Revista Colombiana de Matemáticas
Vol. XXI (1987), págs. 155-166

ENERGY ANALYSIS OF A NONLINEAR SINGULAR DIFFERENTIAL EQUATIONS AND APPLICATIONS

by

Alfonso CASTRO and Alexandra KUREPA

Resumen. Damos condiciones suficientes sobre la función g para que la energía E(t,d) de las órbitas de la ecuación no lineal $u''+\frac{N-1}{t}u'+g(u) = p(t)$, $t\in[0,\infty)$, $u(0) = d$, $u'(0) = 0$ satisfaga $E(t,d) \to \infty$ cuando $d \to \infty$ uniformemente para t en intervalor acotados. Indicamos como usar estos resultados para el estudio de un problema de Dirichlet.

Abstract. In this paper we give sufficient conditions on the function g so that the energy E(t,d) of the solutions to the nonlinear singular equation: $u''+\frac{N-1}{t}u'+g(u) = p(t)$, $t\in[0,\infty)$, $u(0) = d$, $u'(0) = 0$ tends to infinity on bounded intervals as d tends to infinity. We indicate how to apply these results to a superlinear Dirichlet problem.

§1. Introduction. In this paper we study the "energy" of the solutions to the singular initial value problem

$$\begin{aligned} &u'' + \frac{N-1}{t}u' + g(u) = p(t), \quad t \in [0,T] \\ &u(0) = d \\ &u'(0) = 0 \end{aligned} \tag{1.1}$$

where $N > 1$, $g:\mathbb{R} \to \mathbb{R}$ is a locally Lipschitzian function,

$T > 0$, and $p \in L^\infty[0,T]$. Arguments based on the contraction mapping principle show that for each $d \in \mathbb{R}$ problem (1.1) has a unique solution $u(t,d)$ on the inverval $[0,\infty)$, depending continuously on d. The energy of the solutions to (1.1) is defined by

$$E(t,d) = (u'(t,d))^2/2 + G(u(t,d)), \qquad (1.2)$$

where $G(u) = \int_0^u g(v)dv$. Our main results give sufficient conditions so that

$$E(t,d) \to \infty \quad \text{as} \quad d \to \pm\infty \qquad (1.3)$$

uniformly for $t \in [0,T]$. Property (1.3) plays a central role in the study of the oscillations of the solutions to (1.1). The reader is refered to [2] and [3] for the applications to the study of radially symmetric solutions for superlinear boundary value problems. Theorems 3.1 and 3.2 extend the results of section 2 of [2]. This in turn implies the extension of Theorem A of [2] (see Theorem B below).

§2. **Preliminary Lemmas.** Throughout this paper c will denote various positive constants depending on $(N, \|p\|_\infty, g)$. We will assume that g is strictly increasing, and $g(0) = 0$.

For $\kappa \in (0,1]$ we define

$$\Lambda(\kappa,u) := NG(\kappa u) - \frac{N-2}{2} ug(u), \qquad (2.1)$$

$$\Lambda_\pm(\kappa) := \lim_{u \to \pm\infty} \Lambda(\kappa,u)(u/g(u))^{N/2}. \qquad (2.2)$$

The next lemma provides growth conditions of the nonlinearity g closely related to the Sobolev inequalities (see [1]).

LEMMA 2.1. A) *If* $N \geq 3$ *and* $\Lambda(\kappa,u)$ *is bounded below for some* $\kappa \in (0,1]$, *and all* $u \geq 0$ *(respec.* $u \leq 0$*) then*

$$|g(u)| \leq c(|u|^q + 1)$$

for $u \geqslant 0$ *(respec.* $u < 0$*), where* $q := \frac{N+2-(1-\kappa)2N}{N-2}$.

B) *If* $N = 2$ *and* $(u/g(u))G(u) \to \infty$ *as* $u \to \infty$ *(respec.* $u \to -\infty$*) then for any* $j > 0$ *there exists* $c = c(j)$ *such that*

$$G(u) \leqslant c(\exp(u^2/j) + 1)$$

for $u \geqslant 0$ *(respec.* $u < 0$*).*

Proof. A) Let b be such that

$$NG(\kappa u) - \frac{N-2}{2}g(u)u > b \text{ for all } u \geqslant 0. \tag{2.3}$$

Thus $NG(s) - \frac{N-2}{2\kappa}g(s)s \geqslant b$. Hence, multiplying by $(-\frac{2\kappa}{N-2})\ s^{-(2\kappa N+N-2)/(N-2)}$ we obtain

$$(s^{-2\kappa N/(N-2)}G(s))' \leqslant -\frac{2}{N-2}bs^{-(2\kappa N+N-2)/(N-2)}. \tag{2.4}$$

Integrating on $[1,s]$ we infer

$$G(s) < G(1)s^{2\kappa N/(N-2)} + \frac{b}{N} - \frac{b}{N}\ s^{2\kappa N/(N-2)} < cs^{2\kappa N/(N-2)}. \tag{2.5}$$

Thus from (2.3) we have

$$g(u) \leqslant \frac{2(NG(\kappa u)-b)}{(N-2)u} \leqslant \frac{2N}{N-2}\ cu^{(2\kappa N/(N-2))-1} + c \leqslant c(u^{(2\kappa N-N+2)/(N-2)} + 1).$$

B) Given any positive constant j there exists b such that

$$sG(s) - jg(s) \geqslant b. \tag{2.6}$$

Multiplying (2.6) by $-\frac{1}{j}\exp(-s^2/(2j))$ and integrating on $[1,s]$ we obtain

$$\exp(-s^2/(2j))G(s) \leqslant \exp(-1/(2j))G(1) + b\int_1^s(-1/j)\exp(-t^2/(2j))dt,$$

hence

$$G(s) \leqslant c(\exp(s^2/(2j))+1) \leqslant c(\exp(s^2/j)+1), \tag{2.7}$$

which proves the lemma.

For $\kappa \in (0,1)$ and $d > 0$, let $t_1 := t_1(\kappa,d)$ be such that $d \geqslant u(t,d) > \kappa d$ for all $t \in [0,t_1)$ and $u(t_1,d) = \kappa d$. Multiplying (1.1) by r^{N-1} we infer $(r^{N-1}u'(r,d))' = r^{N-1}(-g(u(r,d)) + p(r))$. Therefore

$$u'(t,d) = t^{-N+1} \int_0^t r^{N-1}[p(r) - g(u(r,d))]dr. \tag{2.8}$$

From (2.8) we see that if g is bounded above then u' is bounded below. Hence, for d sufficiently large we have $t_1 > T$. Thus, for all $t \in [0,T]$ we get

$$E(t,d) \geqslant G(\kappa d). \tag{2.9}$$

On the other hand if $g(d) \to \infty$ as $d \to \infty$, then by choosing d such that $g(\kappa d) \geqslant \|p\|_\infty$, from (2.8) we see that

$$u'(t,d) \geqslant t^{-N+1} \int_0^t r^{N-1}[p(r) - g(d)]dr \geqslant [-\|p\|_\infty - g(d)]\frac{t}{N}. \tag{2.10}$$

Integrating over $[0,t_1]$ we find that $\kappa d \geqslant d - (\|p\|_\infty + g(d))t_1^{\,2}/(2N)$. Hence

$$t_1 \geqslant [2N(1-\kappa)d/(\|p\|_\infty + g(d))]^{\frac{1}{2}} \geqslant c(d/g(d))^{\frac{1}{2}} := t_0. \tag{2.11}$$

Similar arguments show that if $d < 0$ then

$$t_1 \geqslant [2N(\kappa-1)d/(\|p\|_\infty - g(d))]^{\frac{1}{2}} \geqslant c(d/g(d))^{\frac{1}{2}} := t_0 .$$

LEMMA 2.2. *If for some* $\kappa \in (0,1)\, \Lambda_+(\kappa) = \infty$ *(respect.* $\Lambda_-(\kappa) = \infty$*), then for* $u > 0$ *(respect.* $u < 0$*) and* d *sufficiently large*

$$\int_0^{t_0} r^{N-1}[NG(u(r)) - \frac{N-2}{2}g(u(r))u(r)]dr \geqslant cg(\kappa d)d[d/g(d)]^{N/2}.$$

Proof. Since g is an increasing function, then for $u > 0$

$$G(u) = \int_0^{\kappa u} g(s)ds + \int_{\kappa u}^{u} g(s)ds \geqslant G(\kappa u) + (1-\kappa)ug(\kappa u).$$

Thus, if $\Lambda_+(\kappa) = \infty$, then there exists $C_1 > 0$ such that for $u \geq C_1$ (C_1 is chosen so that $NG(\kappa u) - \frac{N-2}{2}g(u)u > 0$ for $\kappa u \geq C_1$) we infer

$$NG(u) - \frac{N-2}{2}g(u)u \geq NG(\kappa u) + N(\frac{1-\kappa}{\kappa})\kappa u g(\kappa u) - \frac{N-2}{2}g(u)u$$

$$\geq N(\frac{1-\kappa}{\kappa})\kappa u g(\kappa u) \geq N(\frac{1-\kappa}{\kappa})G(\kappa u) \geq (\frac{1-\kappa}{\kappa})\frac{N-2}{2}g(u)u.$$

This inequality and (2.11) for d sufficiently large yield

$$\int_0^{t_0} r^{N-1}[NG(u(r)) - \frac{N-2}{2}g(u(r))u(r)]dr \geq cg(\kappa d)d(d/g(d))^{N/2},$$

and this concludes the proof of the lemma.

We also observe that since

$$E'(t,d) = u'(t)p(t) - \frac{N-1}{t}(u'(t))^2 \leq |u'(t)|\|p\|_\infty \leq \sqrt{2}\|p\|_\infty\sqrt{E(t,d)},$$

then for $0 \leq t' \leq t < T$ and for $E(t',d)$ sufficiently large we have

$$E(t,d) \leq (\sqrt{E(t',d)} + (\sqrt{2}/2)\|p\|_\infty t)^2 \leq 3E(t',d).$$

§3. Main Results.

In order to state the next theorem we introduce the following notation

$$F(\kappa,\rho,d) := F(d) = (\frac{d}{g(d)})^{N+\rho-1}G(\kappa d), \tag{3.1}$$

where $\rho > 0$.

THEOREM 3.1. *If* $F(d) \to \infty$ *as* $d \to \infty$ *(respectively as* $d \to -\infty$*), then*

$$\lim_{d\to\infty} E(t,d) = \infty, \text{ (respectively } \lim_{d\to-\infty} E(t,d) = \infty)$$

uniformly for $t \in [0,T]$.

Proof. From the definition of energy and (1.1) we have

$$
\begin{aligned}
(E(r,d))' &= -\frac{N-1}{r}(u'(r))^2 + p(r)u'(r) \\
&\geq -\frac{N-1}{r}(r'(r))^2 - \sqrt{r/2\rho}\|p\|_\infty\sqrt{2\rho/r}|u'(r)| \\
&\geq -\frac{N-1}{r}(u'(r))^2 - \frac{r}{4\rho}\|p\|_\infty^2 - \frac{\rho}{r}(u'(r))^2 \qquad (3.2)\\
&= -\frac{2(N-1+\rho)}{r}E(r,d) + \frac{2(N-1+\rho)}{r}G(u(r)) - \frac{r}{4\rho}\|p\|_\infty^2 .
\end{aligned}
$$

Multiplying (3.2) by $r^{2(N-1+\rho)}$ we infer

$$
(r^{2(N-1+\rho)}E(r,d))' \geq r^{2(N-1+\rho)-1}G(u(r)) - \frac{\|p\|_\infty^2}{4\rho} r^{2(N-1+\rho)+1}. \qquad (3.3)
$$

By integrating (3.3) on $[t_o,t]$ and using (2.11) and the fact $G \geq 0$ we obtain

$$
\begin{aligned}
E(t,d) &\geq t^{-2(N-1+\rho)}\{t_o^{2(N-1+\rho)}E(t_o,d) \\
&\quad + \int_{t_o}^{t} r^{2(N-1+\rho)}[2(N-1+\rho)G(u(r)) - \frac{\|p\|_\infty^2}{4\rho} r^2]dr\} \\
&\geq T^{-2(N-1+\rho)}\{c(d/g(d))^{N-1+\rho}G(\kappa d)+m\}, \qquad (3.4)
\end{aligned}
$$

where $m \in \mathbb{R}$. Since by hypothesis $F(d) \to \infty$ as $d \to \infty$, from (3.4) the proof of the theorem follows.

THEOREM 3.2. *Suppose that for some* $a > 0$ $|g(u)| \geq a|u|$ *for* u *sufficiently large. If* $\Lambda(1,u)$ *is bounded below and for some* $\kappa \in (0,1)\Lambda_+(\kappa) = \infty$ *(respectively* $\Lambda_-(\kappa) = \infty$*) then*

$$
\lim_{d\to\infty} E(t,d) = \infty, \qquad \textit{(respectively } \lim_{d\to-\infty} E(t,d) = \infty)
$$

uniformly for $t \in [0,T]$.

Proof. In order to simplyfy the notations we write $u(t) := u(t,d)$. Multiplying (1.1) by $r^N u'$ and integrating over $[\hat{t},t]$, $0 \leq \hat{t} \leq t$, we get

$$
t^N E(t,d) - \hat{t}^N E(\hat{t},d) + \int_{\hat{t}}^{t} [\frac{N-2}{2}r^{N-1}(u'(r))^2 - Nr^{N-1}G(u(r))]dr = \int_{\hat{t}}^{t} p(r)r^N u'(r)dr, \qquad (3.5)
$$

where we have integrated by parts the term

$$\int_{\hat t}^{t} r^N[u''(r)u'(r) + (G(u(r)))']dr.$$

Similarly, multiplying (1.1) by $r^{N-1}u$ and integrating over $[\hat t, t]$ we infer

$$\int_{\hat t}^{t} r^{N-1}(u'(r))^2 dr = u'(t)u(t)t^{N-1} - u'(\hat t)u(\hat t)\hat t^{N-1} + \int_{\hat t}^{t} r^{N-1}[g(u(r))u(r) - p(r)u(r)]dr. \tag{3.6}$$

By replacing (3.6) in (3.5) we obtain

$$t^{N-1}[tE(t,d) + \frac{N-2}{2} u(t)u'(t)] = \hat t^{N-1}[\hat t E(\hat t,d) + \frac{N-2}{2} u(\hat t)u'(\hat t)] + \int_{\hat t}^{t} r^{N-1}[NG(u(r)) - \frac{N-2}{2} g(u(r))u(r)]dr + \int_{\hat t}^{t} p(r)r^{N-1}[ru'(r) + \frac{N-2}{2}u(r)]dr. \tag{3.7}$$

Since $u(t_o) \geq \kappa d$ and $u'(t_o) < 0$ (see (2.11)), after integrating by parts the term $\int_0^{t_o} \|p\|_\infty r^N u'(r)dr$, from (3.7) for d sufficiently large we obtain

$$\begin{aligned} t_o^{N-1}&[t_o E(t_o,d) + \frac{N-2}{2}u(t_o)u'(t_o)] \\ &\geq \int_0^{t_o} r^{N-1}[NG(u(r)) - \frac{N-2}{2} g(u(r))u(r)]dr \\ &\quad + \|p\|_\infty t_o^N u(t_o) - \int_0^{t_o} \|p\|_\infty [N + \frac{N-2}{2} r^{N-1}] u(r)dr \\ &\geq cdg(\kappa d)t_o^N + \|p\|_\infty t_o^N \kappa d - \|p\|_\infty [N + \frac{N-2}{2}]\frac{d}{N} t_o^N \\ &\geq cdg(\kappa d)t_o^N, \end{aligned} \tag{3.8}$$

where we have also used the fact that $g(\kappa d) \to \infty$ as $d \to \infty$.

In order to simply our notation we define:

$$H(t) := \left[tE(t,d) + \frac{N-2}{2} u(t)u'(t)\right]. \tag{3.9}$$

Therefore, from (3.8) we have

$$H(t_o) \geqslant cdg(\kappa d)t_o. \tag{3.10}$$

We claim that

$$|t_o u'(t_o) + \frac{N-2}{2} u(t_o)| < H(t_o). \tag{3.11}$$

If (3.11) does not hold then either

$$\textbf{a)} \quad u(t_o) \geqslant cH(t_o), \quad \text{or} \quad \textbf{b)} \quad |t_o u'(t_o)| \geqslant cH(t_o) \tag{3.12}$$

Since $\Lambda_+(\kappa) = \infty$, and g is increasing we have

$$\frac{N-2}{2N} dg(d) \leqslant G(\kappa d) \leqslant \kappa dg(\kappa d). \tag{3.13}$$

Therefore if (a) holds then by using (3.13) we infer

$$\kappa d > cdg(\kappa d)t_o \geqslant cdg(\kappa d)(d/g(d))^{\frac{1}{2}} \geqslant cdg(\kappa d)(d^2/G(\kappa d))^{\frac{1}{2}}$$
$$\geqslant cd\sqrt{G(\kappa d)},$$

which is a contradiction. Hence (a) cannot hold.

From (2.10) we see that for d sufficiently large $u'(t_o) \geqslant -\frac{2g(d)}{N} t_o$. Therefore if (b) holds then we have

$$\frac{2g(d)}{N} t_o^2 \geqslant |u'(t_o)| t_o \geqslant H(t_o) \geqslant cdg(\kappa d)t_o.$$

Hence,

$$g(d) \geqslant cd(g(\kappa d))^2. \tag{3.14}$$

On the other hand, from (3.13) we get that $g(d) \leqslant G(\kappa d)/d \leqslant cdg(\kappa d)/d \leqslant cd(g(\kappa d))$, which contadicts (3.14). Thus (3.11) hods.

Let now $t > t_o$ be such that for all $r \in [t_o, t]$

$$|ru'(r) + \frac{N-2}{2} u(r)| < H(r). \tag{3.15}$$

Since $\Lambda(1,u)$ is bounded below and $\Lambda_+(\kappa) = \infty$, we see that there exists a constant K such that

$$NG(u) - \frac{N-2}{2}g(u)u > K. \tag{3.16}$$

By combining (3.7) and (3.16) we infer

$$\begin{aligned}(r^{N-1}H(r))' &\geq -\|p\|_\infty r^{N-1}[|ru'(r) + \tfrac{N-2}{2}u(r)|] \\ &\quad + [NG(u(r)) - \tfrac{N-2}{2}g(u(r))u(r)]r^{N-1} \\ &\geq -\|p\|_\infty r^{N-1}H(r) - Kr^{N-1} \geq -r^{N-1}\|p\|_\infty H(r) - KT^{N-1}.\end{aligned} \tag{3.17}$$

Multiplying (3.17) by $e^{\|p\|_\infty r}$ we get

$$[e^{\|p\|_\infty r}\, r^{N-1}H(r)]' \geq -KT^{N-1}e^{\|p\|_\infty T}. \tag{3.18}$$

From (2.11), (3.13) and Lemma 2.1 we have

$$\begin{aligned}dg(\kappa d)t_o^N &\geq cdg(d)t_o^N \geq cdg(d)\left(\frac{d}{g(d)}\right)^{N/2} \geq cd^{1+(N/2)}(g(d))^{1-(N/2)} \\ &\geq cd^{1+(N/2)+q(1-(N/2))}.\end{aligned} \tag{3.19}$$

Hence, by integrating (3.18) on $[t_o,t]$ and using (3.10) and (3.19), for d sufficiently large we obtain

$$t^{N-1}H(t) \geq c[e^{-\|p\|_\infty T}t_o^{(N-1)}H(t_o) - 1] \geq cdg(\kappa d)t_o^N.$$

Therefore, (see (3.9)), we have that either

$$E(t,d) \geq cdg(\kappa d)t^{-N}t_o^N > cdg(\kappa d)T^{-N}t_o^N > cdg(\kappa d)t_o^N. \tag{3.20}$$

or else

$$u(t)u'(t) > cdg(\kappa d)t^{1-N}t_o^N \text{ and } E(t,d) > cdg(\kappa d)t^{-N}t_o^N. \tag{3.21}$$

From (3.21) we infer

$$|u(t)| > c\sqrt{dg(\kappa d)}\; t^{1-(N/2)}t_o^{N/2} > c\sqrt{dg(\kappa d)}\,t_o^{N/2}. \tag{3.22}$$

Since $E(t,d) \geqslant G(u(t)) \geqslant \frac{N-2}{2N} g(u(t))u(t)-K \geqslant c(u(t))^2$, from (3.20) and (3.22) we have

$$E(t,d) \geqslant cdg(\kappa d)t_o^N , \qquad (3.23)$$

for all $t \in [0,T]$ for which (3.15) holds.

If (3.15) does not hold for all $t \in [t_o,T]$, then let $\bar{t} \leqslant T$ be the such that (3.15) holds for all $t \in [t_o,\bar{t}]$ and not for $\bar{t}$. By the continuity of u and u' we have

$$|\bar{t}u'(\bar{t}) + \frac{N-2}{2}u(\bar{t})| = H(\bar{t}), \qquad (3.24)$$

and

$$\bar{t}^{N-1}H(\bar{t}) \geqslant cdg(\kappa d)t_o^N. \qquad (3.25)$$

From (3.24) and (3.25) we see that either

$$2\bar{t}|u'(\bar{t})| \geqslant H(\bar{t}) \geqslant cdg(\kappa d)(\bar{t})^{-N+1}t_o^N \qquad (3.26)$$

or

$$(N-2)|u(\bar{t})| \geqslant H(\bar{t}) \geqslant cdg(\kappa d)(\bar{t})^{-N+1}t_o^N. \qquad (3.27)$$

In either case

$$E(\bar{t},d) \geqslant cd^2(g(\kappa d))^2(\bar{t})^{-2(N-1)}t_o^{2N} \qquad (3.28)$$

where we have also used the fact that $|g(u)| \geqslant a|u|$ for u sufficiently large. By integrating (3.3) on $[\bar{t},t]$ and choosing $\rho \in (0,1)$ sufficiently small, from (3.19) and (3.28), for d sufficiently large, we obtain

$$\begin{aligned} E(t,d) &\geqslant T^{-2(N+\rho-1)}\{(\bar{t})^{2(N+\rho-1)}E(\bar{t},d) + m\} \\ &\geqslant T^{-2(N+\rho-1)}\{c(\bar{t})^{2(N+\rho-1)}d^2(g(\kappa d))^2(\bar{t})^{-2N+2}t_o^{2N} + m\} \\ &\geqslant T^{-2(N+\rho-1)}\{ct_o^2\ (dg(\kappa d)t_o^N)^2 + m\} \qquad (3.29) \\ &\geqslant T^{-2(N+\rho-1)}\{cd\ (g(d))^{-\rho}d^{2+N+q(2-N)} + m\} \\ &\geqslant cd^{\rho+N+2+q(2-\rho-N)}. \end{aligned}$$

Now, from (3.23), (3.19) and (3.29) the proof of the theorem follows.

§4. Applications. The existence of radially symmetric solutions to the boundary value proble

$$\begin{aligned} \Delta u + g(u) &= p(\|x\|) \quad x \in \Omega, \\ u &= 0 \quad x \in \delta\Omega, \end{aligned} \tag{4.1}$$

where Ω is the ball of radius T in $\mathbb{R}^N$ centered at the origin, is equivalent to the existence of solutions to (1.1) satisfying

$$u(T) = 0.$$

Combining the above results with the phase plane analysis in [2] it follows:

THEOREM B. *Suppose* $\lim_{|u|\to\infty} \frac{g(u)}{u} = \infty$. *If*

i) $\Lambda(1,u)$ *is bounded below and* $\Lambda_+(\kappa) = \infty$ *(respectively* $\Lambda_-(\kappa) = \infty$*) for some* $\kappa \in (0,1)$, *or*

ii) $F(d) \to \infty$ *as* $d \to \infty$ *(respectively* $F(d) \to \infty$ *as* $d \to -\infty$*),* *then* (4.1) *has infinitely many radially symmetric solutions with* $u(0) > 0$ *(respectively* $u(0) < 0$*).*

We observe that condition (i) in Theorem B includes cases such as $g(u) = u^{(N+2)/(N-2)}$ for either $u < 0$ or $u > 0$. Theorem B improves Theorem A of [2]. The reader is refered to [2] for details of the proof.

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Department of Mathematics
North Texas State University
Denton, Texas. 76203
U. S. A.

(Recibido en Octubre de 1986, revisado en abril de 1987).