from Šmulian's theorem and the fact that F is closed, that:

(1) *If* E *is a* (GS)-*space, then every weakly relatively countably compact set is weakly relatively sequentially compact.*

The following result was essentially obtained in [1].

(2) A (GS)-*space possesses a bounded sequence spanning an infinite dimensional subspace if and only if there is a*

(*) After Grunau-Šmulian

null sequence $x_n \to 0$ *of nonzero elements of* E *with the property that for any sequence* α_n *of scalars, the set of weak adherent points of* $\{\alpha_n x_n : n = 1,2,\ldots\}$ *is either* $\{0\}$ *or is empty.*

Proof. It is easy to see that a sequence with this property would necessarily have an infinite dimensional span. On the other hand, suppose a bounded linearly independent sequence exists which spans the closed subspace F. If E is a (GS)-space, we can apply the Hahn-Banach theorem to obtain a sequence $x_n \to 0$ in F together with a sequence ϕ_m in F' which separates the points of F and satisfies

$$\phi_m(x_n) \neq 0 \quad \text{if and only if } m = n.$$

This sequence has the desired property. (For more details, see the proof of (3) in [1]). ▲

We first investigate weak topologies. Recall that E is said to be semireflexive if it coincides with the strong dual of its dual E'. In such a space, the bounded sets are relatively weakly compact, so a bounded sequence whose only weak adherent point is 0 must converge weakly to 0. Applying (2), we have:

(3) *Let* E *be a semireflexive space which is a* (GS)*-space. Then* E *with the weak topology is boundedly levered if and only if the bounded sets of* E *have finite dimensional spans.*

It follows from the definition that the weak dual of a barrelled space is semireflexive. So a corollary of (3) is:

(4) *If* E *is a separable* LCS *which is barrelled, then* E' *with the weak topology from* E *is boundedly levered if and only if the bounded sets in* E' *have finite dimensional spans.*

In one important case, at least, the separability assumption in (4) can be dropped.

(5) *If* E *is a Banach space, then its weak dual is boundedly levered if and only if* E *is finite dimensional.*

Proof. If E is infinite dimensional, it follows from the theorem of [3] that there is a bounded sequence ϕ_n in E' which is weakly convergent to 0 and satisfies $\|\phi_n\| > \delta$ for some $\delta > 0$. Our assertion follows from this. ▲

Now, by the obvious device of looking at spaces in which weakly convergent sequences are strongly convergent, one can get results for the strong topology. From (3) we obtain:

(6) *If the Montel space* E *is a* (GS)*-space, it is boundedly levered if and only if the bounded sets have finite dimensional spans.*

The above theorem applies to many of the important (M)-spaces of analysis.

(7) *Let* E *be an* (M)*-space which is either a metrizable space, a strict* (LF)*-space or the strong dual of such a space. Then* E *is a* (GS)*-space.*

Proof. An (FM)-space is separable so its dual is weakly separable. The closed span F of a bounded sequence in a strict (LF)-space is contained in one of the component (F)-spaces, which is an (M)-space if E is, and then F is also an (FM)-space.

Also, a strict inductive limit of separable spaces is separable, so if the dual of E has an <E',E>-admissible topology of this type, then E has a weakly separable dual, and so is a (GS)-space. ▲

If E is a metrizable LCS, it is not difficult to give a necessary and sufficient condition that E be boundedly levered in terms of the seminorms which determine the topology. First, we observe that a metrizable space whose topology is induced by a increasing sequence of seminorms $\|\cdot\|_q$ which are not norms cannot be boundedly levered. All we need do is choose a sequence x_n of nonzero elements such that $\|x_q\|_q = 0$ for all q to obtain a sequence such that $\alpha_n x_n \to 0$ for all scalar sequences α_n.

(8) *A metrizable* LCS, E *is boundedly levered if and only if the topology on* E *is given by an increasing sequence of norms,* $\|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant \ldots$, *and, for every sequence* x_n *in* E, *there is a subsequence* x_n' *of* x_n *such that for some positive integer* p

$$\liminf_n \frac{\|x_n'\|_p}{\|x_n'\|_q} > 0 \qquad (*)$$

holds for all $q \geqslant p$.

Proof. Suppose the condition is satisfied. Let $x_n \to 0$ be given and let the subsequence x_n' and the positive integers p be such that (*) holds for all $q \geqslant p$. For each q, let β_q denote the value of the inferior limit in (*). Define $\alpha_n = 1/\|x_n'\|_p$. Then $\alpha_n x_n'$ does not converge to 0, but since for all $q \geqslant p$,

$$\|\alpha_n x_n'\|_q = \|x_n'\|_q/\|x_n'\|_p \leqslant 2/\beta_q$$

holds when n is sufficiently large, it follows that $\alpha_n x_n'$ is bounded. It follows that E is boundedly levered.

On the other hand, if E is a countably normed space in which the condition fails, then there is a sequence x_n in E with the property that for a given subsequence x_n' and a given p, there is a $q > p$ such that

$$\liminf_n \frac{\|x_n'\|_p}{\|x_n'\|_q} = 0.$$

This would not change if we replaced the sequence x_n by a sequence $\beta_n x_n$, so with no loss of generality we can assume $x_n \to 0$.

Now suppose the sequence α_n is such that $\alpha_n x_n$ is bounded. Then there is a sequence of positive numbers M_q, $q = 1,2,\ldots$, such that $\|\alpha_n x_n\|_q \leq M_q$ holds for all n. Also, given p and a subsequence x_n' of x_n there is a $q > p$ such that for the corresponding subsequence α_n' of α_n

$$0 \leq \lim_n \inf \|\alpha_n' x_n'\|_p \leq M_q \lim_n \inf \frac{\|\alpha_n' x_n'\|_p}{\|\alpha_n' x_n'\|_q} = 0.$$

It follows from this that $\alpha_n x_n \to 0$. Thus E is not boundedly levered. ▲

We remark that we do not know an example of a non-normable metrizable LCS which is boundedly levered.

These results may suggest a serious shortcoming in this generalization of the Fréchet derivative (cf. also Chapter 3 of [5]).

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